ON THE STEFAN PROBLEM: THE VARIATIONAL APPROACH AND SOME APPLICATIONS

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I. Introduction

In this first lecture we give the strong formulation for the classical Stefan problem and introduce the weak formulation.

For simplicity's sake we will consider only the case of two phases, the results described here can be generalized to the case of more than two phases provided there is one single parameter which determines the phase (in this case the temperature). This treatment therefore does not cover the case of super cooling or the multiple-point change of phase.

Let Ω be a bounded domain in $\mathbb{R}^3$. We consider a system in Ω and, as is customary for the Stefan problem, we assume that the changes of volume are very small compared to the energy involved in the change of phase. This generally holds for the usual physical situations provided the temperature does not vary a lot (low external pressure helps too !).

We consider a time interval $[0, T](T > 0)$ on which we study the system; we denote by Q the cylinder $]0, T[\times \Omega$ and in Q we look for two unknowns:

The free boundary S (the section of which in Ω is a moving boundary); it is a hypersurface which represents the interface between the two phases.

The temperature distribution θ in Q .

If θ_0 is the temperature of change of phase, then θ is required to equal θ_0 on S . In order to write the equations for θ we introduce the thermal conductivity $k(\theta)$ (a smooth function of θ except perhaps at θ_0 , but bounded below away from zero), and the specific heat $\alpha(\theta)$, which is also bounded below away from zero and smooth except perhaps at θ_0 .

The heat equation can be written in $Q - S$ as usual:

$$(1) \quad \alpha(\theta) \frac{\partial \theta}{\partial t} - \operatorname{div}(k(\theta) \nabla \theta) = f \quad \text{in } Q - S$$

where f is an internal heating term. On S we have two conditions:

$$(2) \quad \theta = \theta_0$$

(including the "continuity" of the temperature), and the Stefan condition, which is as follows:

$$(3) \quad b \cos(\mathbf{v}, \mathbf{t}) - \sum_{i=1}^3 \left[k(\theta) \frac{\partial \theta}{\partial x_i} \right]_S \cos(\mathbf{v}, \mathbf{x}_i) = 0$$

where $b > 0$ is the latent heat, $\mathbf{v}$ is the unit normal to S and $[]_S$ is the jump along $\mathbf{v}$ across S going from temperatures below θ_0 to temperatures above θ_0 ; $\cos(\mathbf{v}, \mathbf{t})$, $\cos(\mathbf{v}, \mathbf{x}_i)$ are the components of $\mathbf{v}$ in space-time. In the case where S has a local parametrization of the form $t = \varphi(x)$, (3) can be rewritten as

$$(3') \quad b + [k(\theta) \nabla \theta \nabla \varphi]_S = 0.$$

We also assume some boundary conditions on $\partial\Omega \times (0, T)$ (where we assume that $\Gamma = \partial\Omega$ is smooth) and initial condition at $t = 0$. We shall return to these conditions later.

First we use the Kirschhoff transform to reduce the elliptic part to a linear elliptic one:

put $v(\theta) = \int_{\theta_0}^{\theta} k(s) ds$ so that, defining $\varrho(v) = \alpha(\theta)/k(\theta)$, we get

$$(4) \quad \varrho(v) \frac{\partial v}{\partial t} - \Delta v = f \quad \text{in } Q - S,$$

$$(5) \quad v = 0 \quad \text{on } S,$$

$$(6) \quad b \cos(\mathbf{v}, \mathbf{t}) - \left[\sum_j \frac{\partial v}{\partial x_j} \cos(\mathbf{v} \cdot \mathbf{x}_j) \right]_S = 0.$$

As is well known, for the Stefan problem (at least in several dimensions) no smooth solution can be expected, because singularities must appear and/or disappear for the free boundary (after all, an ice-cube eventually disappears when heated up).

So, following several authors (from Kamin, Oleinik, etc. ... to Friedman and Ladyženskaia-Solonnikov-Ural'ceva (see references at the end of this paper), we introduce the notion of a weak solution. In order to do so, we put

$$\gamma(r) = \int_0^r \varrho(s) ds + b \operatorname{sign}^+(r)$$

seen as a maximal monotone graph, and $\beta = \gamma^{-1}$ as its inverse graph, which in this case is locally Lipschitz continuous on $\mathbb{R}$ with a flat part corresponding to the change of phase.

If we assume that a smooth solution (S, v) exists, then, multiplying (4) by a test function w in $C_0^1(\bar{Q})$, we get.

$$\int_{Q-S} \varrho(v) \frac{\partial v}{\partial t} \cdot w = \int_Q \frac{\partial}{\partial t} (\gamma(v)) \cdot w = - \int_Q \gamma(v) \frac{\partial w}{\partial t} + \int_S [\gamma(v)]_S w(s, t) \cos(v, t).$$

Similarly,

$$\int_{Q-S} (-\Delta v \cdot w) = \int_{Q-S} \nabla w \cdot \nabla v - \int_S w \left[\sum_i \frac{\partial v}{\partial x_i} \cos(v, x_i) \right]_S.$$

Therefore the choice of jump-discontinuity b for γ allows us to make use of the Stefan condition (6) to cancel out the surface terms:

$$(7) \quad \int_Q \left(-\gamma(v) \frac{\partial w}{\partial t} \right) + \int_Q \nabla v \cdot \nabla w - \int_Q f w = 0.$$

More generally, assume that the lateral boundary conditions are of the following type:

