ELEMENTARY AND ANALYTIC THEORY OF NUMBERS
BANACH CENTER PUBLICATIONS, VOLUME 17
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

NOTES ON THE METHOD OF FROBENIAN FUNCTIONS WITH APPLICATIONS TO FOURIER COEFFICIENTS OF MODULAR FORMS

R. W. K. ODONI

Department of Mathematics, University of Exeter
Exeter, Great Britain

Introduction

In these notes we describe some number-theoretic techniques which can be used to investigate the behaviour of the Fourier coefficients of certain modular forms, particularly those associated with continuous representations of Gal $(\bar{Q}/Q)$ ($\bar{Q}$ = algebraic closure of the rational field Q) in GL (C). We shall show that the study of these Fourier coefficients often reduces to the study of multiplicative functions whose values may be integers in some algebraic number field K, or else in some commutative monoid. In particular, there is considerable interest in divisibility and congruence properties of these coefficients (see Serre [33], Kolberg [14]), and also in the frequency of occurrence of given values of the coefficients, while much effort has been expended on growth properties (see, e.g. Deligne [6], [7], Deligne–Serre [9]). In some cases we can adapt standard methods of analytic number theory to obtain asymptotic expansions for the number of $n \leqslant x$ for which the corresponding Fourier coefficients a_n have the required properties. In other cases our analysis leads to deep problems associated with the desingularisation of complex algebraic varieties; here we lack a complete solution, although relatively recent advances, due to B. Malgrange and others, allow us to deal with many individual cases. The major focus in the present notes is the "local problem", in which we consider the distribution of n for which a_n takes some fixed value in a number field; it is this case which causes the greatest difficulties. We develop the appropriate machinery in Sections 1–5. In Section 6 we outline the appropriate changes needed to deal with multiplicative functions with values in a finite monoid, in particular when the functions are Frobenian; there is a surprisingly large class of distribution problems associated with algebraic number fields which can be reduced to questions of this type; as a special case we may cite the "norm density" problems treated (by different methods) in some of the author's earlier papers ([26]–[29]).

In Section 7 we set up a general framework for the analysis of divisibility problems for multiplicative functions with values in an algebraic number field. Finally, in Section 8 we explain how the work of Sections 1–7 can be applied to modular forms. In the references we include not only papers directly cited in these notes, but also other work closely related to the topics discussed here.

Notation. Throughout these notes we use the following conventions: K, L – algebraic number fields, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ the fields of rational, real and complex numbers, respectively; $\bar{\boldsymbol{Q}}$ is some fixed algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$. $\boldsymbol{T}$ denotes the torus $\{z \in \boldsymbol{C};\ |z| = 1\}$, $\boldsymbol{T} \cong \boldsymbol{R}/\boldsymbol{Z}$, where $\boldsymbol{Z}$ is the ring of rational integers. $\boldsymbol{Z}_K$ denotes the ring of integers of K. When $y \in \boldsymbol{C}$ we abbreviate $\exp(2\pi i y)$ to $e(y)$. $\boldsymbol{F}_q$ denotes the finite field with q elements. When A is a set we denote its cardinality by $\# A$. $\boldsymbol{N}$ denotes the set of natural numbers $\geqslant 1$.

1. Multiplicative harmonic analysis

Let K be an algebraic number field, $\boldsymbol{Z}_K$ its ring of integers, and let $0 \neq \alpha \in \boldsymbol{Z}_K$. Our aim in this chapter is to express (for ξ a variable in $\boldsymbol{Z}_K$) the function

$$\delta_\alpha(\xi) = \begin{cases} 1 & \text{if } \xi = \alpha, \\ 0 & \text{if not} \end{cases}$$

in terms of multiplicative functions of ξ. Once this has been achieved we shall be able to proceed with the analysis of value-distribution for multiplicative functions with values in $\boldsymbol{Z}_K$.

1A. Normalised valuations on K (Lang [15]). The non-archimedean valuations of K arise as follows. For each non-zero prime ideal $\mathfrak{p}$ in $\boldsymbol{Z}_K$, and non-zero β in K, we define $v_{\mathfrak{p}}(\beta)$ to be the exact exponent of $\mathfrak{p}$ occurring in the prime ideal factorisation of the fractional ideal $(\beta) = \beta\boldsymbol{Z}_K$. We convert $v_{\mathfrak{p}}$ (an additive valuation of K) into a *normalised multiplicative valuation* by putting $|\beta|_{\mathfrak{p}} = (N\mathfrak{p})^{-v_{\mathfrak{p}}(\beta)}$, where $N_{\mathfrak{p}} := \# \boldsymbol{Z}_K/\mathfrak{p}$.

The *archimedean valuations* arise as follows. Choose λ such that $K = \boldsymbol{Q}[\lambda]$. Amongst the $[K:\boldsymbol{Q}]$ distinct algebraic conjugates of λ over $\boldsymbol{Q}$ we assume that r of them (labelled $\lambda_1, \ldots, \lambda_r$) are real and $2c = [K:\boldsymbol{Q}] - r$ of them are non-real. The latter occur in complex conjugate pairs. From each such pair we choose one conjugate; these are labelled $\lambda_{r+1}, \ldots, \lambda_{r+c}$. We now define $r+c$ independent *normalised multiplicative valuations* of K as follows. If $p(x) \in \boldsymbol{Q}[x]$ we write $|p(\lambda)|_j$ for $|p(\lambda_j)|$, where $|\cdot|$ is the ordinary absolute value on $\boldsymbol{R}$ when $j \leqslant r$, and the square of the ordinary absolute value on $\boldsymbol{C}$ when $j > r$. With these definitions we have the product formula

$$(1.1) \qquad \prod_{0 \neq \mathfrak{p} \triangleleft \boldsymbol{Z}_K} |\beta|_{\mathfrak{p}} \cdot \prod_{j \leqslant r+c} |\beta|_j = 1 \quad \text{for all } 0 \neq \beta \in K,$$

in view of the classical formula $N(\beta \boldsymbol{Z}_K) = |N_{K/\boldsymbol{Q}}(\beta)|$ for $0 \neq \beta \in \boldsymbol{Z}_K$.

1B. *T*-units. Returning now to our problem of expressing $\delta_\alpha(\xi)$ multiplicatively, we let T be the finite set consisting of all archimedean valuations of K, together with the (finite) set of $|\cdot|_{\mathfrak{p}}$ with $v_{\mathfrak{p}}(\alpha) > 0$. The subgroup K_T of $K^* := K \setminus 0$, consisting of all $\beta \in K^*$ with $v_{\mathfrak{p}}(\beta) = 0$ for all $\mathfrak{p} \notin T$, is called the *group of T-units of K*. Clearly $\alpha \in K_T$, so that $\xi \in K_T$ is a necessary condition for $\xi = \alpha$. Now the set $K_T \cap \boldsymbol{Z}_K$ is a *saturated multiplicatively closed subset* of $\boldsymbol{Z}_K$, i.e. it has the property that, for all x, y in $\boldsymbol{Z}_K$, $xy \in K_T \cap \boldsymbol{Z}_K$ if and only if both x and $y \in K_T \cap \boldsymbol{Z}_K$. Thus the function f_T: $\boldsymbol{Z}_K \to \boldsymbol{F}_2$,

$$(1.2) \qquad f_T(x) = \begin{cases} 1 & \text{if } x \in K_T \cap \boldsymbol{Z}_K, \\ 0 & \text{if not} \end{cases}$$

satisfies $f_T(xy) = f_T(x) f_T(y)$ for all x, y in $\boldsymbol{Z}_K$. It is clear that $f_T(\xi)$ is a factor of $\delta_\alpha(\xi)$, and we now seek the other factors.

1C. The logarithmic mapping (Lang [15]). We now fix a suitable labelling for the members of T. We label the non-archimedean valuations by listing the $\mathfrak{p}$ with $v_{\mathfrak{p}}(\alpha) > 0$ as $\mathfrak{p}_{r+c+1}, \ldots, \mathfrak{p}_t$, where $t = \# T$; for these we write $\|\beta\|_j$ instead of $|\beta|_{\mathfrak{p}_j}$. For the archimedean valuations we use the same labels as in 1A, writing $\|\beta\|_j$ in place of $|\beta|_j$, for $j \leqslant r+c$. We now map K_T into $\boldsymbol{R}^t$ via $\beta \mapsto \mathbf{log}\, \beta := (\log \|\beta\|_1, \ldots, \log \|\beta\|_t)$. The product formula (1.1) shows that $\log K_T$ is contained in the hyperplane $X_1 + \ldots + X_t = 0$. But in fact we have the much more precise

T-UNIT THEOREM. $\log K_T$ *is a free abelian group of rank* $(t-1)$, *contained in the hyperplane* $X_1 + \ldots + X_t = 0$ *in* $\boldsymbol{R}^t$.

($\mathbf{log}$ is obviously a homomorphism: $K_T \to \{\boldsymbol{R}^t; +\}$; further, $\mathbf{log}\, K_T$ is a *discrete* subspace of $\boldsymbol{R}^t$; this follows from the fact that, for a given $A > 0$ in $\boldsymbol{R}$, only finitely many β in $\boldsymbol{Z}_K$ can have $|\beta|_j < A$ for all $j \leqslant r+c$. This already proves that $\mathbf{log}\, K_T$ is free abelian of rank $\leqslant t-1$; to prove that the rank is actually $t-1$ requires an argument of Minkowski type).

The kernel of $\mathbf{log}$ consists of those elements η of K^* for which $v_{\mathfrak{p}}(\eta) = 0$ for all $\mathfrak{p} \neq 0$ in $\boldsymbol{Z}_K$ (and hence are units in $\boldsymbol{Z}_K$), and which also satisfy $|\eta|_j = 1$ for all $j \leqslant r+c$. The latter equations have only finitely many solutions in $\boldsymbol{Z}_K$, so that η must belong to the (finite cyclic) group of all roots of unity in K. The latter group is obviously contained in ker $\mathbf{log}$, so that they coincide.

Clearly, when $\xi \in K_T \cap \boldsymbol{Z}_K$, a necessary condition for $\xi = \alpha$ is that $\mathbf{log}\, \xi$

$= \log \alpha$ and this condition can be neatly expressed in terms of the characters of $\log K_T$. Let $e_1, \ldots, e_{t-1}$ be any Z-basis of $\log K_T \subseteq R^t$. Regarding $e_1, \ldots, e_{t-1}$ as the rows of a $(t-1)\times t$ matrix $\mathbf{E}$ of rank $t-1$, $\log K_T$ will consist of all row vectors $x = n\mathbf{E}$ in R^t, as n varies over all row vectors in Z^{t-1}. To recover n from x we postmultiply by $\mathbf{E}^*$, where $\mathbf{E}\mathbf{E}^* = \mathbf{I}_{t-1}$ is the $(t-1)$-square identity matrix. Thus $n = x\mathbf{E}^*$. The character group of $\log K_T$ is now obtained by choosing $t-1$ independent variables u_i in $T = R/Z$, and defining $\psi(x) = e(x\mathbf{E}^* u^+)$, where $e(y) = e^{2\pi i y}$ and u^+ is the transpose of u. This gives a 1:1 correspondence between choices of u in T^{t-1} and characters of $\log K_T$, while the orthogonality relations for characters yield

$$(1.3) \qquad \int_{T^{t-1}} \psi(\log \xi)\bar{\psi}(\log \alpha)\, du_1 \ldots du_{t-1} = \begin{cases} 1 & \text{if } \log \xi = \log \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For fixed $u \in T^{t-1}$, the function $\xi \mapsto f_T(\xi)\psi(\log \xi)$ is completely multiplicative on Z_K, and (1.3) shows that we can express the characteristic function for the property $\xi \in K_T \cap Z_K$, $\log \xi = \log \alpha$, as a fixed "linear combination" of such functions.

1D. There now remains the problem: *given that* $\xi \in K_T \cap Z_K$ *and* $\log \xi = \log \alpha$, *when does* $\xi = \alpha$? Since $\xi\alpha^{-1}$ must now lie in ker $\log$, it must be a root of unity, and we need to decide which one it is. We do this by the following simple trick. Choose some fixed prime ideal $\mathfrak{p}^* \neq 0$ in Z_K such that $\mathfrak{p}^* \notin T$, and such that distinct roots of unity in K are incongruent (mod $\mathfrak{p}^*$). (There are infinitely many such $\mathfrak{p}^*$; for the condition on roots of unity is simply that $(0 \neq)\ \Pi(\zeta - \zeta') \notin \mathfrak{p}^*$, where ζ and ζ' run over all pairs of distinct roots of unity in K; there are certainly infinitely many such $\mathfrak{p}^*$, while T is finite). Let χ be a Dirichlet character (mod $\mathfrak{p}^*$); thus a character of $(Z_K/\mathfrak{p}^*)^*$ lifted to $Z_K \backslash \mathfrak{p}^*$, and then extended to Z_K by putting $\chi(\gamma) = 0$ when $\gamma \in \mathfrak{p}^*$. Then we have

$$(1.4) \qquad (N\mathfrak{p}^* - 1)^{-1} \sum_{\chi} \chi(\xi)\bar{\chi}(\alpha) = \begin{cases} 1 & \text{if } \xi \equiv \alpha \ (\text{mod } \mathfrak{p}^*), \\ 0 & \text{otherwise,} \end{cases}$$

where χ runs over all the $N\mathfrak{p}^* - 1$ Dirichlet characters (mod $\mathfrak{p}^*$). If now $\xi \in K_T \cap Z_K$, $\log \xi = \log \alpha$ and also $\xi \equiv \alpha$ (mod $\mathfrak{p}^*$), then $\xi = \alpha\tau$, where τ is a root of unity congruent to 1 (mod $\mathfrak{p}^*$), and so $\tau = 1$ by our hypotheses on $\mathfrak{p}^*$. The individual functions

$$(1.5) \qquad \omega(T, \chi, u; \xi) := f_T(\xi)\chi(\xi)\psi(\log \xi)$$

are completely multiplicative from Z_K into $0 \cup T$, and (1.3) and (1.4) show that $\delta_\alpha(\xi)$ is expressible as a fixed finite linear combination of integrals

$$(1.6) \qquad \int_{T^{t-1}} \omega(T, \chi, u; \xi)\bar{\psi}(\log \alpha)\, du_1 \ldots du_{t-1}.$$

1E. Some variants.

