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ON BANG-BANG PRINCIPLES IN LINEAR-QUADRATIC CONTROL PROBLEMS FOR GENERALIZED ANALYTIC FUNCTIONS

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In a recent paper [4] the author treated the linear-quadratic control problem for generalized analytic functions, which was first studied by C. and Cl. Simionescu (cf. [2]). In this paper we use the results of [4] for deriving some simple bang-bang properties for the optimal controls of such problems. The theorems obtained partly answer a question raised by W. Wendland after the author's lecture at the Conference on Complex Analysis at Halle in October 1980.

1. Distributed control

Let G be a bounded simply connected region in a complex z plane of class $C^1_\varkappa$, $0 < \varkappa \leqslant 1$, with boundary Γ, i.e., for Γ the representation $t = t(s)$, s arc length, holds with derivative $t'(s) \in C_\varkappa(\Gamma)$, the space of Hölder continuous functions on Γ with exponent $\varkappa$ (cf. [3], Chap. I, § 2).
The complex state functions $W(z)$ fulfil the differential equation

$$\partial W/\partial\bar{z} + a(z)W + b(z)\overline{W} = f(z) + \beta(z)V(z) \quad \text{in} \quad G \tag{1}$$

with complex coefficients $a(z)$, $b(z) \in L_p(G)$, $p > 2$, and $f(z)$, $\beta(z) \in L_q(G)$, $q \geqslant 4/3$, and the boundary condition

$$\mathrm{Re}[\overline{\lambda(t)}W] = g(t) \quad \text{on} \quad \Gamma \tag{2}$$

with a real-valued right-hand side $g(t) \in L_\gamma(\Gamma)$, $2 \leqslant \gamma < \infty$, and a complex coefficient $\lambda(t) \in C_\mu(\Gamma)$, $0 < \mu \leqslant 1$, satisfying the normality condition $\lambda(t) \neq 0$ on Γ and possessing a non-negative index $n = \mathrm{ind}\,\lambda = (1/2\pi)\cdot$ $\cdot[\arg\lambda(t)]_\Gamma$. The real-valued control functions $V(z)$, $z \in G$, lie in the admissible control set

$$U_{\mathrm{ad}} = \{V \in L_2(G)\colon |V(z)| \leqslant 1 \text{ a.e. in } G\}, \tag{3}$$

i.e., actually, $V(z) \in L_\infty(G)$. The state functions $W(z)$ belong to $L_r(G)$ and possess boundary values $W(t) \in L_\delta(\Gamma)$, where $r = 2\delta$ and δ with $2 \leqslant \delta < \infty$ is given by

$$\delta = \begin{cases} \min[q/(2-q), \gamma] & \text{as} \quad 4/3 \leqslant q < 2, \\ \gamma & \text{as} \quad q \geqslant 2. \end{cases}$$

We wish to minimize the cost functional

$$J = \int_\Gamma |QW(t) - h(t)|^2 ds, \tag{4}$$

where $h(t) \in L_2(\Gamma)$ is a given complex function and

$$QW(t) = q_1(t)W(t) + q_2(t)\overline{W(t)} \tag{5}$$

with given complex functions $q_1(t)$, $q_2(t) \in L_\alpha(\Gamma)$, $\alpha = 2\delta/(\delta-2)$.

For an optimal control function $U(z)$ with the corresponding optimal state function $W(z)$ the following *necessary and sufficient optimality condition* holds (cf. [4], formula (64)):

$$\iint_G \operatorname{Re}[\beta(z)Z(z)]\,[U(z) - V(z)]\,dx\,dy \geqslant 0 \quad \forall V \in U_{\mathrm{ad}}, \tag{6}$$

where the uniquely determined adjoint state function $Z(z)$ is the solution of the boundary value problem

$$\partial Z/\partial\bar z - a(z)Z - b(z)\bar Z = 0 \quad \text{in} \quad G, \tag{7}$$

$$\operatorname{Re}[\lambda(t)t'(s)Z] = 2\operatorname{Im}[\lambda(t)\overline{\sigma(t)}] \quad \text{on} \quad \Gamma \tag{8}$$

with

$$\sigma(t) = \overline{q_1(t)}\eta(t) + q_2(t)\overline{\eta(t)}, \tag{9}$$

$$\eta(t) = QW(t) - h(t) = q_1(t)W(t) + q_2(t)\overline{W(t)} - h(t). \tag{10}$$

The adjoint state function $Z(z)$ belongs to $L_\varrho(G)$ and possesses boundary values $Z(t) \in L_\nu(\Gamma)$, where $\varrho = 2\nu$ and $\nu = \delta/(\delta-1)$ with $1 < \nu \leqslant 2$.