Let $\Gamma^+ \oplus \Gamma^-$ be a partition of Γ , and denote by $\Sigma^\pm$ the product $\Gamma^\pm \times (0, T)$; let g be a smooth function from $(0, T)$ to $H^1(\Omega)$ and consider the following boundary conditions

$$(8) \quad v - g = 0 \quad \text{on} \quad \Sigma^-,$$

$$(9) \quad \frac{\partial}{\partial n} (v - g) + p(v - g) = 0 \quad \text{on} \quad \Sigma^+,$$

where $p(x)$ is a non-negative measurable function on Γ^+ which measures the permeability to heat of the boundary at the point x (actually g can be any lifting in Q of such boundary conditions on Σ , and so without loss of generality we can assume that $\Delta g = 0$ a.e. in Q). In this situation, if we write

$$(10) \quad a(v, w) = \int_\Omega \nabla v \cdot \nabla w + \int_{\Gamma^+} p v w,$$

equation (7) can be written as

$$(11) \quad \int_Q -\gamma(v) \frac{\partial w}{\partial t} + \int_0^T a(v - g, w) = \int_Q f w + \int_\Omega w(0) \gamma(v(0))$$

for all w 's in $C^1(\bar{Q})$ with $w = 0$ on the lateral boundary and at $t = T$. We shall call such a v satisfying (8) and (11) a *weak solution* for problem (4) (5) (6) (8) (9) with initial data v_0 . However, $u_0 = \gamma(v(0))$ has to be made precise. In the classical case, v_0 is given as well as S_0 , the initial position of the interface, and so u_0 can be constructed as follows:

$$u_0 = \gamma(v_0) \quad \text{for} \quad v_0 \neq 0.$$

For $v_0(x) = 0$, $u_0 = \gamma(0^+) = b$ for x in the initial liquid phase,

$$u_0 = \gamma(0^-) \quad \text{for } x \text{ in the initial solid phase;}$$

this choice will be understood more clearly when we introduce the notion of enthalpy.

Remark. The fact that the free boundary S has disappeared from the formulation is the main advantage of this approach. It is essential to realize that this is possible only because the Stefan condition is a Rankine–Hugoniot type condition associated with equation (4) taken in the distribution sense; in other words, condition (11) can be seen as a “weak” formulation for a *distribution* equation, which we will write in terms of u instead of v , where u is the proper section of the $\gamma(v)$ which works in (11):

$$(12) \quad \begin{aligned} \frac{\partial u}{\partial t} - \Delta(\beta(u) - g) &= f \quad \text{in } Q, \\ u(0) &= u_0 \quad \text{in } \Omega, \\ \beta(u) - g &= 0 \quad \text{on } \Sigma^-, \\ \frac{\partial}{\partial n}(\beta(u) - g) + p(\beta(u) - g) &= 0 \quad \text{on } \Sigma^+. \end{aligned}$$

The function u introduced above represents the so-called *thermodynamical enthalpy*, and what is important is that the enthalpy of the system characterizes its whole state (locally).

In the next section we will see that (12) can be interpreted as a parabolic variational inequality and solved accordingly, by using a method generalizing that of H. Brezis.

II. Variational formulation and existence and uniqueness results

Recall that u is a generalized solution for the Stefan problem whenever we have:

$$\begin{aligned} u &\in L^1(Q), \quad \nabla(\beta(u) - g) \in L^1(Q), \\ p(\beta(u) - g) &\in L^1(\Sigma^+), \quad \beta(u) - g = 0 \quad \text{on } \Sigma^- \end{aligned}$$

and

$$(2.1) \quad - \int_Q u \frac{\partial w}{\partial t} + \int_Q \nabla w \cdot \nabla(\beta(u) - g) + \int_{\Sigma^+} p(\beta(u) - g) w = \int_Q f w + \int_{\Omega} u_0 w(0)$$

holds for every w in $C^1(\bar{Q})$ such that $w(T) = 0$ and $w = 0$ on Σ^- .

We shall now prove the following theorem, for which we need some notation. First we put

$$j(\Gamma) = \int_0^r \beta(s) ds;$$

j is a convex function which is continuous on $\mathbf{R}$ with values in $\mathbf{R}^+$. More generally, we can accept j to be l.s.c. only with values in $\mathbf{R}^+ \cup \{+\infty\}$. Next, for α in $[1, +\infty)$, we define hypothesis $(\mathcal{H}_\alpha)$ as follows (N is the dimension of Ω):

$$\text{if } \alpha \geq \frac{2N}{N+2} \text{ and } \alpha > 1: \liminf_{|r| \rightarrow +\infty} \frac{j(r)}{|r|^\alpha} > 0;$$

if $\alpha \in (1, 2N/(N+2))$, j satisfies the Δ_2 condition at infinity (i.e., $\exists k \forall r$, $f(2r) \leq k(1+f(r))$, which is a growth condition at infinity satisfied for polynomial growth) and

$$\liminf_{|r| \rightarrow +\infty} \frac{j(r)}{|r|^\alpha} > 0;$$

if $\alpha = 1$, j satisfies the Δ_2 condition at infinity and

$$\lim_{|r| \rightarrow +\infty} \frac{j(r)}{|r|} = +\infty.$$

Remarks. $\alpha = 2$ corresponds to the usual physical case, the Δ_2 -condition at infinity can be weakened or even got rid of in the case where Γ^+ is globally the graph of a Lipschitz continuous function or when $\Omega \cup \Gamma^-$ is star-shaped.

(2.2) THEOREM. Under hypothesis $\mathcal{H}_\alpha$, let f be in $L^2(Q)$ and g in $\text{BV}(0, T; L^{\alpha'}(\Omega))$ (where $1/\alpha + 1/\alpha' = 1$).

Let u_0 satisfy $\int_\Omega j(u_0(x)) dx < +\infty$ (and for $\alpha < 2N/(N+2)$ assume that $w \mapsto \int_\Omega w \cdot u_0$ extends to a continuous linear form on the $H^1(\Omega)$ -closure of the set $\{w \in C^1(\bar{\Omega}); w = 0 \text{ on } \Gamma^-\}$).

Then there exists a unique generalized solution u for the Stefan problem in $W^{1,2}(0, T; H^{-1}(\Omega))$; furthermore, u belongs to $L^\infty(0, T; L^\alpha(\Omega))$. The corresponding weak solution v satisfies $v - g \in L^2(0, T; H^1(\Omega))$.