I. The condition $\xi = \alpha$ can be relaxed to $\xi \in A$, A some finite subset of Z_K. Indeed, $\delta_A(\xi) = \sum_{\alpha \in A} \delta_\alpha(\xi)$.

II. We may consider finitely many α_i in Z_K, and the simultaneous equations $\xi_i = \alpha_i$ for all i, which we abbreviate to $\underline{\xi} = \underline{\alpha}$. The characteristic function $\delta_{\underline{\alpha}}(\underline{\xi})$ is now $\prod_i \delta_{\alpha_i}(\xi_i)$.

III. Let S be a finite set of valuations of K including all the archimedeans. We denote by Z_K^S the set of all *S-integers of* K thus the set of all $\gamma \in K$ with $v_\mathfrak{p}(\gamma) \geqslant 0$ when $\mathfrak{p} \notin S$. Now let $\alpha \neq 0$, $\alpha \in Z_K^S$, and consider the problem: *when does the variable* ξ *in* Z_K^S *coincide with* α? To deal with this we replace the set T of § 1A by $T^* = T \cup S$. Then $\alpha \in K_{T^*}$, and the analysis of § 1A–§ 1D goes through, with T replaced by T^* and f_T replaced by $f^*\colon Z_K^S \to F_2$,

$$f^*(\xi) = \begin{cases} 1 & \text{if } \xi \in Z_K^S \cap K_{T^*}, \\ 0 & \text{if not.} \end{cases}$$

IV. We may combine I–III in various obvious ways. Thus, in II, we can take the α_i in different fields. The most general useful variant is the following. Take finitely many fields K_i, and, for each i, a finite set S_i of valuations of K_i, including the archimedeans. For each i, let α_i be a non-zero member of $Z_{K_i}^{S_i}$, and let ξ_i vary over $Z_{K_i}^{S_i}$. Then the methods of Section 1 allow us to express the simultaneous equations $\xi_i = \alpha_i$ ($\forall\, i$) in terms of multiplicative functions of the ξ_i. The same is still true if we replace α_i by a finite set.

2. Harmonic analysis of multiplicative functions

2A. Let $\theta\colon N \to Z_K$ be a multiplicative function, i.e. $\theta(mn) = \theta(m)\theta(n)$ whenever $(m, n) = 1$. We consider the distribution of those $n \in N$, $n \leqslant x$ (x large) such that $\theta(n) = \alpha$, some fixed member of $Z_K \setminus 0$. In the notation of Section 1 the number of such n is $\sum_{n \leqslant x} \delta_\alpha(\theta(n))$. We introduce the Dirichlet series

$$D(\theta; \alpha; s) = \sum_{n=1}^{\infty} \delta_\alpha(\theta(n)) n^{-s}, \tag{2.1}$$

where $s = \sigma + i\tau$, $\sigma, \tau \in R$, $\sigma > 1$. It is holomorphic for $\sigma > 1$ and, moreover, when $a > 1$, we have

$$\sum_{n \leqslant x} \delta_\alpha(\theta(n)) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{s} D(\theta; \alpha; s)\, ds, \tag{2.2}$$

by Perron's summation formula. Our plan is to obtain (where possible) a continuation of D to the left of $\sigma = 1$; provided we can discover enough information about the singularities of D, (2.2) should give an asymptotic expansion for $\sum_{n \leqslant x} \delta_\alpha(\theta(n))$ as $x \to \infty$. In practice it is better to work with the weighted summation formulae

$$\sum_{n \leqslant x} \delta_\alpha(\theta(n)) \log \frac{x}{n} = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} \frac{x^s}{s^2} D(\theta; \alpha; s)\, ds \tag{2.3}$$

or

$$\sum_{n \leqslant x} \delta_\alpha(\theta(n)) \left(1 - \frac{n}{x}\right) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} \frac{x^s}{s(s+1)} D(\theta; \alpha; s)\, ds, \tag{2.4}$$

since the extra factors in the denominators of the integrals ensure absolute convergence, which facilitates the estimation of error terms. One may pass easily from asymptotic expansions of (2.3) or (2.4) to that of (2.2), by means of Tauberian theorems of a rather simple kind. (See e.g. [33]).

2B. We now consider $D(\theta; \alpha; s)$ in the light of Section 1. By (1.6), $\delta_\alpha(\theta(n))$ is a fixed finite linear combination of integrals

$$\int_{T^{t-1}} \bar{\psi}(\log \alpha)\, \omega(T; \chi; \boldsymbol{u}; \theta(n))\, d\boldsymbol{u}, \tag{2.5}$$

with coefficients which are independent of n. Moreover $\omega(T; \chi; \boldsymbol{u}; \theta(n))$ is a *multiplicative function of* n (when the other parameters are fixed). It follows that, for $\sigma > 1$, $D(\theta; \alpha; s)$ is the same linear combination of the integrals

$$\int_{T^{t-1}} \bar{\psi}(\log \alpha)\, \Lambda(T; \theta; \chi; \boldsymbol{u}; s)\, d\boldsymbol{u}, \tag{2.6}$$

where

$$\Lambda(T; \theta; \chi; \boldsymbol{u}; s) := \sum_{n=1}^{\infty} n^{-s} \omega(T; \chi; \boldsymbol{u}; \theta(n)) \tag{2.7}$$

for $\sigma > 1$. Thus, to obtain the asymptotics of (2.3), we need those of

$$I(x; \alpha; \theta; \chi) := \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} \frac{x^s}{s^2} \left\{ \int_{T^{t-1}} \bar{\psi}(\log \alpha)\, \Lambda(T; \theta; \chi; \boldsymbol{u}; s)\, d\boldsymbol{u} \right\} ds. \tag{2.8}$$

We may think of Λ in (2.7) as a sort of "pseudo L-function"; for $\sigma > 1$, Λ has an Euler product $\prod_{p \text{ prime}} \Lambda_p(T; \theta; \chi; \boldsymbol{u}; s)$, where

$$\Lambda_p = 1 + \sum_{k \geqslant 1} p^{-ks} \omega(T; \chi; \boldsymbol{u}; \theta(p^k)), \tag{2.9}$$

and it is this feature which allows us to obtain the analytic continuation of Λ (and hence of D) in some important special cases. Roughly speaking, if the sets of primes for which $\omega(T; \chi; \boldsymbol{u}; \theta(p))$ takes given values are sufficiently "regular", then a continuation will be possible: see § 3.

2C. Some variants.

I. We may consider θ defined on integral ideals of a number field, and satisfying $\theta(\mathfrak{ab}) = \theta(\mathfrak{a})\theta(\mathfrak{b})$ when $(\mathfrak{a}, \mathfrak{b}) = 1$, in which case we seek an asymptotic expansion for $\sum_{N\mathfrak{a} \leqslant x} \delta_\alpha(\theta(\mathfrak{a}))$. More generally, we may restrict the $\mathfrak{a}$ to ideals composed only of prime ideals in some suitably "regular" set.

II. We may consider finitely many θ_i simultaneously, using the function $\prod_i \delta_{\alpha_i}(\theta_i(n))$; the products $\prod_i \omega(T_i; \chi_i; \boldsymbol{u}_i; \theta_i(n))$ are still multiplicative functions of n.

III. The condition $\theta(n) = \alpha$ can be relaxed to $\theta(n) \in A$, with A finite.

IV. As remarked at the end of Section 1, we may replace $\boldsymbol{Z}_K \backslash 0$ by $\boldsymbol{Z}_K^S \backslash 0$, the set of non-zero S-integers of K. The changes needed are straightforward, and the details will be omitted.

3. The Λ-function in the Frobenian case

3A. In a number of important applications the multiplicative function θ in Section 2 is *Frobenian*, that is, there exists an algebraic number field L, Galois over $\boldsymbol{Q}$, such that, for primes p unramified in $L/\boldsymbol{Q}$, the values $\theta(p^n)$ ($n \geqslant 1$) depend solely on the Frobenius conjugacy class Frob $p := \left(\frac{L/\boldsymbol{Q}}{p}\right)$ in Gal $L/\boldsymbol{Q} := G$ (see Odoni [23], [24]). In this case we can obtain some very precise information about the function Λ of Section 2. For simplicity we shall assume that $\theta(p^n) = 0$ whenever p is ramified; the general case is dealt with at the end of this section.

Suppose that θ is Frobenian; then the same is true of $\omega(T; \chi; \boldsymbol{u}; \theta(n))$. Consider the Euler product (2.9). Here Λ_p depends only on Frob p, so that we may write $\Lambda(T; \chi; \boldsymbol{u}; s) = \prod_C \Lambda_C(T; \chi; \boldsymbol{u}; s)$, where C runs over all the (finitely many) conjugacy classes in G, and $\Lambda_C := \prod_{\mathrm{Frob}\, p = C} \Lambda_p$. For $\sigma = \mathrm{Re}\, s > 1$, we can take logarithms (using the branch which attributes real logarithms to positive reals), obtaining

$$\log \Lambda_C = \sum_{\mathrm{Frob}\, p = C} \log \left\{1 + \sum_{k \geqslant 1} p^{-ks} \omega(T; \chi; \boldsymbol{u}; \theta(p^k))\right\}. \tag{3.1}$$

Since θ is Frobenian we may write $\omega(T; \chi; \boldsymbol{u}; \theta(p^k))$ as $\omega_k(C; \boldsymbol{u})$, where we regard T and χ as fixed. Hence

$$\log \Lambda_C = \sum_{\mathrm{Frob}\, p = C} \log \{1 + \sum_{k \geqslant 1} p^{-ks} \omega_k(C; \boldsymbol{u})\}. \tag{3.2}$$

3B. If y_k ($k \geqslant 1$) and x are indeterminates, it is easily seen that, as formal power series,

$$\log \{1 + \sum_{k \geqslant 1} y_k x^k\} \equiv \sum_{m \geqslant 1} x^m P_m(y_1, \ldots, y_m), \tag{3.3}$$

where P_m is a polynomial over $\mathbf{Q}$, involving at most $y_1, \ldots, y_m$. If we substitute uniformly bounded complex numbers for the y_k then (3.3) also gives a convergent power series expansion, provided x is a suitably small complex variable. If we write

$$\omega^{(m)}(C; \boldsymbol{u}) := P_m(\omega_1(C; \boldsymbol{u}), \ldots, \omega_m(C; \boldsymbol{u})), \tag{3.4}$$

then (3.2) and (3.3) yield

$$\log \Lambda_C = \sum_{\mathrm{Frob}\, p = C} \sum_{k \geqslant 1} p^{-ks} \omega^{(k)}(C; \boldsymbol{u}) = \sum_{k \geqslant 1} \omega^{(k)}(C; \boldsymbol{u}) \{ \sum_{\mathrm{Frob}\, p = C} p^{-ks}\}. \tag{3.5}$$

This is, in fact, valid at least for $\sigma > 1$, since the $\omega_k(C; \boldsymbol{u})$ are all of unit modulus and each $p \geqslant 2$. The Chebotarev density theorem ([33]) asserts that

$$\sum_{\mathrm{Frob}\, p = C} p^{-z} = (G:C)^{-1} \log \frac{1}{z-1} + R_C(z) \quad (\mathrm{Re}\, z > 1), \tag{3.6}$$

where $(G:C) = \# G / \# C$ and $R_C(z)$ is holomorphic and grows no faster than $\log \log \{2 + |\mathrm{Im}\, z|\}$ as $|\mathrm{Im}\, z| \to \infty$ in the region

$$\mathrm{Re}\, z \geqslant 1 - \frac{c(L)}{\{\log (|\mathrm{Im}\, z| + 2)\}^{A(L)}}, \tag{3.7}$$

$c(L)$ and $A(L)$ being positive parameters depending only on L. Hence we have

$$\log \Lambda_C = \sum_{k=1}^{\infty} \omega^k(C; \boldsymbol{u}) \left\{ (G:C)^{-1} \log \frac{1}{ks-1} + R_C(ks) \right\}, \tag{3.8}$$

in the first place for $\sigma > 1$, but we may use the right-hand side of (3.8) to extend the definition of $\log \Lambda_C$, as follows. If $\varkappa(C)$ is the smallest integer $k \geqslant 1$ such that $\omega^{(k)}(C; \boldsymbol{u})$ is not identically zero in $\boldsymbol{u}$, the summation in (3.8) reduces to $k \geqslant \varkappa(C)$, and the right-hand side is holomorphic, provided $z = s\varkappa(C)$ satisfies (3.7) and we cut the s-plane leftwards from $(\varkappa(C))^{-1}$ along the real axis, to avoid the branch points which arise from the terms $\log (ks-1)$.

If we now add (3.8) over all conjugacy classes C in G, this yields a continuation of $\log \Lambda$ into the region corresponding to $\varkappa = \min_{C} \varkappa(C)$. In practice this $\varkappa$ is almost invariably 1, the cases $\varkappa \geqslant 2$ usually arising from some degeneracy in θ.