Remark. In the case where $g(t) \in L_2(\Gamma)$, i.e., $\gamma = 2$ with bounded measurable functions $q_k(t)$, $k = 1, 2$, one also has $\delta = 2$ and $\nu = 2$ with functions $\eta(t)$, $\sigma(t) \in L_2(\Gamma)$.

From (6) one gets

$$U(z) = \operatorname{sign}\operatorname{Re}[\beta(z)Z(z)] \quad \text{if} \quad \operatorname{Re}[\beta(z)Z(z)] \neq 0. \tag{11}$$

Therefore, one can obtain *bang-bang assertions* for the optimal control function $U(z)$ by studying the relation

$$\operatorname{Re}[\beta(z)Z(z)] = 0 \tag{12}$$

for the generalized analytic function $Z(z)$ in subregions G_0 of G. A natural assumption for the validity of a bang-bang principle is here not only the known condition $J_{\min} > 0$, i.e., $\eta(t) \neq 0$ on a set of positive measure on Γ, but also the additional condition that

$$K(t) = \operatorname{Im}[\lambda(t)\overline{\sigma(t)}] \neq 0 \tag{13}$$

on a set of positive measure on Γ. Namely, if condition (13) is not fulfilled, in virtue of (7) and (8) one has $Z(z) \equiv 0$ in G because of the assumption $n = \operatorname{ind}\lambda \geqslant 0$. In particular, the situation $K(t) \equiv 0$ on Γ occurs for the trivial functional (4) with $QW(t) = \operatorname{Re}[\overline{\lambda(t)}W]$, where because of the boundary condition (2) the value of the functional is wholly independent of the control.

At first we deal with the important special *case of constant coefficients* $a(z) = a$, $b(z) = b$ in (1) with $\beta(z) = c+id$, where c, d are real constants with $c^2+d^2 > 0$. For $a = b = 0$ with $c = 0$ this case contains the control problem in the theory of whirls considered by Cl. Simionescu [2].

THEOREM 1. *In the control problem* (1)–(5) *with constant coefficients* $a(z) = a$, $b(z) = b$ *and* $\beta(z) \equiv \beta \neq 0$ *let* $J_{\min} > 0$. *Let* $U(z)$ *be an optimal control function with the corresponding optimal state function* $W(z)$ *for which condition* (13) *is satisfied and, moreover, the function* $K(t)$ *in* (13) *is not a multiple of the function*

$$M(t) \equiv \operatorname{Im}[\bar{\beta}\lambda(t)t'(s)]\cdot\exp\{2\operatorname{Re}([a-\bar{b}\cdot e^{2i\arg\beta}]\bar{t})\} \quad \text{on} \quad \Gamma. \tag{14}$$

Then for the corresponding adjoint state function $Z(z)$ *we have the relation*

$$\operatorname{Re}[\beta Z(z)] \neq 0 \quad \text{in} \quad G-G_1 \tag{15}$$

with an exceptional subset G_1 *of* G *containing no interior point. The optimal control function* $U(z)$ *is bang-bang in* $G-G_1$ *and given by formula* (11).

Furthermore, $U(z)$ *is the unique optimal control function of the problem in* $G-G_1$.

Remark. With respect to the uniqueness of the optimal state function $W(z)$ see [4].