There is also a regularizing effect on the initial condition, namely, if u_0 is only in the H^{-1} closure of the set of admissible initial conditions above, then there is a unique generalized solution u in

$$\mathcal{C}([0, T]; H^{-1}(\Omega)) \cap W_{\text{loc}}^{1,2}((0, T]; H^{-1}(\Omega)),$$

which also satisfies

$$u \in L^2(Q), \quad t^{1/2}u \in L^2(0, T; L^2(\Omega)), \quad \sqrt{t} \frac{du}{dt} \in L^2(0, T; H^{-1}(\Omega)),$$

and the corresponding weak solution satisfies

$$\sqrt{t}(v-g) \in L^2(0, T; H^1(\Omega)).$$

The proof of theorem (2.2) will be a direct application of the existence results for abstract evolution problems involving subdifferential operators on a Hilbert space. To do so we define the space

$$V = \{u \in H^1(\Omega); u|_{\Gamma^-} = 0\}$$

with the scalar product

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Gamma^+} puv.$$

Clearly, V is a Hilbert space under the scalar product a (this is actually true provided $\Gamma^- \cup \{x \in \Gamma^+; p(x) > 0\}$ is not a null set of the boundary Γ).

It is easy to check that H_0^1 is closed in V and that $C^1(\bar{\Omega}) \cap V$ is dense in V (because of the regularity of the boundary partition in Γ^+ and Γ^-). We consider the dual space V' of V in the duality which identifies $L^2(\Omega)$ with its dual space, and denote by A the duality mapping from V to V' . The mapping A is the isometry defined by

$$\langle Au, v \rangle_{V', V} = a(u, v) \quad \text{for } u, v \text{ in } V.$$

In this framework V' can be identified with the quotient space of $(H^1(\Omega))'$ by the orthogonal of V . On V' we define a convex function

$$(2.3) \quad \Phi(u) = \begin{cases} \int_{\Omega} (j(u) - g \cdot u) dx & \text{for } u \text{ in } V' \cap L^2(\Omega), \\ +\infty & \text{elsewhere.} \end{cases}$$

Here the t -dependence of g is not indicated and neither is that of Φ . Note that, since $g \in L^2(\Omega)$, $j(u) - gu$ is always semi-integrable below, so that its integral makes sense in $\mathbb{R} \cup \{+\infty\}$.

Remark. For $\alpha \geq 2N/(N+2)$, it is easy to check (by the duality of Sobolev's imbedding of $H^1(\Omega)$) that $L^2 \subset V'$. For $\alpha < 2N/(N+2)$, $u \in V' \cap L^2$ is to be understood as follows: there is a sequence U_n in $L^2(\Omega)$ which converges to u both in L^2 and in V' .

The functional Φ satisfies the following under the hypothesis $\mathcal{H}_\alpha$:

(2.4) PROPOSITION. Φ is l.s.c. convex and proper on V' .

This is due to the coerciveness of j indicated in $\mathcal{H}_\alpha$ (for $\alpha = 1$ one uses

the Dunford-Pettis criterion of weak compactness in $L^1(\Omega)$) and to the use of Fatou's lemma.

In the subsequent proofs the following lemma will play a crucial role:

(2.5) LEMMA. For $1 \leq \alpha < 2N/(N+2)$ and u in $D(\Phi)$ there exists a sequence in $L^\alpha(\Omega)$ such that $u_n \rightarrow u$ in $L^\alpha(\Omega)$, in V' , and almost everywhere with $\Phi(U_n) \rightarrow \Phi(u)$.

This lemma is proved by truncation and regularization, making use of the fact that j is convex and of the Δ_2 condition at infinity, or its replacement as in the remark following (2.1), (the case $\alpha \geq 2N/(N+2)$ is also true but much simpler).

We want to determine the subdifferential on V' of Φ , and here we insist on the fact that we do not want to consider this subdifferential as a mapping from V' to V . First we recall that the scalar product on V' is the one lifted back from V via A :

$$\langle u, v \rangle_{V'} = \langle A^{-1}u, v \rangle_{V, V'} = \langle A^{-1}v, u \rangle_{V, V'} = a(A^{-1}u, A^{-1}v).$$

PROPOSITION. $v \in \partial\Phi(u) \Leftrightarrow u \in D(\Phi), \exists w \in V$ with $w + g = \beta(u)$ a.e. in Ω and $Aw = v$.

In other words, $v = A(\beta(u) - g)$.

Proof. We recall first that $v \in \partial\Phi(u)$ is by definition equivalent to $\forall w \in V', \Phi(w) - \Phi(u) \geq \langle w - u, v \rangle_{V'}$.

It Φ^* is the conjugate of Φ on V' , defined by

$$\Phi^*(v) = \sup_{u \in V'} \langle u, v \rangle_{V'} - \Phi(u),$$

then

$$\Phi(u) + \Phi^*(v) \geq \langle u, v \rangle_{V'}$$

always holds and equality holds if and only if $v \in \partial\Phi(u)$ (Young-Fenchel's characterization of the subdifferential). So we first compute $\Phi^*(v)$, using $A^{-1}v = w$:

$$\Phi^*(v) = \sup_{u \in V'} \langle u, w \rangle_{V, V'} - \Phi(u).$$

By lemma (2.5)

$$\Phi^*(v) = \sup_{u \in L^\alpha(\Omega)} \langle u, w \rangle_{V, V'} - \Phi(u) = \sup_{u \in L^\alpha(\Omega)} \int_\Omega u \cdot w + ug - j(u).$$

Here, making use of a celebrated result of R.T. Rockafellar, which allows us to commute integration and supremum in the last line, we get

$$\Phi^*(v) = \int_\Omega \sup_{r \in \mathbb{R}} (r(w+g)(x) - j(r)) dx = \int_\Omega j^*(w+g)(x) dx$$

where j^* is the Fenchel conjugate of j on $\mathbf{R}$. Now $\Phi(u) + \Phi^*(v) = \langle u, v \rangle_{V'}$ becomes

$$\int_{\Omega} j(u) + j^*(w+g) - g \cdot u = \langle u, v \rangle_{V'} = \langle u, w \rangle_{V', V}.$$