3C. Remarks and examples.

I. In the above we assumed that $\theta(p^n) = 0$ whenever p is ramified in L/Q. If this is not the case then the Euler product (2.9) also involves factors Λ_p corresponding to ramified p, and these Λ_p may not $= 1$. However, there are only finitely many ramified p. It is then easy to see that the corresponding function $\prod_{p \text{ ramified}} \Lambda_p$ is holomorphic for $\sigma > 0$, and thus offers no obstruction to the continuation of Λ, since the relevant singularities of the Λ_C, given by (3.8), are to the right of $\sigma = 0$; it is these which give rise to the dominant terms in the asymptotic expansions which we shall derive later.

II. Some examples of Frobenian multiplicative functions:

1. Let θ be a Dirichlet character derived from $(Z/qZ)^*$; we take $L = Q(e(1/q))$. That θ is Frobenian is a direct translation of Artin's reciprocity theorem in classfield theory, admittedly applied to a simple special case!

2. Let $\{f_i(\boldsymbol{x})\}_{i \in I}$ be any family of polynomials in $Z[\boldsymbol{x}] = Z[x_1, \ldots, x_n]$. We let $\theta(n) = 1$ if there exists a solution $y \in Z^n$ of the simultaneous congruences $f_i(y) \equiv 0 \pmod{n}$, $\forall i \in I$, and put $\theta(n) = 0$ otherwise. θ is multiplicative, by the Chinese remainder theorem, while a theorem of J. Ax (see Serre [35], Odoni [25]) shows the existence of a finite Galois extension L/Q such that θ is Frobenian relative to L/Q.

3. Let L/Q be any finite Galois extension and let D be some multiple of the discriminant of L/Q. Suppose that ϱ is some representation of Gal L/Q into GL (n, C). Let $L(s)$ be the (trivially altered) Artin L-series

$$\prod_{p \nmid D} \det \{\mathbf{I}_n - p^{-s} \varrho(\text{Frob } p)\}^{-1} \quad (\sigma = \operatorname{Re} s > 1).$$

We can expand this as a Dirichlet series $\sum_{n \geqslant 1} \theta(n) n^{-s}$, and the coefficients $\theta(n)$ are clearly Frobenian multiplicative, with values in the ring $Z[e(1/q)]$, where q is the exponent of Gal L/Q.

4. Let $\tau(n)$ be Ramanujan's function, given by the identity

$$\sum_{n \geqslant 1} \tau(n) z^n \equiv z \prod_{k \geqslant 1} (1 - z^k)^{24},$$

and let p be a fixed prime. $\tau(n)$ is a multiplicative with values in Z, so that θ: $n \mapsto \tau(n) \pmod{p}$ is certainly multiplicative, with values in Z/pZ. Deep results due to Deligne imply that θ is also Frobenian, relative to a certain finite Galois extension L_p/Q (see Serre [33]).

5. Frobenian multiplicative functions also occur when examining density

questions associated with norms of algebraic integers, or of ideals in various orders (such as integral group rings). For particular instances see Section 6. In such cases it is often convenient to allow θ to take values in some commutative monoid M. The following properties of such are of considerable utility:

(i) *If* $\theta\colon N \to M_1$ *is Frobenian multiplicative, and* $f\colon M_1 \to M_2$ *is a monoid morphism, then* $n \mapsto f(\theta(n))$ *is also Frobenian multiplicative.*

(ii) *If* $\theta_i\colon N \to M_i$ $(1 \leqslant i \leqslant k)$ *are Frobenian multiplicative, then so is* $\boldsymbol{\theta}\colon n \mapsto (\theta_i(n)) \in M_1 \times \ldots \times M_k$.

Indeed, if θ_i is Frobenian relative to L_i/Q (finite Galois) then $\boldsymbol{\theta}$ is Frobenian relative to $\prod L_i/Q$, where $\prod L_i$ is the compositum of the L_i. For there exists a natural injection $\mathrm{Gal} \prod L_i/Q \hookrightarrow \bigoplus_i \mathrm{Gal}\, L_i/Q$, and projections $\mathrm{Gal} \prod L_i/Q \twoheadrightarrow \mathrm{Gal}\, L_i/Q$, which respect the Frobenius classes of primes p unramified in $\prod L_i/Q$.

We can use (i) and (ii) to deal with "simultaneous distribution problems", e.g. to handle $\# \{n;\ 1 \leqslant n \leqslant x,\ \theta_i(n) = \mu_i \in M_i\}$ or $\# \{n;\ 1 \leqslant n \leqslant x,\ \theta_1(n)\theta_2(n) = \mu\}$, with $\theta_1,\ \theta_2\colon N \to M$ Frobenian multiplicative.

4. Asymptotics for $\theta(n) = \alpha$ in the Frobenian case

4A. As remarked in Section 2 the sum $\sum_{n \leqslant x} \delta_\alpha(\theta(n))$ and its asymptotics are closely related to the integrals

$$(4.1)\qquad I(x;\alpha;\theta;\chi) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2}\left\{\int_{T^{t-1}} \bar{\psi}(\log \alpha)\Lambda(T;\theta;\chi;\boldsymbol{u};s)\,d\boldsymbol{u}\right\} ds.$$

When θ is Frobenian we may use the analysis of Section 3 to obtain asymptotic expansions, at least "generically", as we shall see later in this section. Our first step here is to use Cauchy's theorem to deform the vertical s-contour from $a-i\infty$ to $a+i\infty$ into one which exploits the analytic continuation and singularities of the Λ-function. We begin by interchanging the order of integration in (4.1); we study for fixed $\boldsymbol{u} \in \boldsymbol{T}^{t-1}$ the integral

$$(4.2)\qquad J(x;T;\theta;\chi;\boldsymbol{u}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2}\Lambda(T;\theta;\chi;\boldsymbol{u};s)\,ds.$$

We note that (3.8) implies that Λ in (4.2) is holomorphic and uniformly bounded as $|\tau| \to \infty$ in the closed half-plane $\varkappa\sigma \geqslant 1+\varepsilon$ ($\varepsilon > 0$ arbitrary), where $\varkappa$ is defined below (3.8). Thus we can certainly replace a by any real number $>\varkappa^{-1}$. But we can do much better than this. We cut the s-plane along the real semi-axis $s \leqslant \varkappa^{-1}$, and deform the vertical contour from $a-i\infty$ to $a+i\infty$ into the union of 5 contours $\mathscr{C} = \mathscr{C}_1^- \cup \mathscr{C}_2^- \cup \mathscr{C}_3 \cup \mathscr{C}_2^+ \cup \mathscr{C}_1^+$, where $\mathscr{C}_3$ is the anticlockwise circle $|s-\varkappa^{-1}| = \varepsilon$, $\left\{\begin{matrix}\mathscr{C}_2^+\\ \mathscr{C}_2^-\end{matrix}\right.$ is the $\left\{\begin{matrix}\text{upper}\\ \text{lower}\end{matrix}\right.$ edge of the cut, taken in the sense $\left\{\leftrightarrows\right.$ for $1-\dfrac{c(L)}{(\log 2)^{A(L)}} \leqslant \varkappa s \leqslant 1-\varkappa\varepsilon$, with the notation of (3.7), and $\left\{\begin{matrix}\mathscr{C}_1^+\\ \mathscr{C}_1^-\end{matrix}\right.$ is the contour $\varkappa\sigma = 1-\dfrac{c(L)}{\{\log(2+\varkappa|\tau|)\}^{A(L)}}$ for $\left\{\begin{matrix}\tau \geqslant 0\\ \tau \leqslant 0\end{matrix}\right.$ and increasing. Here $\varepsilon > 0$ is arbitrarily small.

By (3.8) we can write

$$A(T;\theta;\chi;\boldsymbol{u};s) = \exp\left\{\Omega(\boldsymbol{u})\log\frac{1}{\varkappa s-1}\right\} H(s;\boldsymbol{u}) \tag{4.3}$$

for $z = \varkappa s$ satisfying (3.7), where

$$\Omega(\boldsymbol{u}) := \sum_C (G:C)^{-1}\,\omega^{(\varkappa)}(C;\boldsymbol{u}) \tag{4.4}$$

and $H(s;\boldsymbol{u})$ is holomorphic when $z^* := (1+\varkappa)s$ satisfies (3.7) with z^* replacing z, and so certainly on $\mathscr{D}_0 := \mathscr{C}_2^- \cup \mathscr{C}_3 \cup \mathscr{C}_2^+$.

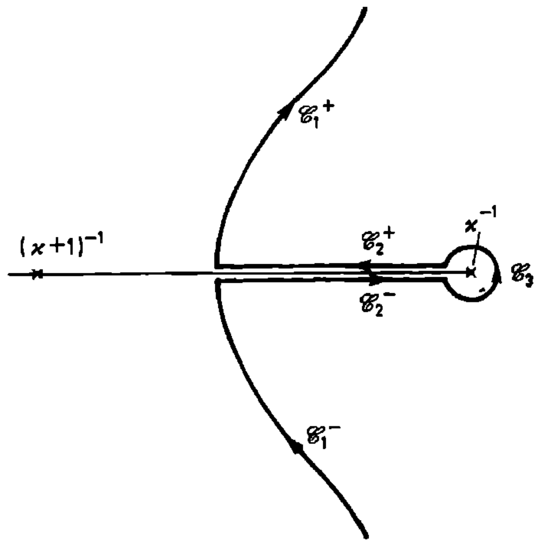

Fig. 1

Moreover, the $\omega^{(m)}(\boldsymbol{u};C)$ are, by (3.4), trigonometric polynomials in $\boldsymbol{u}$, so that $H(\sigma+i\tau,\boldsymbol{u})$ is holomorphic in $\boldsymbol{u}\in T^{t-1}$ and grows no faster than a uniformly bounded power of $\log|\tau|$ as $|\tau|\to\infty$ on $\mathscr{C}$. Standard arguments (analogous to the estimation of the remainder term in the proof of de la Vallée-Poussin version of the prime number theorem (Ingham [13])) now show that

$$\int_{\mathscr{C}_1^- \cup \mathscr{C}_1^+} \frac{x^s}{s^2} \Lambda(T; \theta; \chi; u; s)\, ds = O\big(x^{\varkappa^{-1}} \exp(-B\sqrt{\log x})\big), \tag{4.5}$$

where $B = B(T, \theta, \chi)$ is independent of u, as is the implied constant in $O(\ldots)$. We shall see that (4.5) is, in general, negligible compared to the integral along $\mathscr{D}_0$.

4B. We now deal with the integral along $\mathscr{D}_0$. We first write $\varkappa s = 1 + \varkappa w$, so that, by (4.3),

$$\int_{\mathscr{D}_0} \frac{x^s}{s^2} \Lambda(T; \theta; \chi; u; s)\, ds = \int_{\mathscr{D}} \frac{x^{\varkappa^{-1}+w}}{(w+\varkappa^{-1})^2} \Lambda(T; \theta; \chi; u; w+\varkappa^{-1})\, dw \tag{4.6}$$

$$= x^{\varkappa^{-1}} \int_{\mathscr{D}} x^w (\varkappa^{-1}+w)^{-2} H(\varkappa^{-1}+w, u) \exp\{\Omega(u)(\log w^{-1} + \log \varkappa^{-1})\}\, dw,$$

where $\mathscr{D}$ is $\mathscr{D}_0$ translated by $-\varkappa^{-1}$. We now write

$$G(w; u) = (w+\varkappa^{-1})^{-2} H(\varkappa^{-1}+w, u) \exp\{\Omega(u) \log \varkappa^{-1}\}. \tag{4.7}$$

This function is certainly holomorphic in w in some closed disc whose interior contains the whole of $\mathscr{D}$, and so we have the Taylor series

$$G(w; u) = \sum_{m=0}^{\infty} \frac{w^m}{m!} G^{(m)}(0; u), \tag{4.8}$$

certainly uniformly and absolutely convergent for $w \in \mathscr{D}$. The integral (4.6) is now

$$x^{\varkappa^{-1}} \int_{\mathscr{D}} x^w \exp(-\Omega(u) \log w) \sum_{m=0}^{\infty} \frac{w^m}{m!} G^{(m)}(0; u)\, dw \tag{4.9}$$

$$= x^{\varkappa^{-1}} \sum_{m=0}^{\infty} \frac{G^{(m)}(0; u)}{m!} \int_{\mathscr{D}} x^w w^m \exp(-\Omega(u) \log w)\, dw$$

$$= x^{\varkappa^{-1}} \sum_{m=0}^{\infty} \frac{G^{(m)}(0; u)}{m!} \int_{\mathscr{D}} \exp\big(w \log x + (m - \Omega(u)) \log w\big)\, dw.$$

4C. Let $\mathscr{D}^+$ consist of $\mathscr{D}$, extended out to $-\infty$ along each edge of the w-cut (which runs along the real semi-axis $w \leqslant 0$). It is a well-known theorem due to Hankel (see Copson [4]) that

$$\lambda^{-1} e^{z\log\lambda} \Gamma(z)^{-1} = \frac{1}{2\pi i} \int_{\mathscr{D}^+} e^{\lambda w - z\log w} dw \tag{4.10}$$

when $\lambda > 0$, where Γ is the classical gamma-function. Let $-\delta$ denote the endpoints of $\mathscr{D}$. Then, along either edge of the cut, we have, for large $\lambda > 0$, and bounded Im z, with Re z bounded below,

$$\begin{aligned}\Big| \int_{w \leqslant -\delta} e^{\lambda w - z\log w} dw \Big| &\leqslant A_1 \int_{\delta}^{\infty} e^{-\lambda y} e^{-\operatorname{Re} z(\log y \pm i\pi)} dy \leqslant A_2 \int_{\delta}^{\infty} e^{-\lambda y} y^{-\operatorname{Re} z} dy \\ &\leqslant A_2 \int_{\delta}^{\infty} e^{-\lambda y/2} \sup_{y \geqslant \delta} \{e^{-\lambda y/2} y^{-\operatorname{Re} z}\} dy \leqslant A_3 e^{-\lambda y/4},\end{aligned}$$

where the A_i are positive constants; here $\delta > 0$.