Proof. Suppose that, to the contrary, for the adjoint state function $Z(z)$ the relation $\operatorname{Re}[\beta Z(z)] = 0$ holds in a subset G_0 of G containing an interior point. Together with the differental equation (7) this yields

$$Z(z) = Ci\bar{\beta}\exp\left\{2\operatorname{Re}\left(\left[a-\bar{b}\cdot\frac{\beta}{\bar{\beta}}\right]\bar{z}\right)\right\} \tag{16}$$

with an arbitrary real constant C, say in a disk of a sufficiently small positive radius lying in the interior of G_0. Owing to the unique continuation property of generalized analytic functions (cf. [3], Chap. III, § 4, Th. 3.5), relation (16) holds in the whole region G. The assumptions (13) and (14) now lead to a contradiction with the boundary condition (8) for $Z(z)$ on Γ. Therefore, inequality (15) is proved.

To prove the uniqueness of $U(z)$ we have to show that all optimal control functions must be bang-bang in a set of type $G-G_1$. Then the uniqueness of $U(z)$ follows in a well-known way from the fact that the set of optimal control functions is convex. Now suppose that $U_1(z)$, $W_1(z)$ is a pair of optimal control and state function, where $U_1(z)$ does not possess the bang-bang property, i.e., for $U_1(z)$, $W_1(z)$ the corresponding function $K_1(t)$ of (13) is (identically zero or) a multiple of the function $M(t)$ in (14). But then the function $K_2(t) = (1/2)[K(t)+K_1(t)]$ belonging to the (non bang-bang) optimal control function $U_2(z) = (1/2)[U(z)+U_1(z)]$ with $W_2(z) = (1/2)[W(z)+W_1(z)]$ does not possess this property because $K(t)$ does not have it. This means that, in view of the first part of the theorem, $U_2(z)$ and hence $U_1(z)$ must in fact be bang-bang. This completes the proof.

Remarks 1. In the case of real-valued functions $h(t)$ and $QW(t)$, i.e., if $q_2(t) = \overline{q_1(t)}$, one has $\sigma(t) = 2\overline{q_1(t)}\eta(t)$ with a real-valued function $\eta(t)$ and $K(t) = 2\eta(t)\mathrm{Im}[\lambda(t)q_1(t)]$. Then (13) takes the form

$$(13') \qquad \eta(t)\cdot\mathrm{Im}[\lambda(t)q_1(t)] \neq 0$$

on a set of positive measure on Γ. In particular, for the important case $QW(t) = \mathrm{Im}[\overline{\lambda(t)}W]$ condition (13) coincides with the condition $J_{\min} > 0$.

2. A theorem analogous to Theorem 1 holds in the case of a piecewise constant coefficient $\beta(z)$ taking constant values $\beta_k \neq 0$ in subdomains G_k, $k = 1, \ldots, m$ of a finite decomposition of G, where condition (14) is satisfied for each corresponding function $M_k(t)$. Further, a corresponding statement to that in Theorem 1 is true if we define the control function $V(z)$ on a subdomain G_0 of G only putting formally $\beta(z) \equiv 0$ outside G_0 in (1).

The following simple examples show the necessity of the additional assumptions (13) and (14) and also give an application of the theorem.

EXAMPLE 1. For the equation

$$(17) \qquad \partial W/\partial\bar{z} = iV(z) \quad \text{in the unit disk } G\colon |z| < 1$$

with the boundary condition

$$(18) \qquad \mathrm{Re}\,W = \sin s \quad \text{on} \quad \Gamma\colon |t| = 1,\ t = e^{is},$$

and the functional

$$(19) \qquad J = \int_\Gamma |W(t)|^2 ds$$

the functions $U(z) = 1/2$ and $W(z) = (i/2)(\bar{z}-z)$ are obviously optimal — we have $\sigma = \eta = \sin s$ and $K(t) \equiv 0$. Condition (13) is not fulfilled and $U(z)$ is not bang-bang.

EXAMPLE 2. For equation (17) with the boundary condition

$$\text{(20)} \qquad \operatorname{Re} W = \cos s \quad \text{on} \quad \Gamma\colon |t| = 1, \quad t = e^{is},$$

and functional (19) the functions $U(z) \equiv 0$ and $W(z) = z$ are optimal. Namely, $\sigma = \eta = t$ and $K(t) = -\sin s$, and therefore $Z(z) = 2$ and $\operatorname{Re}[\beta Z(z)] \equiv 0$ in G so that the optimality condition (6) is satisfied. Condition (13) is fulfilled, but $K(t)$ is a multiple of $M(t) = \sin s$ and $U(z)$ is not bang-bang.