But, using lemma (2.5) again, we get for some sequence U_n in L^∞ converging to u in L^2 and V' :

$$\langle U_n, w \rangle_{V', V} = \int_{\Omega} U_n w,$$

so that

$$\int_{\Omega} j(U_n) + j^*(w+g) - (g+w) U_n \rightarrow 0.$$

Since the integrand above is everywhere non-negative, by taking a subsequence it converges a.e. to zero, i.e., $j(u) + j^*(w+g) - (g+w)u = 0$ a.e., which is equivalent to $g+w \in \beta(u)$ a.e. The converse is obtained as follows: we assume that $j(u) + j^*(w+g) - (w+g)u = 0$ a.e. and $u \in D(\Phi)$. Hence $u \cdot w$ is bounded below by an integrable function. Using the approximation lemma (2.5) and Fatou's lemma, we get $\int_{\Omega} w \cdot u \leq \langle v, u \rangle_{V'}$ and, integrating the former inequality, we obtain

$$\Phi(u) + \Phi^*(v) = \int_{\Omega} w \cdot u \leq \langle v, u \rangle_{V'};$$

hence equality holds.

We now can turn back to (2.1) and interpret it in the sense of vector-valued distributions (from $(0, T)$ to V') as

$$(2.6) \quad \frac{du}{dt} + \partial\Phi(t, u) = f, \quad u(0) = u_0,$$

for which we can apply the results of abstract Cauchy problems in Hilbert spaces involving subdifferentials. In our particular case $D(\Phi(t, \cdot))$ is independent of t , and so the hypothesis most amenable to this problem is the one introduced in Attouch-Damlamian, which guarantees the existence and uniqueness of the solution (and exactly the regularity stated in theorem (2.2)):

$$\forall u \in D(\Phi(s)) \quad \forall 0 \leq s < t \leq T$$

$$\Phi(t, u) \leq \Phi(s, u) + (a(t) - a(s))(\Phi(s, u) + |u| + 1),$$

for some a non-decreasing on $[0, T]$ and independent of u . This can be checked in our situation from the fact that

$$\Phi(t, u) - \Phi(s, u) = \int_{\Omega} (g(t) - g(s))u \leq \|g(t) - g(s)\|_{L^{2'}(\Omega)} \|u\|_{L^2(\Omega)}$$

and from the other estimate (due to $\mathcal{H}_\alpha$)

$$|u|_{L^2(\Omega)}^2 \leq C(\phi(s, u) + \text{meas } \Omega) + |g|_{L^\infty(0, T; L^2(\Omega))} \cdot |u|_{L^2(\Omega)}.$$

This completes the proof of theorem (2.2).

III. Approximation and regularity

In this section we consider two refinements of the results of the previous section they are both based on the possibility of approximating problem (12) by regular approximations on which a priori estimates of various types can be applied and carried out to the limit.

Recently Irena Pawlow and Marek Niezgodka have used this type of technique for much more general situations and have obtained very nice results, generalizing in particular the conditions under which the problem is well posed and there is regular dependence upon the data.

(3.1) PROPOSITION. *Let γ_ϵ be a sequence of smooth monotone functions which converge to γ in the sense of the graphs (this is equivalent to saying that $(\text{Id} + \gamma_\epsilon)^{-1}$ converges to $(\text{Id} + \gamma)^{-1}$ everywhere) and such that $(\gamma_\epsilon)^{-1} \equiv \beta_\epsilon$ is also smooth. Assume that $\alpha \geq 2N/(N+2)$ and β_ϵ satisfies also $(\mathcal{H}_\alpha)$.*

Consider then the (classical) solutions u_ϵ of the problem

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} - \Delta \beta_\epsilon(u_\epsilon) &= f_\epsilon \quad \text{in } \Omega_\epsilon \times (0, T), \\ \beta_\epsilon(u_\epsilon) - g_\epsilon &= 0 \quad \text{on } \Sigma_\epsilon^-, \\ \frac{\partial(\beta_\epsilon(u_\epsilon) - g_\epsilon)}{\partial \nu} + p_\epsilon(\beta_\epsilon(u_\epsilon) - g_\epsilon) &= 0 \quad \text{on } \Sigma_\epsilon^+, \\ u_\epsilon(0) &= u_{\epsilon,0}, \end{aligned}$$

where Ω_ϵ , g_ϵ , p_ϵ , $u_{\epsilon,0}$, $\Sigma_\epsilon^\pm$ are smooth approximations for Ω , g , p , u_0 , $\Sigma^\pm$ which converge in the obvious sense as ϵ tends to zero.

Then, as ϵ tends to zero, u_ϵ converges to the generalized solution u of the limit Stefan problem, and $v_\epsilon = \beta_\epsilon(u_\epsilon)$ converges to the weak solution $v = \beta(u)$ in the following spaces:

$$\begin{aligned} u_\epsilon &\rightharpoonup u \quad \text{in } W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \\ v_\epsilon - g_\epsilon &\rightharpoonup v - g \quad \text{in } L^2(0, T; H^1(\Omega)) \end{aligned}$$

(with some more conditions due to the boundary term on Σ_ϵ^-).

The proof is quite elementary as soon as one realizes that the estimates obtained in the previous section hold here uniformly with respect to ϵ . By weak compactness we can assume (for a subsequence which by uniqueness

for the limit problem will be the whole sequence a posteriori) that $u_\varepsilon \rightharpoonup u$, $v_\varepsilon \rightharpoonup v$ in the weak topologies above. By Aubin's lemma we find that u_ε converges strongly to u in $C([0, T]; H^{-1}(\Omega))$ and all we are left to show is that $v = \beta(u)$ a.e. in Q . In order to do so we use the maximal monotonicity of the natural extension of β to $L^r(0, T; L^2(\Omega)) \times L^1(0, T; L^r(\Omega))$. Therefore it is enough to guarantee that

$$\int_{\Omega'} uv \geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega'} u_\varepsilon v_\varepsilon$$

for every $\Omega' \subset \Omega$ to obtain $v = \beta(u)$ a.e. in Q . This in turn is obtained simply because the convergences are in topologies that are in duality.