It follows from this and (4.10) that

$$\frac{1}{2\pi i} \int_{\mathscr{D}} \exp\{w \log x + (m - \Omega(\boldsymbol{u})) \log w\} dw = \frac{e^{(\Omega(\boldsymbol{u}) - m)\log\log x}}{\log x \Gamma(\Omega(\boldsymbol{u}) - m)} + O(x^{-\delta'}), \tag{4.11}$$

where $O(\ldots)$ is uniform for $\boldsymbol{u} \in \boldsymbol{T}^{t-1}$ and $\delta' > 0$. Consequently (4.9) and (4.5) yield an asymptotic expansion

$$\begin{aligned} &J(x; T; \theta; \chi; \boldsymbol{u}) \\ &= \frac{x^{\varkappa - 1} e^{\Omega(\boldsymbol{u})\log\log x}}{\log x} \left\{ \sum_{m=0}^{M} \frac{G^{(m)}(0; \boldsymbol{u})(\log x)^{-m}}{m!\, \Gamma(\Omega(\boldsymbol{u}) - m)} + O(\log x)^{-1-M} \right\}, \end{aligned} \tag{4.12}$$

the functions $x^{-\delta'}$ and $\exp(-B\sqrt{\log x})$ being asymptotically negligible, relative to the scale $(\log x)^{-m}$ $(m \geqslant 0)$.

4D. To obtain the asymptotics of I, we now need to multiply (4.12) by $\bar{\psi}(\log \alpha)$, and then integrate over all $\boldsymbol{u} \in \boldsymbol{T}^{t-1}$, and the problem reduces to finding the asymptotics of

$$\int_{\boldsymbol{T}^{t-1}} \bar{\psi}(\log \alpha) \exp\{\Omega(\boldsymbol{u}) \log\log x\} \frac{G^{(m)}(0; \boldsymbol{u})}{\Gamma(\Omega(\boldsymbol{u}) - m)} d\boldsymbol{u} \tag{4.13}$$

as $x \to \infty$. We write $\lambda = \log\log x$ and call the integral in (4.13) $\mathscr{I}_m(\lambda)$.

We now recall that $\Omega(\boldsymbol{u})$ is a *trigonometric polynomial*, i.e. a member of the ring $\boldsymbol{C}[z_1, \ldots, z_{t-1}, z_1^{-1}, \ldots, z_{t-1}^{-1}]$ of Laurent polynomials in the variables $z_j = e(u_j)$. Moreover (4.7) and (4.3) also show that $G^{(m)}(0, \boldsymbol{u})$ is an entire function of the variables $\omega^{(n)}(\boldsymbol{u}; C)$ and hence holomorphic in $z_1, \ldots, z_{t-1}$, except for possible singularities where $\prod z_j = 0$. Also $(\Gamma(w))^{-1}$ is an entire function of $w \in C$, so that $(\Gamma(\Omega(u)-m))^{-1}$ is holomorphic in z, except where $\prod z_j = 0$. Thus the change of variables $z_j = e(u_j)$ in (4.13) leads to the consideration of the asymptotics of integrals of the type

$$\mathscr{J}(\lambda) = \int_{\underbrace{S^1 \times \ldots \times S^1}_{t-1}} e^{\lambda R(z)} F(z)\, dz_1 \wedge \ldots \wedge dz_{t-1} \tag{4.14}$$

where S^1 is the unit circle (oriented anticlockwise), $R(z)$ is a polynomial in z, divided by a monomial, and $F(z)$ is holomorphic for $\prod z_j \neq 0$.

5. The asymptotics of $\mathscr{J}(\lambda)$

It is clearly desirable to obtain an asymptotic expansion for $\mathscr{J}(\lambda)$ of (4.14) in terms of a scale of simpler functions of λ, such as exponentials or powers. The sheer generality of $\mathscr{J}(\lambda)$ makes this a very difficult problem; indeed, to date, there is no complete solution available. However, thanks to the efforts of B. Malgrange and B. A. Vasil'ev, based on Hironaka's theory of resolution of singularities, one can at least solve the problem for "most" $R(z)$.

5A. $\mathscr{J}(\lambda)$ represents a generalization to several complex variables of the classical *saddle-point integrals*

$$\int_{\mathscr{C}} e^{\lambda f(z)} g(z)\, dz \qquad (\lambda \to +\infty), \tag{5.1}$$

where $\mathscr{C}$ is a contour in C, lying in some domain where f and g are both holomorphic. In mathematical physics integrals such as (5.1) are of very frequent occurrence, and their asymptotics are often accessible via a technique due to Riemann [32] (rediscovered and extended by P. Debye [5] in 1909). Briefly, one proceeds by deforming $\mathscr{C}$ so as to pass through certain "saddle-points" (i.e. critical points z_0 where $f'(z_0) = 0$); the direction of passage through a saddle-point is preferably along a curve on which $\operatorname{Im} f$ is constant, and such that $\operatorname{Re} f$ has a local maximum at the saddle-point (relative to the curve). When λ is large and positive $\lambda \operatorname{Re} f$ will have a sharp peak at the saddle-point, and the problem reduces to one of a type first treated by Laplace in his discussion of Stirling's formula for $n!$.

5B. For $\mathscr{J}(\lambda)$ the analogous critical points (where now $\partial R/\partial z_j = 0$ for $1 \leqslant j < t$) are also of crucial importance. The $t-1$ simultaneous equations $\partial R/\partial z_j = 0$ ($z_j \neq 0$) amount to a system of $t-1$ algebraic equations in $t-1$ unknowns z_j, subject to the constraints $z_j \neq 0$ (necessary since R may be undefined when $\prod z_j = 0$). Consider a family of rational functions of the type $R_\mu(z) = \sum_{\alpha} \mu_\alpha z_1^{\alpha_1}, \ldots, z_{t-1}^{\alpha_{t-1}}$, where α runs over some sufficiently large finite set of vectors in Z^{t-1}. For our purposes we may regard as equivalent any pair of such functions which differ by some non-zero constant factor. Thus we may regard μ as a point which runs over complex projective space P^q of suitably large finite dimension. For each μ let V_μ denote the variety in z-space consisting of all points with $\partial R_\mu/\partial z_j = 0 \neq z_j$ for all j. Then V_μ is generically of dimension 0, in the sense that the set of μ for which $\dim V_\mu \neq 0$ will be a proper subvariety in P^q. Now zero-dimensional algebraic varieties consist of finite sets of points; almost all generic μ will have the same value for $\# V_\mu$; those μ yielding exceptional values for $\# V_\mu$ will constitute a proper subvariety of P^q. Furthermore, the points of V_μ will be *non-degenerate* for generic μ, that is, at each point of V_μ, the Hessian matrix $(\partial^2 R/\partial z_i \partial z_j)$ will be non-singular. Further, there will be no "missing critical points" lying on $\prod z_j = 0$.

If R corresponds to a generic value of μ, in the above sense, then the asymptotics of $\mathscr{I}(\lambda)$ can be obtained by a direct imitation of the one-variable saddle-point method; we obtain a finite sum (over points in V_μ) of expansions of the type

$$e^{\lambda R(z^0)}\lambda^{-1/2}\sum_{n\geqslant 0} a_n(F)\lambda^{-n/2}, \tag{5.2}$$

where the coefficients $a_n(F)$ vary "smoothly" with F, in the sense of the m-adic topology on the local ring of germs of functions holomorphic near z^0 (see Malgrange [19]).

5C. In non-generic cases this is a rich diversity of pathological behaviour which can occur. However, since R is a rational function, there is a redeeming feature which allows us to treat many of the remaining cases. The set of critical values of R is *finite*, i.e. regardless of how many critical points R may have, there is only a finite set of $w \in C$ such that the variety $R(z) = w$ in C^{t-1} has any singular points at all (this is a mild generalization of a theorem of H. Whitney – see Milnor [20]). We emphasise that this result holds for rational functions; it is easy to produce counterexamples for non-rational R. Of course, it is possible that there are no critical values at all for certain R (they may "escape out to ∞" or onto the forbidden variety $\prod z_j = 0$). When this degenerate case occurs the corresponding $\mathscr{I}(\lambda)$ can have very pathological behaviour, as the following example shows.

5D. Example of R without critical values. This even occurs in one complex variable. We consider

$$\mathscr{I}(\lambda) = \int_{S^1} e^{\lambda z} e^{z^{-q}}\,dz \qquad (q \geqslant 1 \text{ in } Z), \tag{5.3}$$

which might well occur as a special case of (4.13). Here $R(z) = z$ obviously has no critical values. However, we can obtain an asymptotic expansion for $\mathscr{I}(\lambda)$ in this case, by a special trick. The contour S^1 can be replaced by any concentric circle, by Cauchy's theorem. We make a linear change of variable $z = \mu w$, $dz = \mu dw$, obtaining

$$\mathscr{J}(\lambda) = \mu \int_{S^1} \exp\{\lambda \mu w + \mu^{-q} w^{-q}\}\, dw.$$

We now "balance" the exponents by requiring that $\lambda\mu = \mu^{-q}$, $\lambda = \mu^{-1-q}$, and write $N = \lambda\mu = \lambda^{1-1/(q+1)}$, so that $N \to +\infty$ with λ. Thus

$$\mathscr{J}(\lambda) = \mu \mathscr{J}^*(N) = \mu \int_{S^1} \exp\{N(w+w^{-q})\}\, dw.$$

The function $w+w^{-q}$ has critical points where $1-qw^{-q-1} = 0$, i.e. at the points $w_r = q^{1/(q+1)} e\left(\dfrac{r}{q+1}\right)$, $0 \leqslant r < q+1$. The steepest descent paths here are tangential to the circle S: $|w|^{q+1} = q$, so that S will be an optimal contour for the application of the saddle-point method. Amongst these saddle-points, w_0 yields the largest value of $\operatorname{Re}(w+w^{-q})$, and will give the main contribution to $\mathscr{J}^*(N)$. We find that

$$\mathscr{J}^*(N) \sim N^{-1/2} \exp\{N(w_0+w_0^{-q})\} \sum_{n\geqslant 0} a_n N^{-n/2},$$

(5.4)

$$\mathscr{J}(\lambda) \sim \lambda^{-1/(q+1)} \exp\{\lambda^{1-1/(q+1)}(w_0+w_0^{-q})\} \sum_{n\geqslant 0} a_n (\lambda^{1-1/(q+1)})^{-(n+1)/2}.$$

Notice here that $w+w^{-q}$ exhibits all the features of the generic R_μ in § 5B; the critical points are all non-degenerate, and their number and nature are stable under small perturbations of the rational function.

5E. When, in (4.14), R does have critical values, the natural method of attack is to fibre $\mathbf{C}^{t-1} \setminus \{\prod z_j = 0\}$ into the union of the varieties V_w: $R(z) = w$, as w varies over $\mathbf{C}$. Since the critical values of R form a finite set, the corresponding *critical varieties* V_w are separated. In order to obtain the asymptotics of $\mathscr{J}(\lambda)$, we aim to deform the cycle $\underbrace{S^1 \times \ldots \times S^1}_{t-1}$ into another, $\mathfrak{z}$ say, which is smooth, compact, and homotopic to it relative to $\mathbf{C}^{t-1} \setminus \{\prod z_j = 0\}$, and passing through or near certain of the critical varieties V_w, in such a way that the neighbourhoods of the latter give the dominant contribution; in particular, $\operatorname{Re} z$ ought to have local maxima on $\mathfrak{z} \cap V_w$ for appropriate critical w.

We consider now the contributions to $\mathscr{J}(\lambda)$ from the portions of $\mathfrak{z}$ away from critical V_w. Suppose that γ is a compact, smooth chain of integration in $\mathbf{C}^{t-1} \setminus \{\prod z_j = 0\}$, not meeting any of the critical V_w. We can dissect γ into a finite number of pieces γ_{jr}, on which some fixed partial derivative $\partial R/\partial z_j$ is bounded away from 0. On such a γ_{jr} we can change variables in

(5.5)
$$\int_{\gamma_{rj}} e^{\lambda R(z)} F(z)\, dz_1 \wedge \ldots \wedge dz_{t-1}$$

by replacing the z_j for which $\partial R/\partial z_j$ fails to vanish on γ_{jr} by R; that is, we put $w_j = R$, and $w_k = z_k$ for all $k \neq j$. Then (5.5) becomes

$$\int_{\gamma^*_{jr}} e^{\lambda w_j} F^*(w)\, dw_1 \wedge \ldots \wedge dw_{t-1}, \tag{5.6}$$

where $F^*(w)\,dw_1 \wedge \ldots \wedge dw_{t-1} = F(z)\,dz_1 \wedge \ldots \wedge dz_{t-1}$, with F^* holomorphic, and γ^*_{jr} is the chain corresponding to γ_{jr} in w-space. By parametrising γ^*_{jr} with local real parameters, we can reduce (5.6) to a finite number of integrals of the type

$$\int_{t=a}^{b} e^{\lambda w_j(t)} G(t) \frac{dw_j}{dt}\, dt \tag{5.7}$$

(the number of these being independent of λ), where $G(t)$ is a C^∞-function of t, and dw_j/dt is non-zero and C^∞ for $a \leqslant t \leqslant b$. We can now apply integration by parts, integrating $e^{\lambda w_j(t)} \cdot dw_j/dt$ and differentiating G. This process is applied repeatedly, yielding an asymptotic expansion of the type

$$\left[e^{\lambda w_1(t)} \sum_{n=1}^{\infty} \lambda^{-n} A_n(t)\right]_{t=a}^{t=b} \tag{5.8}$$

for (5.7). If we have chosen $\mathfrak{z}$ in the way described above, the expansions (5.8), will, ideed, be negligible compared with those arising from the neighbourhoods of the critical varieties.