EXAMPLE 3. For equation (17) with the boundary condition

$$\text{(21)} \qquad \operatorname{Re} W = -\sin s \quad \text{on} \quad \Gamma\colon |t| = 1, \quad t = e^{is},$$

and the functional

$$\text{(22)} \qquad J = \int_{\Gamma} |W(t) - 2i\bar{t}|^2 ds$$

the functions $U(z) = 1$ and $W(z) = i\bar{z}$ are optimal. Namely, $\sigma = \eta = -i\bar{t}$ and $K(t) = \cos s$, and therefore $Z(z) = -2i$ and $\operatorname{Re}[\beta Z(z)] = 2$ in G so that (6) is satisfied. Further, $M(t) = \sin s$ and Theorem 1 applies. $U(z)$ is bang-bang and is the unique optimal control function.

We now turn to the general case of *non-constant coefficients* $a(z)$, $b(z)$ and $\beta(z) \neq 0$ a.e. in G. A solution $Z(z)$ of equation (7) satisfying relation (12) in a subregion G_0 of G is also a solution of the equation

$$\text{(23)} \qquad \partial Z/\partial \bar{z} - \left[a(z) - \frac{\overline{\beta(z)}}{\beta(z)}\overline{b(z)}\right] Z = 0 \quad \text{in} \quad G_0,$$

which by means of the well-known Theodorescu formula (cf. [3], Chap. III, § 4, (4.6)) has the general solution

$$\text{(24)} \qquad Z(z) = A(z)\Phi(z)$$

with

$$\text{(25)} \qquad A(z) = \exp\left(-\frac{1}{\pi}\iint_{G} \frac{1}{\zeta - z}\left[a(\zeta) - \frac{\overline{\beta(\zeta)}}{\beta(\zeta)}\overline{b(\zeta)}\right] d\xi\, d\eta\right)^{*}$$

and an arbitrary holomorphic function $\Phi(z)$. Therefore, relation (12) for $Z(z)$ is equivalent to the relation

$$\text{(26)} \qquad \operatorname{Re}[\gamma(z)\Phi(z)] = 0 \quad \text{in} \quad G_0, \quad \gamma(z) = A(z)\beta(z),$$

* Obviously, in (25) the integral may be extended over the whole region G instead of G_0.

for the holomorphic function $\Phi(z)$. A solution Φ of (26), and therefore a solution Z of (12) are uniquely determined apart from a real constant factor because from $\operatorname{Re}[\gamma(z)\Phi_k(z)] = 0$ in G_0 for $k = 1, 2$ it follows that the meromorphic function $\Phi_1(z)/\Phi_2(z)$ in G_0 must be a real-valued function.

In general the solution $Z(z)$ of relation (12) depends on the subregion G_0 under consideration and it seems difficult to obtain a general expression for the solution of (12) according to G_0. We therefore confine ourselves to the following special case.

ASSUMPTION D. The function $\gamma(z) = A(z)\beta(z)$ with $A(z)$ given by (25) allows a factorization of the form $\gamma(z) = \gamma_1(z)\cdot\gamma_2(z)$, where the function

$$\varphi(z) = \frac{\overline{\gamma_2(z)}}{\gamma_1(z)} \tag{27}$$

is a (not necessarily regular) analytic function in G.

Then, of course, the (uniquely determined) solution Φ of (26) for all subregions G_0 of G is simply $\Phi(z) = Ci\varphi(z)$ and correspondingly the solution Z of (12) has the form

$$Z(z) = CiA(z)\varphi(z) \tag{28}$$

with an arbitrary real constant C.