Remark. For $\alpha < 2N/(N+2)$, $L^\alpha(\Omega)$ is not longer compact in H^{-1} , anymore, but the same type of method works under the Δ_2 condition at infinity, which allows us to use the Orlicz spaces associated with f and f^* instead of the spaces L^2 and $L^r(\Omega)$.

Making use of the above proposition, we can prove the following

(3.2) THEOREM. *If β is Lipschitz continuous, then the weak solution v of the Stefan problem satisfies*

$$\frac{\partial v}{\partial t} \in L^2(Q), v \in L^\alpha(0, T; H^1(\Omega))$$

provided

$$g \in W^{2,1}(0, T; L^2(\Omega)) \cap L^\alpha(0, T; H^1(\Omega)) \quad \text{and} \quad v_0 - g(0) \in V$$

(i.e., $v_0 - g(0)$ vanish on Γ^-).

To prove this result, because of proposition (3.1) we only need to show the same results uniformly in ε on some proper approximation. We will not write the dependence with respect to ε as it will be obvious. Multiplying the equation by $\partial/\partial t(v-g)$, we have

$$\int_{\Omega} \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} + a\left(v-g, \frac{\partial}{\partial t}(v-g)\right) = \int_{\Omega} f \cdot \frac{\partial}{\partial t}(v-g) + \int_{\Omega} \frac{\partial u}{\partial t} \cdot \frac{\partial g}{\partial t}.$$

All we do now is integrate on $[0, T]$ and by parts for the last term. If k is the lipschitz constant for β , we have

$$\frac{\partial u}{\partial t} \cdot \frac{\partial v}{\partial t} \geq \frac{1}{k} \left| \frac{\partial v}{\partial t} \right|^2,$$

whence the results by standard arguments.

Other applications of proposition (3.1) are the following comparison results, inspired by the works of Philippe Benilan.

(3.3) THEOREM. *The assumptions are those of proposition (3.1). Let $\hat{f}$, $\hat{g}$, $\hat{u}_0$ be another set of data for the corresponding solution $\hat{u}$. Assume further that $\hat{g} = g$ on Σ^- ; then*

$$\begin{aligned} |u(t) - \hat{u}(t)|_{L^1(\Omega)} \leq & |u_0 - \hat{u}_0|_{L^1(\Omega)} + \int_0^t \|f - \hat{f}\|_{L^1(\Omega)} dr + \\ & + \int_0^t \|(\partial/\partial n + p)(g - \hat{g})\|_{L^1(\Gamma^+)} dr. \end{aligned}$$

This type of estimate is due to the formal accretiveness of the operator $-\Delta\beta$ in $L^1(\Omega)$. Physically the L^1 integral of u is the total energy of the system, so that it is natural to obtain this estimate.

Proof. Using a test function w for the smooth approximations, we get (again dropping the ε -dependence)

$$\int_{\dot{Q}} \frac{\partial(u - \hat{u})}{\partial t} w + \int_0^T \int_{\Omega} a(v - \hat{v}, w) = \int_{\Sigma^+} (G - \hat{G}) w + \int_{\dot{Q}} (f - \hat{f}) w$$

where $G = (\partial/\partial n + p)g|_{\Sigma^+}$ and similarly for $\hat{G}$.

We now take

$$w(G, x) = \theta(v(G, x) - \hat{v}(G, x)) 1_{[0, t]}(r) \quad \text{where } \theta' \text{ is in } D^+(\mathbf{R}) \text{ and } \theta(0) = 0.$$

By a simple computation we get

$$\begin{aligned} \int_0^t \int_{\Omega} \frac{\partial(u - \hat{u})}{\partial t} \theta(v - \hat{v}) + \int_0^t \int_{\Gamma^+} p(v - \hat{v}) \theta(v - \hat{v}) \\ \leq \int_0^t \int_{\Gamma^+} (G - \hat{G}) \theta(v - \hat{v}) + \int_0^t \int_{\Omega} (f - \hat{f}) \theta(v - \hat{v}). \end{aligned}$$

We now let θ converge to the sign function and use the fact that $\text{sign}(v - \hat{v}) = \text{sign}(u - \hat{u})$ because the approximating β 's are strictly increasing. Therefore we get

$$\int_0^t \int_{\Omega} \frac{\partial(u - \hat{u})}{\partial t} \text{sign}(u - \hat{u}) + \int_0^t \int_{\Gamma^+} p|v - \hat{v}| \leq \int_0^t \int_{\Gamma^+} |G - \hat{G}| + \int_0^t \int_{\Omega} |f - \hat{f}|.$$

Finally the first terms is easily identified with $\int_0^t d/dt \int_{\Omega} |u - \hat{u}|$, which yields the result for the approximate solutions and is carried over to the weak and generalized solutions.

A similar theorem holds using T-accretiveness:

(3.4) THEOREM. *Under the same hypotheses as those of theorem (3.3) but assuming only $g - \hat{g} \leq 0$ on Σ^- , we have*

$$\|(u(t) - \hat{u}(t))^+\|_{L^1(\Omega)} \leq \|(u_0 - \hat{u}_0)^+\|_{L^1(\Omega)} + \int_0^t \|(G - \hat{G})^+\|_{L^1(\Gamma^+)} + \int_0^t \|(f - \hat{f})^+\|_{L^1(\Omega)}.$$

In particular, if $u_0 \leq \hat{u}_0$ a.e. in Ω , $G \leq \hat{G}$ a.e. on Σ^+ , $f \leq \hat{f}$ a.e. in Q , then $u \leq \hat{u}$ a.e. in Q , and the same holds also for $v \leq \hat{v}$.

The proof is the same as above except for θ , which is taken to converge to sign^+ instead of sign .