5F. We now face the major task of determining the contributions to $\mathscr{J}(\lambda)$ arising from the neighbourhoods of the critical varieties. We shall make use of the following fundamental

THEOREM 5.1 (Malgrange [19], Vasil'ev [38]). *Let $f\colon U \to \boldsymbol{C}$ be a function holomorphic in some neighbourhood U of O in $\boldsymbol{C}^n$, with $f(O) = O$, and such that $df = 0$ at O. Suppose that Δ is a smooth, compact n-chain contained in U, such that $\partial\Delta$ is contained in $\{z \in U; \operatorname{Re} f(z) < 0\}$. Then, for any $\varphi\colon U \to \boldsymbol{C}$ holomorphic in U, we have*

$$\int_{\Delta} e^{\lambda f(z)} \varphi(z)\, dz_1 \wedge \ldots \wedge dz_n \sim \sum_{P} \sum_{Q} a_{P,Q}(f, \Delta, \varphi)\, \lambda^{-P} (\log \lambda)^Q. \tag{5.9}$$

Here $Q \in \{0, 1, \ldots, n-1\}$, P runs through a finite set of non-negative rational arithmetic progressions $\{\alpha m + \beta;\, m \geqslant 0 \text{ in } \boldsymbol{Z}\}$, and the coefficients $a_{P,Q}(f, \Delta, \varphi)$ depend only on f, Δ and φ. The sets of P and Q which occur are dictated by the topological properties of the set $U \cap \{f(z) = 0\}$.

The proof of Theorem 5.1 is based on Hironaka's theory of resolution of singularities of complex analytic varieties, applied locally to $U \cap \{f(z) = 0\}$, and does not assume that the point O is an isolated singularity. However, if

O is isolated (i.e. there exists an open ball B, of positive radius, centred on O, such that $z \in B$, $df(z) = 0$ implies that $z = O$), then the sets of P and Q were related (by Malgrange) to monodromy groups of the singularity. Vasil'ev obtains information on the first non-vanishing $a_{P,Q}$, by studying the Newton polyhedra of the Taylor series of f and φ.

5G. Assuming now that R of (4.14) has critical values of appropriate type for the application of (5.8) and (5.9), we see that $\mathscr{I}(\lambda)$ of (4.14) can be expected to have an asymptotic expansion consisting of a finite sum of expansions of the type

$$e^{\lambda R_0} \sum_P \sum_Q a_{P,Q}(R, F) \lambda^{-P} (\log \lambda)^Q \tag{5.10}$$

(R_0 a critical value of R), and hence that $\mathscr{I}_m(\lambda)$ of (4.13) has the same type of expansion. This, in turn, yields an expansion for $I(x; T; \theta; \chi)$ of (2.8) = (4.1), consisting of a finite sum of expansions of the type

(5.11)
$$x^{\kappa-1} (\log x)^{\Omega_0 - 1} \sum_{m \geq 0} \sum_P \sum_Q b_{P,Q,m} (\log\log x)^{-P} (\log\log\log x)^Q (\log x)^{-m},$$

with Ω_0 some critical value of Ω (with respect to the variables $z_j = e(u_j)$).

When R fails to have critical values, the example of § 5D suggests the possibility of more exotic expansions, such as the product of (5.11) by such factors as $\exp\{A(\log\log x)^\beta\}$, or even stranger functions. I have not been able to determine the general type of function which one might expect to occur.

5H. Remarks. Our analysis in Sections 4–5 has been based on the behaviour of a single Frobenian multiplicative function θ with values in Z_K. However, if we consider the remarks in § 2C and § 3C, it is clear that, with straight-forward modifications, we may extend our analysis to cover the case where θ takes values in Z_K^S, the ring of S-integers of K, and also the simultaneous problems $\#\{n; 1 \leq n \leq x, \theta_i(n) = \alpha_i\}$, with θ_i: $N \to Z_{K_i}^{S_i}$ Frobenian multiplicative, I being some finite index set.

6. Frobenian multiplicative functions with values in a finite monoid

If, in place of the difficult "local" problem $\theta(n) = \alpha$, to which we have devoted Sections 2–5, we weaken the problem to $\theta(n) \in$ some suitably chosen large subset, we frequently obtain asymptotics of a rather simpler kind. This often happens when considering divisibility problems.

6A. A subset S, of a commutative ring R with 1, is said to be a *saturated multiplicatively closed subset* of R if it has the property $r_1 \cdot r_2 \in S$ iff both r_1

and $r_2 \in S$. It is an interesting exercise in the use of Zorn's Lemma to show that every SMCS S in R is necessarily of the form $\bigcap_{\mathfrak{p} \in \Sigma} (R \setminus \mathfrak{p})$, where Σ is the set of all prime ideals $\mathfrak{p}$ in R with $\mathfrak{p} \cap S = \emptyset$. (In fact this is set as a student exercise in Atiyah, MacDonald [1]!). For example, if K is an algebraic number field and $R = Z_K$, then $\Sigma = \emptyset$ corresponds to R itself, while $\Sigma = \operatorname{spec} R$ corresponds to the group of units, and Σ finite (including 0) corresponds to $K_T \cap Z_K$, where K_T is the group of T-units of K, in the sense of Section 1. Let S be any SMCS in R. We define a function g: $R \to F_2$, whereby

$$g(\xi) = \begin{cases} 1, & \xi \in S, \\ 0, & \text{if not}. \end{cases}$$

If now θ: $N \to R$ is multiplicative, then so is $\tilde{\theta}(n) = g(\theta(n))$, and if θ is Frobenian, then so is $\tilde{\theta}$, the latter taking values in the finite monoid $\{F_2, \cdot\}$.

6B. Let $K_1, \ldots, K_r$ be algebraic number fields, $\mathfrak{f}_i$ a conductor in K_i, and let $\mathfrak{H}_i$ be the maximal divisor class group of K_i, relative to $\mathfrak{f}_i$ (thus, the group of fractional ideals of Z_{K_i} prime to $\mathfrak{f}_i$, modulo the subgroup generated by all principal fractional ideals (α) with $\alpha \equiv 1 \ (\mathrm{mod}^{\times} \mathfrak{f}_i)$ and α totally positive – see Hasse [11]). Let $\mathfrak{H}$ be the direct product of all the $\mathfrak{H}_i$ (thus a finite abelian group), and let $X = 2^{\mathfrak{H}}$ be the set of all subsets of $\mathfrak{H}$. We make X into a finite commutative monoid by putting $A \cdot B = \{ab; a \in A, b \in B\}$ for all $A, B \in X$ (The identity element is $\mathfrak{H}$, and we define $A \cdot \emptyset = \emptyset$ for all $A \in X$). We now define θ: $N \to X$ by writing $\theta(n) = \{(h_1, \ldots, h_r) \in \mathfrak{H}$: $\exists$ ideals $\mathfrak{a}_i \in h_i$ with $N\mathfrak{a}_i = n$ for all $i \leqslant r\}$. It is easily seen to be multiplicative. Moreover θ is also Frobenian; let L/Q be the Galois hull over Q of the compositum $H_1 \ldots H_r$, where H_i/K_i is the classfield corresponding to the divisor class group $\mathfrak{H}_i$. By use of the Artin reciprocity map, and the standard properties of Frobenius symbols, one can show that, for p unramified in L/Q, $\theta(p^n)$ is determined for all $n \geqslant 1$ by Frob p in Gal L/Q (see Odoni [23]).

6C. As we mentioned in § 3C, the function θ: $n \mapsto \tau(n) \pmod p$, where τ is Ramanujan's function and p is prime, is Frobenian multiplicative with values in $\{Z/pZ, \cdot\}$.

6D. In § § 6A–6C we have produced three concrete examples of Frobenian multiplicative functions with values in a finite commutative monoid. One can produce many other examples in which it is of interest to obtain the asymptotics of $\# \{n; 1 \leqslant n \leqslant x, \theta(n) = \mu\}$, where θ is Frobenian multiplicative into M a finite commutative monoid and μ is a fixed element of M. I have explained the analysis of such problems elsewhere ([23], [30]) and will be content here merely to sketch the main features. Effectively, we need some procedure for separating the points of M by means of "harmonic analysis". However, there is no simple, direct description of the characters of M, unless it happens to be a group. We proceed instead by means of formal power series associated with a "presentation of M as a quotient of a free commutative monoid". Let $M = \{\mu_1, \ldots, \mu_n\}$. We introduce indeterminates z_j, one for each μ_j. For a fixed $\mu \in M$ we introduce the formal power series

$$G(\mu; z) = \sum_{\nu \geqslant 0}' z^\nu \quad (z^\nu = z_1^{\nu_1} \ldots z_n^{\nu_n}), \tag{6.1}$$

summed over all vectors $\nu = (\nu_1, \ldots, \nu_n)$ of non-negative integers such that $\mu_1^{\nu_1} \ldots \mu_n^{\nu_n} = \mu$. Consider, for a fixed $\gamma \in M$, the sequence $1 = \gamma^0, \gamma^1, \ldots, \gamma^k, \ldots$ ($k \geqslant 0$ in $\mathbf{Z}$). Since M is finite, the sequence γ^k is recurrent. Let $a = a(\gamma)$ be the least $k \geqslant 0$ such that $\gamma^a = \gamma^m$ for some $m > a$. We write $b = b(\gamma)$ for the least such m. Then, if $c(\gamma) = b(\gamma) - a(\gamma)$, we shall have $\gamma^k = \gamma^{k+c}$ for all $k \geqslant a$. This produces the "tadpole diagram" below:

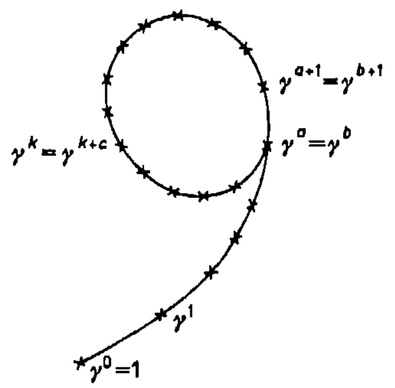

Fig. 2

Let the set $\mathscr{R}(\mu)$ of all ν occurring in (6.1) be called the *relation set* of μ. If, in some $\nu \in \mathscr{R}(\mu)$, one of the ν_i is at least $b_i = b(\mu_i)$, we may obtain a new relation $\nu' \in \mathscr{R}(\mu)$ by subtracting $c_i = c(\mu_i)$ from ν_i.

The process may be repeated until ν_i has been reduced to lie in the interval $[a_i, b_i)$, and, moreover, we can apply this process to each i for which $\nu_i \geqslant b_i$. We thus obtain a new relation ν^+ in which every component $\nu_i^+ < b_i$; such ν^+ are called *reduced relations* for μ. The set of all such reduced relations is clearly finite. Conversely, given any reduced relation ν^+, to each component $\nu_i^+ \geqslant a_i$ we may add an arbitrary non-negative multiple of c_i. It is now easy to see that $G(\mu; z)$ of (6.1) must be expressible as a rational function of z, with a denominator which is a divisor of $\prod_i (1 - z_j^{c_j})$. If we now treat the z_j as independent complex variables, then $G(\mu; z)$ is certainly holomorphic for $\max_j |z_j| < 1$, and writing it as $P(\mu; z)/\prod_j (1 - z_j^{c_j})$ with $P(\mu; z)$ a polynomial, we also have a continuation of G to a meromorphic function on $\mathbf{C}^n$.

We now introduce the Euler product

$$\Lambda^*(M;\theta;z;s)=\prod_p\{1+\sum_j z_j\sum_{\substack{k\geq 1\\ \theta(p^k)=\mu_j}}p^{-sk}\}\qquad(\sigma>1),\tag{6.2}$$

an analogue of Λ of Section 3. If we expand Λ^* as a Dirichlet series $\sum_{k\geq 1} e_k k^{-s}$, the coefficient $e_k=e_k(z)$ will be $z^{N(k)}$, where $N(k)$ is the vector whose jth component is the number of prime powers $p^v \| k$ having $\theta(p^v)=\mu_j$. If we take each $|z_j|<1$ we have the formula

$$\sum_{\theta(k)=\mu}k^{-s}=(2\pi i)^{-n}\oint\cdots\oint_{|z_j|=p<1}\Lambda^*(M;\theta;z^{-1};s)\,G(\mu;z)\prod_{j\leq n}\frac{dz_j}{z_j}\qquad(\sigma>1)\tag{6.3}$$

using the so-called *Hadamard convolution* of power series. It follows that

$$\sum_{\substack{k\leq x\\ \theta(k)=\mu}}\log x/k\tag{6.4}$$

$$=(2\pi i)^{-n-1}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s^2}\left\{\oint\cdots\oint_{|z_j|=p<1}\Lambda^*(M;\theta;z^{-1};s)\,G(\mu;z)\prod_{j\leq n}\frac{dz_j}{z_j}\right\}ds$$

when $a>1$. The function $\Lambda^*(M;\theta;z^{-1};s)$ is holomorphic in z for $\prod z_j\neq 0$, and is holomorphic at ∞ in each z_j. Moreover the poles of $G(\mu;z)$ are separable variable-by-variable. We may therefore apply Cauchy's residue theorem, one variable at a time, moving the circles of integration out to infinity, and expressing (6.3) as a sum of residues which depend holomorphically on s. The poles of G occur as simple poles where the z_j are various roots of unity, while the numerator P of G can cause multiple poles where some of the z_j are infinite. Thus (6.3) reduces to a finite sum of terms

$$\sum_\zeta k(\zeta)\,\Lambda^*(M;\theta;\zeta^{-1};s)\tag{6.5}$$

over vectors ζ of roots of unity, together with a sum over poles γ at ∞ of the form

$$k(\gamma)\,\partial\Lambda^*(M;\theta;z^{-1};s)|_{z=\gamma},\tag{6.6}$$

where ∂ is some mixed derivative with respect to z.