In the same way as Theorem 1 we now get

THEOREM 2. *In the control problem* (1)–(5), *where* $a(z), b(z)$ *and* $\beta(z) \neq 0$ *in* G *fulfil Assumption* D, *let* $J_{\min} > 0$. *Let* $U(z)$ *be an optimal control function with a corresponding optimal state function* $W(z)$ *for which condition* (13) *is satisfied.*

Moreover, if the analytic function $\varphi(z)$ *in* (27) *is holomorphic in* G *with* $\varphi(z) \in L_\varrho(G)$ *and boundary values* $\varphi(t) \in L_\nu(\Gamma)$, *the function* $K(t)$ *in* (13) *shall not be a multiple of the function*

$$M(t) \equiv \operatorname{Im}[\lambda(t)t'(s)A(t)\varphi(t)] \tag{29}$$

on Γ. *Then for the corresponding adjoint state function* $Z(z)$ *we have relation* (15). $U(z)$ *is given by formula* (11) *and it is the unique optimal control function of the problem.*

Remarks. 1. If $\varphi(z)$ is not holomorphic in G with $\varphi(z) \in L_\varrho(G)$ and $\varphi(t) \in L_\nu(\Gamma)$, in particular, if $\varphi(z)$ has poles in $\bar{G}$ or essential singularities in G, assumption (13) alone is sufficient for the validity of the assertion of Theorem 2.

2. In the case of Theorem 1 we have

$$A(z) = \exp\left(\left[a - \frac{\beta}{\bar{\beta}}\bar{b}\right]\bar{z}\right)\cdot A_0(z) \tag{30}$$

with a holomorphic function $A_0(z)$ in G, which is continuous and different from zero in $\bar{G}$. Therefore, we may put

$$\gamma_1(z) = A_0(z), \quad \gamma_2(z) = \beta \exp\left(\left[a - \frac{\beta}{\bar{\beta}} \bar{b}\right] \bar{z}\right)$$

and obtain (14).

3. Particular classes of functions $\gamma(z)$ satisfying Assumption D are meromorphic functions, functions with meromorphic $\overline{\gamma(z)}$, and the products of such functions. Some simple concrete examples are given by the functions

(31) $\gamma_1(z) = z^{\lambda/2}, \quad \gamma_2(z) = \bar{z}^{\lambda/2}, \quad \lambda \geqslant 0;$

(32) $\gamma_1(z) = z^{n/2}, \quad \gamma_2(z) = (\bar{z})^{-n/2}, \quad n \in \boldsymbol{N};$

(33) $\gamma_1(z) = e^{\beta_1 z}, \quad \gamma_2(z) = e^{\beta_2 \bar{z}}, \quad \beta_1, \beta_2 \in \boldsymbol{C};$

(34) $\gamma_1(z) = e^{\frac{1}{2z}(i\mu_1 - \mu_2)}, \quad \gamma_2(z) = e^{\frac{1}{2\bar{z}}(i\mu_1 + \mu_2)}, \quad \mu_1, \mu_2 \in \boldsymbol{R};$

respectively.

2. Boundary control

Let the domain G be as before. The state functions $W(z)$ now satisfy the differential equation

(35) $$\partial W / \partial \bar{z} + a(z) W + b(z) \overline{W} = f(z) \quad \text{in} \quad G$$

with complex coefficients $a(z)$, $b(z) \in L_p(G)$, $p > 2$, and $f(z) \in L_q(G)$, $q \geqslant 4/3$, and the boundary condition

(36) $$\operatorname{Re}[\overline{\lambda(t)} W] = g(t) + \delta(t) v(t) \quad \text{on} \quad \Gamma$$

with real-valued functions $g(t)$, $\delta(t) \in L_\gamma(\Gamma)$, $2 \leqslant \gamma < \infty$, and a complex coefficient $\lambda(t) \in C_\mu(\Gamma)$, $0 < \mu \leqslant 1$, where $\lambda(t) \neq 0$ on Γ and $n = \operatorname{ind} \lambda \geqslant 0$. The (real-valued) control functions $v(t)$, $t \in \Gamma$, are taken from the control set

(37) $$U_{\mathrm{ad}} = \{v \in L_2(\Gamma): |v(t)| \leqslant 1 \text{ a.e. on } \Gamma\},$$

i.e., actually, $v(t) \in L_\infty(\Gamma)$. Again the state functions $W(z)$ belong to $L_r(G)$ and possess boundary values $W(t) \in L_\delta(\Gamma)$. The cost functional J is the same as that given above by (4) with (5).