As a consequence we can give an application to non-linear Lipschitz perturbations of the Stefan problem:

(3.5) PROPOSITION. *Let ϱ be Lipschitz continuous from $\mathbf{R}$ to itself. Assume that f , g , u_0 and β are the same as in theorem (3.2); then there is a unique generalized solution u for the problem whose strong formulation is*

$$\frac{\partial u}{\partial t} - \Delta \beta(u) = f + \varrho(u),$$

with the usual parabolic boundary conditions. Furthermore, u can be obtained as the limit of the standard iteration scheme.

Proof. Denote by Su the (generalized) solution of the problem

$$\frac{\partial(Su)}{\partial t} - \Delta \beta(Su) = f + \varrho(u),$$

with boundary conditions. Then we have

$$\|Su - S\hat{u}\|_{L^\infty(0,t;L^1(\Omega))} \leq K \|u - \hat{u}\|_{L^1(0,t;L^1(\Omega))},$$

where K is the Lipschitz constant of ϱ .

From this it is standard to infer that some power of S is a strictly contracting mapping in $L^\infty(0, T; L^1(\Omega))$, which ensures the existence and uniqueness for the fixed point of the closure of S in $L^\infty(0, T; L^1(\Omega))$.

It is then necessary to show that the fixed point obtained in this fashion is actually a fixed point of S itself, which is done when β has an affine growth at infinity (or when an *a priori* L^∞ bound is known). As a final application we can see how to treat a non-coercive problem via a simple perturbation argument. Consider a pure Neumann boundary condition for the Stefan problem; the previous theory does not apply directly because the bilinear form $a(v, w) = \text{grad } v \cdot \text{grad } w$ is not coercive on the whole of $H^1(\Omega)$. If, instead, we take $a(v, w) = \int_\Omega (\text{grad } v \cdot \text{grad } w + v \cdot w)$, which is the natural

scalar product on $H^1(\Omega)$, then the strong formulation of the problem which we can solve in a weak sense is

$$\frac{\partial u}{\partial t} - \Delta \beta(u) + \beta(u) = f \quad \text{in } Q,$$

with the parabolic boundary conditions. Provided β is Lipschitz continuous, we can apply a result similar to that of (3.5) and perturb this problem by $\beta(u)$ in order to recover the original Stefan problem in a non-coercive case. This is in agreement with the fact (well known in linear theory) that semi-coerciveness is enough for parabolic problems to have existence and uniqueness.

IV. The Stefan problem with variable lateral boundary conditions

The results of this section are taken from a joint work of the present author and N. Kennochi, to which we refer for details. In this framework, the partition of the boundary $\Sigma = (0, T) \times \Gamma$ is not time independent as in the previous section. Here the main difficulty will arise from the fact that Σ is partitioned in a skew fashion, namely $\Sigma = \Sigma^- \oplus \Sigma^+$ where $\Sigma^- = \bigcup_{t \in (0, T)} \{t\} \times \Gamma(t)$, $\Gamma(t)$ now depending upon t . We look for a generalized solution u :

$$\left\{ \begin{array}{l} \forall \varphi \in C^1(\bar{Q}) \quad \text{with } \varphi|_{\Sigma^-} = 0 \text{ and } \varphi(T) = 0, \\ - \int_Q \frac{\partial \varphi}{\partial t} u + \int_0^T a(v, \varphi) = \int_Q f \cdot \varphi + \int_{\Omega} u_0 \varphi(0), \\ v = \beta(u) - g \text{ in } Q \text{ (we have changed the notation here),} \\ v = 0 \text{ on } \Sigma^-, \end{array} \right.$$

where as usual $a(v, \varphi) = \int_{\Omega} \nabla v \cdot \nabla \varphi + \int_{\Gamma} p v \varphi$.

This problem is much more complicated even in the linear case (cf. Baiocchi, who solves the linear case completely).

Here we will assume that β is Lipschitz continuous and has a linear growth at infinity and we will look for solutions satisfying the following conditions:

- a) u is weakly continuous from $[0, T]$ to $L^2(\Omega)$, with $u(0) = u_0$;
- b) $v = \beta(u) - g$ is in $L^2(0, T; H^1(\Omega))$ with $v(t) = 0$ on $\Gamma(t)$, a.e. t ;
- c) (4.1) is satisfied for every φ in the space

$$W(Q) = \{\varphi \in H^1(Q) \mid \varphi|_{\Sigma^-} = 0, \varphi(T) = 0\},$$

In order to apply a variational method we introduce

$$V(t) = \{z \in H^1(\Omega); z|_{\Gamma(t)} = 0\}, \quad \mathcal{X} = L^2(0, T; H^1(\Omega)),$$

$$\mathcal{V} = \{\varphi \in \mathcal{X}; \varphi(t) \in V(t) \text{ a.e. } t\} \quad \text{and} \quad \mathcal{W} = \{\varphi \in \mathcal{V}; \varphi' \in \mathcal{X}'\}.$$

We therefore look for v in $\mathcal{V}$, which is clearly a closed subspace of $\mathcal{X}$. The difficulty here is due to the variations of $V(t)$ with respect to time, which are in no way controllable in $H^1(\Omega)$. But things are even worse for the dual space $V'(t)$, which is not naturally imbedded in a fixed space. Here again we shall regularize γ by taking $\beta_v = \beta + v \text{Id}$, which corresponds to a non-degenerate problem, leaving for the last step the limiting process for $v \rightarrow 0^+$.

DEFINITION. The "weak d/dt " operator L_{u_0} is defined by its graph:

$$f \in L_{u_0} u \Leftrightarrow u \in \mathcal{V}, f \in \mathcal{X}' \text{ and}$$

$$\forall \Phi \in \mathcal{W} \quad -\langle \Phi', u \rangle_{\mathcal{X}', \mathcal{X}} = \langle f, \Phi \rangle_{\mathcal{X}', \mathcal{X}} + (u_0, \Phi(0))_{L^2(\Omega)}.$$

This operator L_{u_0} will play the part of d/dt , incorporating also the constraints on the boundary.