Hence (6.4) reduces to the problem of obtaining an asymptotic expansion for an integral of the type

$$\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s^2}\,\partial\Lambda^*(M;\theta;z^{-1};s)\,ds,\tag{6.7}$$

where ∂ is some mixed z-derivative (possibly of order 0), and γ is some vector of roots of unity or infinities. Since θ is Frobenian, we can imitate the analysis of Section 3, working with Λ^* in place of Λ; the analogue of (3.8) is

$$\log \Lambda^* = \sum_C \log \Lambda_C^* + \log \Lambda_{\mathrm{ram}}^*, \tag{6.8}$$

where

$$\log \Lambda_C^* = \sum_{m=1}^{\infty} P_m(z_{1j}, \ldots, z_{mj}) \sum_{\mathrm{Frob}\, p = C} p^{-ms}. \tag{6.9}$$

Here $z_{kj} = z_{kj}(C)$ is the unique corresponding z_j such that $\theta(p^k) = \mu_j$, and $P_m(\ldots)$ is as in (3.3). Using the Chebotarev density theorem once again, we find that

$$\begin{aligned} &\log \Lambda_C^*(M; \theta; z^{-1}; s) \\ &\quad = \sum_{m=1}^{\infty} P_m(z_{1j}^{-1}, \ldots, z_{mj}^{-1}) \left\{ (G:C)^{-1} \log \frac{1}{ms-1} + R_C(ms) \right\}. \end{aligned} \tag{6.10}$$

If we let $\varkappa$ be the least $m \geqslant 1$ for which

$$\Omega(z) := \sum_C (G:C)^{-1} P_m(z_{1j}^{-1}, \ldots, z_{mj}^{-1}) \tag{6.11}$$

is not identically zero, we obtain

$$\log \Lambda^*(M; \theta; z^{-1}; s) = \log \Lambda_{\mathrm{ram}}^* + \Omega(z) \log \frac{1}{\varkappa s - 1} + R(s), \tag{6.12}$$

with $R(s)$ a suitably holomorphic function similar to $\log H$ in (4.3), and the asymptotics for (6.7) are given by

$$x^{\varkappa^{-1}} (\log x)^{\Omega^*(\gamma) - 1} \sum_{r=0}^{h} \sum_{n=0}^{\infty} a_{r,n} (\log \log x)^r (\log x)^{-n} \tag{6.13}$$

where $\Omega^*(\gamma)$ is some z-derivative of Ω at γ. The exponent $h = h(\gamma)$ will be 0 unless some component of γ is infinite, and we usually find that $\Omega^*(1, \ldots, 1)$ is the dominant value amongst the $\Omega^*(\gamma)$.

7. Divisibility of multiplicative functions

In this section we consider a general multiplicative function $\theta \colon N \to Z_K$. Choose a finite set S of non-zero prime ideals $\mathfrak{p}$ in Z_K. We consider two new types of problem:

(A) *Given fixed integers $r(\mathfrak{p})$ ($\forall\, \mathfrak{p} \in S$), what is the distribution of the $n \in N$, $n \leqslant x$, having $v_{\mathfrak{p}}(\theta(n)) = r(\mathfrak{p})$ ($\forall\, \mathfrak{p} \in S$)?*

(B) *What is the "average order of divisibility" by the* $\mathfrak{p}$ *in* S *of the* $\theta(n)$, $n \leqslant x$, *relative to the set of all* $n \leqslant x$ *with* $\theta(n) \neq 0$?

More precisely, let us regard $\{n \in N,\ n \leqslant x,\ \theta(n) \neq 0\}$ as a finite probability space, in which every subset is assigned the measure equal to its relative size. Choosing n "at random" in this space, we consider the vector random variable $\boldsymbol{v}(\theta(n))$. Then we seek normalising vector functions $\boldsymbol{M}(x)$, $V(x)$ such that $\{\boldsymbol{v}(\theta(n)) - \boldsymbol{M}(x)\}/V(x)$ has a limiting distribution as $x \to \infty$.

7A. Analysis of Problem A. For each $\mathfrak{p} \in S$ we choose a complex variable. Then $\psi(z; n) := \prod_{\mathfrak{p} \in S} z(\mathfrak{p})^{v_{\mathfrak{p}}(\theta(n))}$ is a multiplicative function on $H := \{n \in N,\ \theta(n) \neq 0\}$. We construct the Euler product

$$\Psi(z; s) := \prod_{p} \{1 + \sum_{p^n \in H} p^{-ns} \psi(z; p^n)\} \quad (\sigma > 1) \tag{7.1}$$
$$= \sum_{n \in H} \psi(z; n) n^{-s}.$$

If we expand the Dirichlet series in (7.1) as a power series in z (it is certainly holomorphic when each $|z(\mathfrak{p})| \leqslant 1$), the coefficient of $z^r = \prod_{\mathfrak{p}} z(\mathfrak{p})^{r(\mathfrak{p})}$ is precisely $\sum' n^{-s}$, summed over the $n \in H$ with $\boldsymbol{v}(\theta(n)) = \boldsymbol{r}$, so that the latter Dirichlet series coincides with some mixed z-derivative of $\Psi(z; s)$ at $z = \boldsymbol{O}$.

When θ is Frobenian we may imitate the analysis in Section 3 and Section 6, obtaining a continuation in s of $\Psi(z; s)$, uniform and holomorphic in z for $\max_{\mathfrak{p}} |z(\mathfrak{p})| \leqslant 1$, since $\psi(z; n)$ is also Frobenian (¹) multiplicative. Hence, in this case, we obtain for $\# \{n \in H;\ n \leqslant x,\ \boldsymbol{v}(\theta(n)) = \boldsymbol{r}\}$ an asymptotic expansion essentially of the same type as (6.13).

7B. Problem B is handled rather differently. For each $\mathfrak{p} \in S$ we introduce a real variable $u(\mathfrak{p})$, and consider the multiplicative functions $\lambda(\boldsymbol{u}; n)$ defined on H by $\lambda(\boldsymbol{u}; n) = e\big(\sum_{\mathfrak{p}} u(\mathfrak{p}) v_{\mathfrak{p}}(\theta(n))\big)$. We consider the Euler product

$$\mathscr{L}(\boldsymbol{u}; s) := \prod_{p} \{1 + \sum_{\substack{p^k \in H \\ k \geqslant 1}} p^{-ks} \lambda(\boldsymbol{u}; p^k)\} = \sum_{n \in H} \lambda(\boldsymbol{u}; n) n^{-s} \quad (\sigma > 1). \tag{7.2}$$

Then

$$\sum_{\substack{n \in H \\ n \leqslant x}} \lambda(\boldsymbol{u}; n) \log(x/n) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} \frac{x^s}{s^2} \mathscr{L}(\boldsymbol{u}; s)\, ds, \tag{7.3}$$

with $a > 1$.

(¹) See the argument between (7.3) and (7.4).

We now assume that θ is Frobenian. Then so is each $\lambda(\boldsymbol{u}; n)$ with $n \in H$. (The characteristic function of H is itself Frobenian multiplicative). We decompose $\mathscr{L}(\boldsymbol{u}; s)$ into $\mathscr{L}_{\mathrm{ram}}(\boldsymbol{u}; s) \prod_C \mathscr{L}_C(\boldsymbol{u}; s)$, with C running over all conjugacy classes in Gal $L/\boldsymbol{Q}$. We have

$$(7.4) \qquad \log \mathscr{L}_C(\boldsymbol{u}; s) = \sum_{m \geqslant 1} \lambda_m^*(\boldsymbol{u}; C) \Big\{ \sum_{\substack{\mathrm{Frob}\, p = C \\ p^m \in H}} p^{-ms} \Big\} \qquad (\sigma > 1)$$

where $\lambda_m^*(\boldsymbol{u}; C) := P_m(\lambda(\boldsymbol{u}; p), \ldots, \lambda(\boldsymbol{u}; p^m))$ for all Frob $p = C$, in the notation of (3.3). We now define $\varkappa$ to be the least $m \geqslant 1$ such that $\mu_m^*(\boldsymbol{u}) := \sum_C (G:C)^{-1} \lambda_m^*(\boldsymbol{u}; C)$ does not vanish identically. Imitating Section 4, we obtain for (7.3) an asymptotic expansion

$$(7.5) \qquad x^{\varkappa^{-1}} \exp \{(\mu_\varkappa^*(\boldsymbol{u}) - 1) \log \log x\} \sum_{n \geqslant 0} a_n(\boldsymbol{u})(\log x)^{-n},$$

by analogy with (4.12). The coefficients $a_n(\boldsymbol{u})$ are actually holomorphic, if we regard the $u(\mathfrak{p})$ as complex variables in some compact polydisc about $\boldsymbol{O}$. The weights $\log(x/n)$ can be stripped from (7.3) without changing the general form of (7.5), other than to replace the $a_n(\boldsymbol{u})$ by possibly different holomorphic coefficients $a_n^*(\boldsymbol{u})$.

We now observe that the substitution $\boldsymbol{u} = \boldsymbol{O}$ in (7.3), (7.5) yields an asymptotic expansion for $\# \{n \in H; n \leqslant x\}$. Hence, if we divide

$$(7.6) \qquad x^{\varkappa^{-1}} \exp \{(\mu_\varkappa^*(\boldsymbol{u}) - 1) \log \log x\} \sum_{n \geqslant 0} a_n^*(\boldsymbol{u})(\log x)^{-n}$$

by the same quantity with $\boldsymbol{u} = \boldsymbol{O}$, we obtain the empirical expected value $\mathscr{E}_x(\lambda(\boldsymbol{u}; n))$ of $\lambda(\boldsymbol{u}; n)$, regarded as a random variable on $H \cap [1, x]$. On performing the division we find that the quotient is asymptotically

$$(7.7) \qquad \exp \{(\mu_\varkappa^*(\boldsymbol{u}) - \mu_\varkappa^*(\boldsymbol{O})) \log \log x\} \sum_{n \geqslant 0} b_n(\boldsymbol{u})(\log x)^{-n},$$

at least for $\boldsymbol{u}$ in some small polydisc about $\boldsymbol{O}$. In the neighbourhood of 0 we have

$$(7.8) \qquad \mu_\varkappa^*(\boldsymbol{u}) - \mu_\varkappa^*(\boldsymbol{O}) = \boldsymbol{a} \cdot \boldsymbol{u} - \frac{1}{2!} \boldsymbol{u}' \mathbf{Q} \boldsymbol{u} + O(\boldsymbol{u}^3),$$

where the constant matrix $\mathbf{Q}$ is easily seen to be positive definite symmetric, in general. Now let $\boldsymbol{t}$ be some fixed vector, and make the substitution $\boldsymbol{u} = \boldsymbol{t}/B\sqrt{\log \log x}$, where B is some positive constant to be chosen later. If we write $\boldsymbol{W}_n$ for the random vector $\{\boldsymbol{v}(\theta(n)) - \boldsymbol{a} \log \log x\}/B\sqrt{\log \log x}$, then we find, as $x \to \infty$, that (7.7) and (7.8) yield the relation

$$(7.9) \qquad \mathscr{E}_x(e(\boldsymbol{t} \cdot \boldsymbol{W}_n)) \to C(B, \mathbf{Q}) \exp(-\boldsymbol{t}' \mathbf{Q} \boldsymbol{t}/B),$$

where $\mathscr{E}_x$ denotes expectation relative to the probability space $H \cap [1, x]$. Now the right-hand side of (7.9) is the Fourier transform of some Gaussian probability distribution on $\boldsymbol{R}^S$ with mean $\boldsymbol{O}$, while the convergence in (7.9) is certainly uniform for all $\boldsymbol{t}$ in any fixed compact box of positive measure in $\boldsymbol{R}^S$. It follows from a classical theorem of P. Lévy (see Lévy [17]) that the empirical distributions of the $\boldsymbol{W}_n$ converge to the corresponding Gaussian law. Whether or not the individual components of the $\boldsymbol{W}_n$ are correlated random variables is dictated by the precise form of $\mathbf{Q}$; independence is equivalent to the diagonality of $\mathbf{Q}$.

We should point out that certain choices of the set S may cause $\mathbf{Q}$ to be singular, in which case we may obtain a limiting distribution which is concentrated on some linear subspace of positive codimension.