The *necessary and sufficient optimality condition* for an optimal control function $u(t)$ with the corresponding optimal state function $W(z)$ now reads (cf. [4], formula (65)):

(38) $$\int_\Gamma \delta(t) \zeta(t) [v(t) - u(t)] ds \geqslant 0 \qquad \forall v \in U_{\mathrm{ad}},$$

where

$$\zeta(t) = \frac{1}{|\lambda(t)|^2}\left\{\frac{1}{2}\operatorname{Im}[\lambda(t)t'(s)Z(t)] + \operatorname{Re}[\lambda(t)\overline{\sigma(t)}]\right\} \tag{39}$$

and $Z(z)$ is again the solution of the adjoint boundary value problem (7), (8) with (9), (10).

From (38) one obtains the expression

$$u(t) = -\operatorname{sign}[\delta(t)\zeta(t)] \quad \text{if} \quad \delta(t)\zeta(t) \neq 0 \tag{40}$$

for the optimal control function $u(t)$. The form of the function $\zeta(t)$ in (39) suggests the following simple *bang-bang statement.*

THEOREM 3. *If for an optimal control function $u(t)$ with the corresponding optimal state function $W(z)$ the relations*

$$\operatorname{Im}[\lambda(t)\overline{\sigma(t)}] \equiv 0 \text{ a.e. on } \Gamma \tag{41}$$

and

$$\operatorname{Re}[\lambda(t)\overline{\sigma(t)}] \neq 0 \quad \text{on} \quad \gamma \subset \Gamma \text{ with } \operatorname{mes}\gamma > 0 \tag{42}$$

are fulfilled, then $\zeta(t) \neq 0$ a.e. on γ and, if $\delta(t) \neq 0$ on γ, expression (40) *holds for $u(t)$ on γ. In particular, for $\gamma = \Gamma$ the optimal control function $u(t)$ is bang-bang.*

Proof. Assumption (41) implies $Z(z) \equiv 0$ in G for the solution of (7), (8) and so in virtue of assumption (42) the function $\zeta(t)$ in (39) is different from zero on γ.

The following theorem yields a *weak bang-bang property* of the optimal control function.

THEOREM 4. *If for an optimal control function $u(t)$ with the corresponding optimal state function $W(z)$ the function*

$$\psi(t) = \overline{it'(s)\sigma(t)} \quad \text{on} \quad \Gamma \tag{43}$$

does not represent the limit function of a (regular) solution $Z(z)$ of equation (7) *in G, then we have $\zeta(t) \neq 0$ on a subset γ of Γ with positive measure and, if further $\delta(t) \neq 0$ a.e. on Γ, expression* (40) *holds for $u(t)$ on γ.*

Proof. If $\zeta(t) = 0$ a.e. on Γ, the relation

$$\operatorname{Im}[\lambda(t)t'(s)Z(t)] = -2\operatorname{Re}[\lambda(t)\overline{\sigma(t)}] \quad \text{on} \quad \Gamma$$

follows, which together with the boundary condition (8) yields $Z(t) = \overline{-2it'(s)\sigma(t)}$ on Γ, i.e., function (43) has to be the limit value of a solution $Z(z)$ of equation (7) in G.

Remark. A necessary and sufficient condition for a (Hölder continuous) function $\psi(t)$ on Γ to be the limit value of a generalized analytic function $Z(z)$ in G is given in [3], Chap. III, § 14, 14.2.