It turns out that it can be defined equivalently as follows (see Kenmochi-Nagai for the proof):

PROPOSITION. $f \in L_{u_0} u \Leftrightarrow \exists u_n \in \mathcal{W}$ such that:

$$u_n(0) \rightarrow u_0 \text{ in } L^2(\Omega), u_n \rightarrow u \text{ in } \mathcal{V}, \text{ and } u'_n \rightarrow f \text{ in } \mathcal{X}'/\mathcal{V}^\perp$$

(with respect to this semi-norm only).

The properties of L_{u_0} are listed in.

(4.2) **PROPOSITION.** (a) $u \in D(L_{u_0}) \Rightarrow u \in \mathcal{C}([0, T]; L^2(\Omega))$ and $u(0) = u_0$;

b) if $u \in D(L_{u_0})$, $v \in D(L_{r_0})$ then we have the formula of integration by parts for $f \in L_{u_0}(u)$, $g \in L_{r_0}(v)$,

$$(u(t), v(t))_{L^2(\Omega)} - (u(s), v(s))_{L^2(\Omega)}$$

$$= \int_s^t \langle g(\sigma), u(\sigma) \rangle_{(H^1)_V, H^1} d\sigma + \int_s^t \langle f(\sigma), v(\sigma) \rangle_{(H^1)_V, H^1} d\sigma;$$

consequently L_{u_0} is monotone from $\mathcal{X}$ to $\mathcal{X}'$;

c) the graph of L_0 is a closed linear subspace of $\mathcal{X} \times \mathcal{X}'$ and that of L_{u_0} is a translate of L_0 's provided it is not empty.

Now these operators can be empty and what is needed is a condition which guarantees that L_{u_0} or L_0 contains enough elements to be useful. The proper condition is not known explicitly, but implicitly it is the following.

PROPOSITION. Denote by F the duality mapping from $\mathcal{X}$ to $\mathcal{X}'$ (it is just the natural extension of the duality mapping from $H^1(\Omega)$ to its dual space).

Assume that there exists a dense subset D of $\mathcal{X}'$ such that for each f in D , there exists a (strong) solution u to

$$u' + Fu = f, \quad u(0) = 0 \quad \text{in } \mathcal{X} \text{ (hence in } \mathcal{Y}).$$

Then L_0 is maximal monotone from $\mathcal{X}$ to $\mathcal{X}'$ and (so are the L_{u_0} 's provided they are not empty).

Proof. Recall F. Browder's characterization of maximal monotonicity of an operator L from a reflexive space $\mathcal{X}$ to its dual $\mathcal{X}'$: $R(R+L) = \mathcal{X}'$ for the duality mapping F . Thus, what the hypothesis says is that the range of $F + L_0$ is dense in $\mathcal{X}'$. On the other hand, it is also a fact that the range of $F + L_0$ is closed (by using monotonicity $(F + L_0)^{-1}$ is a contradiction and the graph of L_0 is linear and thus weakly closed).

When looking for explicit conditions, we only found the following hypothesis and variant, which imply maximal monotonicity:

(H) There exists a Lipschitz continuous arc θ from $[0, T]$ into the manifold of C^1 diffeomorphisms of $\bar{\Omega}$ such that $\Gamma(t) = \theta(t)(\Gamma(0))$.

(H') There is a piecewise Lipschitz continuous arc θ from $[0, T]$ into the same set such that for θ Lipschitz continuous on $[s, t]$ we have

$$\Gamma(t) = \theta(t) \circ \theta(s)^{-1}(\Gamma(s)),$$

and for every jump point t_i of M we have $\Gamma(t_i^-) \supset \Gamma(t_i^+)$.

Now, using the maximal monotonicity of L_{u_0} , we can solve the approximate problem (with β_v), making use of the L_{u_0} pseudo-monotonicity properties of B^v (as defined below), which guarantee (via a result of Brezis) that $L_{u_0} + B^v$ is onto $\mathcal{X}'$. Here we take $B^v u = A(\beta_v(u) - g_v)$, where A is the operator on $H^1(\Omega)$ associated with the bilinear form a , and g_v is a regularization of g .

PROPOSITION. Assume $v > 0$, u_0 in $L^2(\Omega)$, fin $L^2(Q)$ and g_v in $\mathcal{X}$. Assume also that H or H' holds. Then here exists a unique solution u_v for $L_{u_0} u_v + A(B_v(u_v) - g_v) \ni f$.

Furthermore, one has the following estimates, which are independent of v and depend only upon $|u_0|_{L^2(\Omega)}$, $|f|_{L^2(Q)}$ and $|g|_{W^{1,2}(0,T;L^2(\Omega))}$:

$$|\beta_v(u_v) - g|_V \leq C_0, \quad |u|_{L^\infty(0,T;L^2(\Omega))} \leq C_0.$$

Finally, one can pass to the limit as v tends to zero (along a well-chosen subsequence) by applying a compactness lemma which generalized Aubin's well-known lemma to the case of L_{u_0} (instead of d/dt):

LEMMA. Let $(u_n, h_n) \in \mathcal{X} \times \mathcal{X}'$ and $u_{0,n} \in L^2(\Omega)$ such that $h_n \in L_{u_{0,n}} u_n$. Assume that h_n is bounded in $\mathcal{X}'$ and that u_n converges weakly to some u in $L^2(Q)$. Then

$$\sup_{t \in [0,T]} |u_n(t) - u(t)|_{V(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

In particular, this implies that u_v converges to u in $\mathcal{C}([0, T]; H^{-1}(\Omega))$, so that u is weakly continuous from $[0, T]$ to $L^2(\Omega)$.

This allows us also to show (by using Lipschitz continuity and affine growth of β) that $\beta_v(u_v)$ converges to $\beta(u)$ in $L^2(Q)$ and finally to pass to the limit in the weak formulation to obtain (4.1).

The uniqueness of the solution in this framework is not known and seems to be a very interesting question.