8. Modular forms (Serre [35], Deligne–Serre [9])

8A. Modular groups. Let $\mathrm{SL}(2, \boldsymbol{Z})$ be the group of all invertible 2×2 matrices $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over $\boldsymbol{Z}$, with determinant $+1$. $\mathrm{SL}(2, \boldsymbol{Z})/\{\pm I\}$ acts as a discrete group of transformations on $\mathscr{H} := \{z \in \boldsymbol{C}; \operatorname{Im} z > 0\}$ via $z \mapsto Tz = (az+b)/(cz+d)$. For $N \geqslant 1$ in $\boldsymbol{Z}$ we consider subgroups

$$\Gamma(N) \leqslant \Gamma_1(N) \leqslant \Gamma_0(N) \leqslant \mathrm{SL}(2, \boldsymbol{Z}) = \Gamma(1), \tag{8.1}$$

where

$$\begin{aligned} \Gamma(N) &= \{T; T \equiv I \pmod N\}, \\ \Gamma_1(N) &= \{T; a \equiv d \equiv 1, c \equiv 0 \pmod N\}, \\ \Gamma_0(N) &= \{T; c \equiv 0 \pmod N\}. \end{aligned}$$

Any subgroup $\Gamma \leqslant \Gamma(1)$ with $\Gamma \geqslant \Gamma(N)$ for some N is said to be *of congruence type*; its *level* (Stufe) is the least such N.

8B. Modular forms. Let $f(z)$ be defined for all $z \in \mathscr{H}$. Let $k \geqslant 0$ in $\boldsymbol{Z}$, and let $T \in \Gamma(1)$. We write $f|_k T$ for the function $z \mapsto (cz+d)^{-k} f(Tz)$ $(\forall z \in \mathscr{H})$. Note that $f|_k TU = (f|_k T)|_k U$ for all $T, U \in \Gamma(1)$. Now let $\Gamma \geqslant \Gamma(N)$ be of congruence type. A function f defined on $\mathscr{H}$ is called a *modular form of weight k on Γ* if

(i) f is holomorphic on $\mathscr{H}$;

(ii) $f|_k T = f$ for all $T \in \Gamma$;

(iii) f is "holomorphic at cusps", that is, $f|_k T$ has a convergent power series expansion in $e(z/N)$, $\forall T \in \Gamma(1)$.

If, further $f|_k T$ vanishes at ∞, $\forall T \in \Gamma(1)$, f is called a *cusp form of weight k on Γ*.

The set $\mathfrak{M}(k, \Gamma)$ of all modular forms of weight k on Γ is a finite-dimensional C-vector space, in which the cusp-forms comprise a subspace $\mathfrak{M}^0(k, \Gamma)$ which has codimension 1, if k is sufficiently large. Note that an $f \in \mathfrak{M}(k, \Gamma(N))$ will also lie in $\mathfrak{M}(k, \Gamma_1(N))$ if and only if $f(z+1) = f(z)$ for all $z \in \mathscr{H}$, i.e., if and only if it has a "Fourier expansion" $\sum_{n \geq 0} a_n q^n$ $(q = e(z))$.

8C. Forms of type (k, ε). If $f \in \mathfrak{M}(k, \Gamma_1(N))$ and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma_0(N)$, then $f|_k T$ depends only on $d \pmod N$. Moreover, $T \mapsto d \pmod N$ is a homomorphism of $\Gamma_0(N)$ onto $(Z/NZ)^*$, while, if $d \equiv -1 \pmod N$, we have $f|_k T = (-1)^k f$. We thus obtain an action of (Z/NZ) on $\mathfrak{M}(k, \Gamma_1(N))$, and so the latter decomposes as a direct sum of eigenspaces, according to the characters ε of $(Z/NZ)^*$; the eigenspace $\mathfrak{M}(k, \varepsilon, \Gamma_0(N))$ consists of those $f \in \mathfrak{M}(k, \Gamma_1(N))$ such that $f|_k T = \varepsilon(d) f$ for all $T \in \Gamma_0(N)$; such f are called *modular forms of type* (k, ε) *on* $\Gamma_0(N)$.

We say that ε is $\begin{cases} \text{even} \\ \text{odd} \end{cases}$ according as $\varepsilon(-1) = \begin{cases} (-1)^{\text{even}} \\ (-1)^{\text{odd}} \end{cases}$. Since $f|_k T = (-1)^k f$ when $d \equiv -1 \pmod N$, we see that $\mathfrak{M}(k, \varepsilon, \Gamma_0(N)) = 0$ unless k and ε have the same parity, and we shall assume the latter from now on.

8D. Hecke operators on forms of type (k, ε) (see Li [18]). It is possible to generalize the classical notion of Hecke operators on $\mathfrak{M}(k, \Gamma(1))$ to operators on $\mathfrak{M}(k, \varepsilon, \Gamma_0(N))$, so that the cusp subspace $\mathfrak{M}^0(k, \varepsilon, \Gamma_0(N))$ is equipped with a basis of a very convenient type. We begin by defining an analogue of the Petersson inner product; specifically, for $f, g \in \mathfrak{M}^0(k, \varepsilon, \Gamma_0(N))$, we define

$$\langle f, g \rangle = \frac{1}{(\Gamma_1(N):\Gamma_0(N))} \iint_D f(z) \overline{g(z)} \, y^{k-2} \, dx \, dy, \tag{8.2}$$

where D is a fundamental domain for $\Gamma_1(N)$, acting on $\mathscr{H}$. The *Hecke operators* are defined as follows. Assuming that f has the Fourier expansion $\sum_{n \geq 0} a_n q^n$, we write

$$\begin{aligned} f \mid T_p &= \sum_{n \geq 0} a_{pn} q^n + \varepsilon(p) p^{k-1} \sum_{n \geq 0} a_n q^{pn}, && \text{when } p \nmid N; \\ f \mid U_p &= \sum_{n \geq 0} a_{pn} q^n, && \text{when } p \mid N. \end{aligned} \tag{8.3}$$

These operators map $\mathfrak{M}(k, \varepsilon, \Gamma_0(N))$ into itself, while $\mathfrak{M}^0(k, \varepsilon, \Gamma_0(N))$ is invariant for each operator. As operators on the latter, the T_p constitute a family of pairwise commuting ε-Hermitian operators, with respect to the inner product (8.2). As was shown in [18] it follows from this that $\mathfrak{M}^0(k, \varepsilon, \Gamma_0(N))$ has a basis consisting of forms $f_i(z) = g_i(zd_i)$, where

$g_i \in \mathfrak{M}^0(k, \varepsilon, \Gamma_0(N_i))$, d_i divides N/N_i, and N_i is a divisor of N such that ε is defined (mod N_i); these g_i can be chosen to be simultaneous eigenfunctions for all the T_p and U_p acting on $\mathfrak{M}^0(k, \varepsilon, \Gamma_0(N_i))$. Further, we can also insist that the g_i should be *primitive*, in the sense that their Fourier expansions begin with $q+O(q^2)$. If f is primitive with $N = N_i$ then its Fourier coefficients a_n will be multiplicative; the Dirichlet series $\sum_{n \geqslant 1} a_n n^{-s}$ will converge for sufficiently large σ, and will have the Euler product

$$\prod_{p|N} (1-p^{-s}a_p)^{-1} \prod_{p \nmid N} \left(1-p^{-s}a_p+\varepsilon(p)p^{k-1-2s}\right)^{-1}. \tag{8.4}$$

The a_p are, in fact, certain eigenvalues of the T_p of U_p, and can easily be shown to be algebraic integers generating some finite extension of $\boldsymbol{Q}$; thus the same is true of all the a_n. If we consider the special case $N = 1$, ε trivial, $k = 12$, we have the example

$$f(z) = q \prod_{n \geqslant 1} (1-q^n)^{24} = \sum_{n \geqslant 1} \tau(n) q^n, \qquad \tau = \text{Ramanujan's function}, \ \tau(n) \in \boldsymbol{Z}.$$

We can also obtain a good basis for $\mathfrak{M}(k, \varepsilon, \Gamma_0(N))$ by adjoining to our basis of $\mathfrak{M}^0(k, \varepsilon, \Gamma_0(N))$ one of Hecke's "Eisenstein series of Nebentypus"; it was shown in [12] that such a function can also be taken to be a simultaneous eigenfunction for the T_p and U_p of (8.3).

8E. Forms of type $(1, \varepsilon)$. In the case of forms of type $(1, \varepsilon)$ we have the following remarkable

CHARACTERIZATION THEOREM (Deligne–Serre [9]). *The primitive f in $\mathfrak{M}^0(1, \varepsilon, \Gamma_0(N))$, where $\varepsilon(-1) = -1$, are of the type $\sum_{n \geqslant 1} a_n q^n$, where $\sum_{n \geqslant 1} a_n n^{-s}$ is the Dirichlet series expansion of a certain Artin L-function. More precisely, given such an f, there is a unique corresponding irreducible continuous representation ϱ of* Gal $\bar{\boldsymbol{Q}}/\boldsymbol{Q}$ *into* GL $(2, \boldsymbol{C})$, *with conductor N and with* det ϱ $= \varepsilon$, *such that the p-factors (for $p \nmid N$) of the Euler product* (8.4) *are precisely*

$$\det\left(\mathbf{I}_2 - p^{-s}\varrho(\text{Frob } p)\right)^{-1},$$

and the p-factors for $p \mid N$ are the standard Artin factors for ramified primes.

Another account of the proof was given in [34]. One corollary of this theorem is a special case of the Artin conjecture: *for the ϱ occurring in the above theorem, the Artin L-functions are entire functions.*

We also remark that the Hecke "Eisenstein series of Nebentypus" in weight 1 correspond in the context of the above theorem to reducible two-dimensional representations, so that their Dirichlet series are expressible as the product of two ordinary Dirichlet L-series.

8F. Weight $k > 1$. The situation here is not so simple; $\sum_{n \geq 1} a_n n^{-s}$ cannot now be an Artin L-series, since its coefficients grow too rapidly with n. However, let f be primitive in $\mathfrak{M}^0(k, \varepsilon, \Gamma_0(N))$. Its coefficients are integers in some field of finite degree over Q. Let $\mathfrak{p}$ be any non-zero prime ideal in the ring Z_F of integers of this field F. Then there exists an Artin L-function, of conductor N^* dividing (norm $\mathfrak{p}$)N, corresponding to some irreducible representation ϱ with $\det \varrho = \varepsilon$, whose Dirichlet series $\sum_{n \geq 1} b_n n^{-s}$ satisfies $b_n \equiv a_n \pmod{\mathfrak{p}}$ for all $n \geq 1$. (Here ϱ is two-dimensional.)

8G. Problems on coefficients of modular forms of type (k, ε) on $\Gamma_0(N)$. In [32] Serre raised some interesting questions on the Fourier coefficients of the "typical" $f \in \mathfrak{M}(k, \varepsilon, \Gamma_0(N))$.

PROBLEM I. *Suppose that $f \in \mathfrak{M}(k, \varepsilon, \Gamma_0(N))$ has all of its coefficients a_n in Z_F, F some algebraic number field. Choose a non-zero ideal $\mathfrak{a} \lhd Z_F$ (when $k > 1$ we require $\mathfrak{a}$ to be prime). How do the a_n distribute themselves* (mod $\mathfrak{a}$)? *Thus, given $\beta \in Z_F$, obtain the asymptotics of*

$$\# \{n;\ 1 \leq n \leq x,\ a_n \equiv \beta \pmod{\mathfrak{a}}\}. \tag{8.5}$$

PROBLEM II (Weight 1 only). *Obtain the asymptotic expansion of*

$$\# \{n;\ 1 \leq n \leq x,\ a_n \neq 0\}. \tag{8.6}$$

PROBLEM III (Weight 1 only). *For fixed $\alpha \neq 0$ in C, obtain an asymptotic expansion for*

$$\# \{n;\ 1 \leq n \leq x,\ a_n = \alpha\}. \tag{8.7}$$

(The reason for the restriction to weight 1 in Problems II and III is that a_n tends to infinity "in general", when $k > 1$).

We may add a further pair of problems of analogous type, motivated by the results of Section 7. Under the hypotheses of Problem I, choose a finite set S of non-zero primes in Z_F.

PROBLEM IV. *Find the asymptotics of*

$$\# \{1 \leq n \leq x;\ v_{\mathfrak{p}}(a_n) = k_{\mathfrak{p}},\ \forall\, \mathfrak{p} \in S\}, \tag{8.8}$$

where each $k_{\mathfrak{p}} \geq 0$ is fixed.

PROBLEM V. *Consider the n with $a_n \neq 0$, $n \leq x$, as random variables. Decide whether some suitably normalised linear combination of the random variables $v_{\mathfrak{p}}(a_n)$ ($\mathfrak{p} \in S$) has a limiting probability distribution as $x \to \infty$.*

The method of Frobenian multiplicative functions which we have developed in Sections 3–7 enables us to solve these problems, except in certain "pathological" cases.