EXAMPLE 4. For the equation

$$\partial W/\partial\bar z = 0 \quad \text{in the unit disk } G\text{: } |z|<1 \tag{44}$$

with the boundary condition

$$\operatorname{Re} W = v(t) \quad \text{on} \quad \Gamma\text{: } |t| = 1, \quad t = e^{is}, \tag{45}$$

and the functional

$$J = \int_\Gamma |W(t)-\bar t|^2 ds \tag{46}$$

the functions $u(t) \equiv 0$ with $W(z) \equiv 0$ are optimal. Namely, we have $\sigma = \eta = -\bar t$; therefore $Z(z) = 2$ in G and $\zeta(t) \equiv 0$ on Γ so that the optimality condition (38) is fulfilled. The function $\psi(t) = -1$ on Γ is the limit value of the holomorphic function $Z_0(z) = -1$ in G and $u(t)$ does not possess the weak bang-bang property.

We now deal with the situation where the boundary Γ consists of two disjoint measurable parts Γ_1 and Γ_2, say unions of finitely many arcs. The control functions $v(t)$ are defined on Γ_1, i.e.,

$$U_{ad} = \{v \in L_2(\Gamma_1)\text{: } |v(t)| \leqslant 1 \text{ a.e. on } \Gamma_1\} \tag{37'}$$

and $\delta(t) \equiv 0$ on Γ_2 in (36). The cost functional is extended over Γ_2 only, i.e.,

$$J = \int_{\Gamma_2} |QW(t)-h(t)|^2 ds, \tag{4'}$$

and so $\sigma(t) \equiv 0$ on Γ_1. The optimality condition (38) then takes the form

$$\int_{\Gamma_1} \delta(t)\zeta(t)\,[v(t)-u(t)]ds \geqslant 0 \qquad \forall v \in U_{ad} \tag{38'}$$

with

$$\zeta(t) = \frac{1}{2|\lambda(t)|^2}\operatorname{Im}[\lambda(t)t'(s)Z(t)] \quad \text{on} \quad \Gamma_1, \tag{39'}$$

where in the boundary condition (8) for the adjoint state function $Z(z)$ the right-hand side vanishes on Γ_1.

The following *strong bang-bang principle* holds:

THEOREM 5. *In the control problem* (35), (36), (37′), (4′) *with* $\delta(t) \equiv 0$ *on* Γ_2 *and* $\delta(t) \neq 0$ *a.e. on* Γ_1 *let* $J_{\min} > 0$. *Let* $u(t)$ *be an optimal control function with the corresponding optimal state function* $W(z)$ *for which the* *condition*

$$K(t) = \mathrm{Im}[\lambda(t)\overline{\sigma(t)}] \neq 0 \tag{13'}$$

on a set of positive measure on Γ_2 is fulfilled.

Then we have $\zeta(t) \neq 0$ a.e. on Γ_1 and the optimal control function $u(t)$ is bang-bang, given by (40), *and uniquely determined.*

Proof. If $\zeta(t) = 0$ on a subset γ_1 of Γ_1 with positive measure, by (39') and the boundary condition (8) on Γ_1, this implies $Z(t) = 0$ on γ_1 for the boundary values of $Z(z)$. The well-known Vekua-Privalov theorem for generalized analytic functions (cf. [3], Chap. III, § 4, Th. 3.6 and [1], Chap. X, § 2, Th. 1) yields $Z(z) \equiv 0$ in G and the boundary condition (8) on Γ_2 contradicts assumption (13'). The uniqueness of the optimal control function $u(t)$ can be proved as in Theorem 1.

Example 5. We consider the homogeneous equation (35) with $g(t) = 0$ on a subset γ_2 of Γ_2 with positive measure and either $g(t) \neq 0$ on a subset of positive measure on Γ_2 or $|g(t)/\delta(t)| > 1$ on a subset of positive measure on Γ_1 in the boundary condition (36). The cost functional is

$$J = \int_{\Gamma_2} |\mathrm{Im}[\overline{\lambda(t)}W]|^2 ds. \tag{47}$$

Obviously, the corresponding optimal state function $W(z)$ cannot vanish identically in G. Therefore, taking into account the boundary condition (36) on Γ_2, one obtains $\eta(t) = \mathrm{Im}[\overline{\lambda(t)}W] \neq 0$ a.e. on γ_2. Further, we have $\sigma(t) = i\lambda(t)\eta(t)$ for the functional (47). Hence, assumption (13') is satisfied and Theorem 5 applies.

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