8H. Reduction of Serre's problems to Frobenian functions. As pointed out in § 8D, we can choose a C-basis for $\mathfrak{M}(k, \varepsilon, \Gamma_0(N))$ consisting of functions $g_i(zd_i)$, where the g_i are either primitive eigenfunctions in $\mathfrak{M}^0(k, \varepsilon, \Gamma_0(N_i))$ or else an Eisenstein series of Nebentypus on $\Gamma_0(N)$. Thus, if $f \in \mathfrak{M}(k, \varepsilon, \Gamma_0(N))$, it can be written uniquely as

$$f(z) = \sum_i \lambda_i g_i(zd_i) \tag{8.9}$$

with the λ_i in $\boldsymbol{C}$. We write the Fourier expansions of the g_i as $\sum_{n \geqslant 0} b_i(n) e(nz)$. Then

$$a(n) = \sum_i \lambda_i b_i(n/d_i) \tag{8.10}$$

when $f(z) = \sum_{n \geqslant 0} a(n) e(nz)$, where we make the convention that $b_i(n/d_i) = 0$ if $d_i \nmid n$. Each $n \in \boldsymbol{N}$ is uniquely expressible in the form md, where $(m, D) = 1$ and d is composed only of primes dividing D. Here we take $D = N$, except in Problem I, where it is (norm $\mathfrak{a}$) N. The coefficients $b_i(n)$ are multiplicative for $n \geqslant 1$, by our choice of the g_i. Thus

$$a(md) = \sum_i \lambda_i b_i(d/d_i) b_i(m). \tag{8.11}$$

The results of § 8E and § 8F show that, for $k = 1$, b_i *is Frobenian multiplicative from* $\boldsymbol{N}_D$ (natural numbers prime to D) *into* $\boldsymbol{Z}_{K_0}$, where K_0 is some finite cyclotomic extension of $\boldsymbol{Q}$. Also, for $k > 1$, $b_i \pmod{\mathfrak{a}}$ *is Frobenian multiplicative from* $\boldsymbol{N}_D$ *into the finite monoid* $\boldsymbol{Z}_{K_0}/\mathfrak{a} \cap \boldsymbol{Z}_{K_0}$, *provided that* $\mathfrak{a}$ *is prime.*

Let $\mathscr{D}$ be the set of natural numbers composed only of prime factors of D. Let these prime factors be $p_1, \ldots, p_t$. Then the number of $d \in \mathscr{D}$ with $d \leqslant x$ is the same as the number of t-tuples of non-negative integers $y_1, \ldots, y_t$ satisfying $\sum_{i \leqslant t} y_i \log p_i \leqslant \log x$, and thus, by a simple calculation of volumes, satisfies

$$\sum_{\substack{d \in \mathscr{D} \\ d \leqslant x}} 1 \sim \frac{(\log x)^t}{t!} \prod_{i \leqslant t} (\log p_i)^{-1} \qquad (x \to \infty). \tag{8.12}$$

The fact the $\mathscr{D}$ is such a thin subset of $\boldsymbol{N}$ makes it feasible to convert the counting problems in Problems I–V into problems with $d \in \mathscr{D}$ fixed, and m running up to x/d, and then to sum the resulting asymptotic expansions over all d up to, say, some large power of $\log x$. For the $d > (\log x)^T$ we estimate the corresponding number of m with some suitable trivial upper bound. For example, in many cases it is sufficient to use the upper bound x/d. As for the error term thus produced, we note that

$$x \sum_{\substack{d \in \mathscr{D} \\ x \geqslant d \geqslant (\log x)^T}} d^{-1} \ll x \sum_{n \geqslant (\log x)^T} n^{-1}(D_n - D_{n-1}) \ll x \sum_{n \geqslant (\log x)^T} \frac{D_n}{n^2} \ll \frac{x \log \log x}{(\log x)^T},$$

where $D_n = \# \{d \in \mathscr{D}; d \leqslant n\}$. The counting functions for the m can "generically" be obtained from the results of Sections 4–6. Thus we can deal with all cases of Problem I; for the others, we need to make the hypothesis that the non-zero λ_i in (8.9) are linearly independent over the algebraic number field generated by all the b_i. (Since the latter field is countable, while C is not, this really does deal with "most" f!)

EXAMPLES. 1. *Ramanujan's function.* We take $k = 12$, $N = 1$, ε trivial, and consider the cusp-form $\sum_{n \geqslant 1} \tau(n) q^n = q \prod_{n \geqslant 1} (1-q^n)^{24}$. For Problem I with $\mathfrak{a}$ a prime ideal pZ in Z, the method of Frobenian functions on a finite monoid yields another proof of the results of C. Radoux [31]. Radoux's results were also obtained by Serre [33]; by applying further work of his own (Sém. Delange–Pisot–Poitou 14 (1967/8)) and of Swinnerton-Dyer (*Modular functions of one variable* III (Springer Lecture Notes 350 (1973), pp. 1–55)), he shows, in effect, that $n \mapsto \tau(n) \pmod k$ is Frobenian multiplicative for any k. Our methods of Section 6 yield complete asymptotic expansions of the type (6.13) (with no terms in log log x) for $\# \{n; 1 \leqslant n \leqslant x, \tau(n) \equiv \beta \pmod k\}$. Radoux and Serre were content to obtain leading terms only, making use of Delange's Tauberian theorem ([33]). This is sufficient, for example, to decide when $\tau(n)$ is uniformly distributed over invertible residue classes (mod k). Our methods also allow us to impose further Frobenian conditions on n. For example, let $h \in N$ and let χ be a Dirichlet character (mod h). Then $n \mapsto (\tau(n) \pmod k, \chi(n))$ is Frobenian multiplicative, with values in $Z/kZ \times C_e$, where e is the exponent of $(Z/hZ)^*$. From this we are able to extend the Radoux–Serre results to the cases where n is restricted to any fixed arithmetic progression. Slightly more sophisticated ideas will yield the distribution of $\tau(n) \pmod k$ for n a sum of two squares, or a value of a binary quadratic form, or, more generally, a norm of an element of a full Z-module in an arbitrary Z_K, by appropriate combination of Frobenian multiplicative functions with values in a finite monoid.

2. *A cusp form of type* $(1, \varepsilon)$ *discussed by Hecke.* The Fourier expansion

$$q \prod_{m \geqslant 1} \{1-q^{12m}\}^2 = \sum_{n \geqslant 1} a_n q^n = \sum\sum_{\substack{a,b \in Z, a \equiv 1, b \equiv 0(3) \\ a+b \equiv 1(2)}} (-1)^b q^{a^2+b^2}$$

belongs to a primitive cusp form f of type $(1, \varepsilon)$ on $\Gamma_0(144)$, where $\varepsilon(n)$ is the Jacobi symbol $\left(\frac{-4}{n}\right)$. The Dirichlet series $\sum a_n n^{-s}$ $(\sigma > 1)$ is the Artin L-

function corresponding to a certain irreducible 2-dimensional representation ϱ of Gal $(\boldsymbol{Q}(i, \sqrt[4]{12})/\boldsymbol{Q}) \cong D_4$, the symmetry group of the square, having det $\varrho = \varepsilon$. This was already noticed by Hecke (*Gesammelte Werke*, pp. 426, 448); see also Serre [33], [34]. Since f here is primitive the discussion of § 8G, H is directly applicable. Thus, in agreement with Serre [33], we find that

$$\# \{n;\ 1 \leqslant n \leqslant x,\ a_n \neq 0\} \sim x(\log x)^{-3/4}\Big(\sum_{k \geqslant 0} c_k(\log x)^{-k}\Big).$$

For the quantity $\# \{n;\ 1 \leqslant n \leqslant x,\ a_n = \alpha\}$, with $\alpha \neq 0$ fixed in $\boldsymbol{Z}$, we do not really need the heavy machinery of Section 5. The corresponding integrals $\mathscr{J}(\lambda)$ of (4.14) involve only one complex variable (since $K = \boldsymbol{Q}$ here), while $R(z)$ turns out to be a polynomial and $F(z)$ has a multiple pole at $z = 0$. Thus the saddle-point method becomes irrelevant here, and we need only apply the Cauchy residue theorem, obtaining for $\# \{n;\ 1 \leqslant n \leqslant x,\ a_n = \alpha\}$ and expansion of the type (6.13).

A direct application of Section 6 gives an expansion of type (6.13) for $\# \{n;\ 1 \leqslant n \leqslant x,\ a_n \equiv \beta \pmod{k}\}$. The corresponding divisibility problems fit into the typical cases of Section 7.

3. Let F be an imaginary quadratic number field, $\mathfrak{f}$ some conductor in F, and let χ be a non-principal character on ideal classes (mod$^{\times}\mathfrak{f}$). The Hecke L-function

$$\sum_{\mathfrak{a}} \chi(\mathfrak{a})\, N\mathfrak{a}^{-s} = \prod_{\mathfrak{p} \nmid \mathfrak{f}} \{1 - N\mathfrak{p}^{-s} \cdot \chi(\mathfrak{p})\}^{-1} = \sum_{n \geqslant 1} a_n n^{-s} \quad (\text{where } a_n = \sum_{N\mathfrak{a} = n} \chi(\mathfrak{a}))$$

corresponds to a modular form of type $(1, \varepsilon)$ on some $\Gamma_0(N)$; in fact f is a cusp form, and, indeed, an eigenfunction for all the T_p and U_p. In this case the a_n lie in some general cyclotomic field. When f is primitive, § 8G, H will apply, and this time there does not appear to be any reason to think that the general cases of Section 5 can be avoided, when discussing $\# \{n;\ 1 \leqslant n \leqslant x,\ a_n = \alpha\}$; however, the other Serre problems yield to the methods of Sections 6 and 7.

References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, 1969, p. 44.

[2] A. O. L. Atkin and J. Lehner, *Hecke operators on $\Gamma_0(m)$*, Math. Ann. 185 (1970), 134-160.

[3] E. Combet, *Intégrales exponentielles*, Lecture Notes in Mathematics 937, Springer-Verlag, 1982.

[4] E. T. Copson, *An introduction to the theory of functions of a complex variable*, Oxford 1935, Ch. 9.

[5] P. Debye, *Näherungsformeln für die Zylinderfunktionen für grosse Werte des Arguments und unbeschränktveränderliche Werte des Index*, Math. Ann. 67 (1909), 535-558.

[6] P. Deligne, *Formes modulaires et représentation de* GL(2), Springer Lecture Notes 349 (1973), 55–105.

[7] –, *La conjecture de Weil I*, Publ. Math. I. H. E. S. 43 (1974), 273–307.

[8] –, *Formes modulaires et représentations* l*-adiques*, Séminaire Bourbaki # 355, 1968/9.

[9] P. Deligne et J.-P. Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) 7 (1974), 507–530.

[10] G. H. Hardy, *Ramanujan*, Cambridge 1940.

[11] H. Hasse, *Bericht über neuere Untersuchungen ... algebraischen Zahlkörper*, Physica Verlag, Würzburg 1970, Vol. I, 4–6.

[12] E. Hecke, *Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik*, Gesammelte Werke # 24.

[13] A. E. Ingham, *The distribution of prime numbers*, Cambridge Tract # 30, 1932.

[14] O. Kolberg, *Congruences for the coefficients of the modular invariant* $j(\tau)$, Math. Scand. 10 (1962), 173–181.

[15] S. Lang, *Algebraic number theory*, Addison–Wesley, 1968, Ch. 5.

[16] –, *Introduction to modular forms*, Springer Grundlehren 222, 1976.

[17] P. Lévy, *Calcul des probabilités*, Gautheir–Villars, Paris 1925, Ch. 4.

[18] W. I. Li, *Newforms and functional equations*, Math. Ann. 212 (1975), 285–315.

[19] B. Malgrange, *Intégrales asymptotiques et monodromie*, Ann. Sci. École Norm. Sup (4) 7 (1974), 405–430.

[20] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Stud. 61, Princeton 1968.

[21] T. Miyske, *On automorphic forms on* GL(2) *and Hecke operators*, Ann. of Math. 94 (1971), 174–189.

[22] W. Narkiewicz, *Divisibility properties of a class of multiplicative functions*, Colloq. Math. 18 (1967), 219–232, MR 36 6365.

[23] R. W. K. Odoni, *Solution of some problems of Serre on modular forms: the method of Frobenian functions*; in: *Recent Progress in Analytic Number Theory* (ed. H. Halberstam, C. Hooley), Academic Press, 1981, Vol. 2, pp. 159–169.

[24] –, *The distribution of integral and prime integral values of systems of full-norm polynomials and affine-decomposable polynomials*, Mathematika 26 (1979), 80–87.

[25] –, *A proof by classical methods of a theorem of Ax on polynomial congruences modulo a prime*, Bull. London Math. Soc. 11 (1979), 55–58.

[26] –, *The Farey density of norm subgroups of global fields, I*, Mathematika 20 (1973), 155–169.

[27] –, *On the norms of algebraic integers*, ibid. 22 (1975), 71–80.

[28] –, *A new equidistribution property for norms of ideals in given classes*, Acta Arith. 33 (1977), 53–63.

[29] –, *Global norm density results from an extended Čebotarev density theorem*; in: *Algebraic Number Fields* (ed. A. Fröhlich), Academic Press, 1977.

[30] –, *Généralisations du théorème de Landau: la méthode de fonctions frobéniennes*, Journées Arithmétiques d'Orsay, Juin 1980.

[31] C. Radoux, *Répartition des valeurs de la fonction τ de Ramanujan modulo un nombre premier*, Soc. Sci. Bruxelles 89 (IV) (1975), 434–438.

[32] B. Riemann, *Sullo suolgimento del quotiente di due serie ipergeometriche in frazione continu a infinita*, Gesammelte Werke, 424–430.

[33] J.-P. Serre, *Divisibilité de certaines fonctions arithmétiques*. L'Enseignement Math. 22 (1976), 227–260.

[34] –, *Modular forms of weight* 1 *and Galois representations*; in: *Algebraic Number Fields* (ed. A. Fröhlich), Academic Press, 1977, pp. 193–268.

[35] –, *Cours d'arithmétique*, Presses Universitaires de France, 1970.

[36] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, Benjamin, New York 1968.
[37] —, *Quelques applications du théorème de densité de Chebotarev*, Publ. Math. I.H.E.S. 54 (1981), 126–201.
[38] V. A. Vasil'ev, *Asymptotic behaviour of exponential integrals in complex domain*, Funktsional Anal. i Prilozhen. 13 (4) (1979), 1–12; English translation by Plenum Publ. Co.

Presented to the Semester
Elementary and Analytic Theory of Numbers
September 1–November 13, 1982