V. The homogenization of the Stefan problem

In many physical problems one is interested in the macroscopic behaviour of a medium or a mixture of media. In this chapter we will deal with the case of a mixture of two or more media, which have differing properties and are arranged in a periodic fashion. We show that the limit problem, when the size of the periodic mesh tends to zero is a homogeneous problem with new constitutive laws, which are easier to study numerically than the periodic problem with small periodic mesh.

The details of the proof can be found in a recent article (A. Damlamian SIAM J. Math. Analysis 1981).

Position of the model problem. Let Ω be a regular domain in R^n and, for each $\varepsilon \in]0, \varepsilon_0[$, let Ω be covered by a periodic grid whose period is isometric to εY where Y is a parallelotope in R^n . For simplicity's sake we shall assume the presence of only two media, each one undergoing a change of phase between solid and liquid at a given temperature. The distribution in Ω of the two media is periodic according to the distribution given in Y , that is, we assume that Y is partitioned into Y_1, Y_2 , corresponding to medium M_1 and medium M_2 . The common boundary of Y_1 and Y_2 is denoted by Σ .

Similarly, therefore, Ω is partitioned into $\Omega_{1,\varepsilon}$ and $\Omega_{2,\varepsilon}$ with a common boundary Σ_ε . As regards $\partial\Omega = \Gamma$, it is the union of $\Gamma_{1,\varepsilon}$ and $\Gamma_{2,\varepsilon}$. Σ_ε is supposed to be rigid and heat conducting. For each medium denote by α_i the specific heat (which is assumed to be a positive function of the temperature) and by k_i the heat conductivity ($i = 1, 2$). We shall assume each k_i to be independent of the temperature. The reader will easily understand that the Kirschhoff transform cannot be applied here to reduce a quasilinear diffusion operator to a linear one, because we would need one transform for each medium and therefore lose the "continuity" condition on the new temperature across the boundary Σ_ε , as is used below.

Denote by θ_i the temperature of the change of phase and b_i the latent heat of change of phase for each medium.

By using the usual approximation of constant volume, the problem can be written in the following way: In each medium, i.e., $(0, T) \times \Omega_{i,\varepsilon}$, there may

be a free boundary denoted by $S_{i,\varepsilon}$, with t -section $S_{i,\varepsilon}(t)$. One looks for the temperature $v^\varepsilon(t, x)$ and the two surfaces $S_{i,\varepsilon}$ satisfying

– in $(0, T) \times \Omega_{i,\varepsilon} \setminus S_{i,\varepsilon}$

$$(5.1) \quad \alpha_i(v^\varepsilon) \frac{\partial v^\varepsilon}{\partial t} - \operatorname{div}(k_i \nabla v^\varepsilon) = f,$$

where f is a given right-hand side;

– on $(0, T) \times \Sigma_\varepsilon$ – continuity of v^ε ,
– continuity of the heat flux, namely

$$(5.2) \quad [k_i \nabla v^\varepsilon \cdot \mathbf{n}]_{\Sigma_\varepsilon} = 0,$$

where the above notation indicates the jump across Σ_ε , $\mathbf{n}$ being the unit normal to Σ_ε .

– on $S_{i,\varepsilon}$ – $v^\varepsilon = \theta_i$

$$(5.3) \quad b_i \cos(\mathbf{n}, \mathbf{t}) - \sum_j \left[k_i \frac{\partial v^\varepsilon}{\partial x_j} \cos(\mathbf{n}, \mathbf{x}_j) \right]_{S_{i,\varepsilon}} = 0,$$

$\mathbf{n}$ unit normal to $S_{i,\varepsilon}$ in space-time,

$[]_{S_{i,\varepsilon}}$ indicating the jump across $S_{i,\varepsilon}$ along $\mathbf{n}$.

– Boundary conditions on $(0, T) \times \partial\Omega$ for v^ε .

– Initial conditions for v^ε and $S_{i,\varepsilon}(0)$ (which have to be compatible).

Condition (5.3) is the standard Stefan condition expressed for each medium.

As in the single medium case, it turns out that conditions (5.2) and (5.3) are the Rankine–Hugoniot type conditions for the energy balance equation taken in the sense of distributions in $Q = (0, T) \times \Omega$ (which reduces to (5.1) within each phase of each medium). In order to write the energy balance condition one needs to introduce the enthalpy function, and in order to do so – more compact notation.

For each i , we denote by γ_i a maximal monotone graph satisfying

$$\gamma'_i(\theta) = \alpha_i(\theta) + b_i \delta(\theta - \theta_i).$$

γ_i is defined up to a constant and represents the enthalpy for each medium (the γ_i 's generalize the γ used in the single medium case). Now for y in Y define

$$\gamma(y, \theta) = \gamma_i(\theta) \quad \text{for } y \text{ in } Y_i \ (i = 1, 2).$$

Also define

$$k(y) = k_i \quad \text{for } y \text{ in } Y_i \ (i = 1, 2).$$

Then (5.1), (5.2), (5.3) reduce to the form:

$$(5.4) \quad \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(k(x/\varepsilon) \nabla v^\varepsilon) = f,$$

$$(5.5) \quad u^\varepsilon(t, x) \in \gamma(x/\varepsilon, v^\varepsilon(t, x)),$$

equation (5.4) being satisfied in the distribution sense in Q . The initial condition at $t = 0$ can be expressed as an initial condition u_0^ε for u^ε . As regards the boundary conditions, it is known in the homogenization theory that their nature (provided they are of variational type) does not influence the limit problem. For simplicity we shall take then them to be of Dirichlet type.

Here too, as in the single medium case, we can give a variational formulation:

$$(5.6) \quad \int_Q -\varphi' u^\varepsilon + \int_0^T a^\varepsilon(v^\varepsilon, \varphi) = \int_Q f\varphi + \int_\Omega \varphi(0) u_0^\varepsilon$$

for all φ in $C^1(Q)$, $\varphi(T) = 0$, $\varphi = 0$ on Γ , where a^ε is the bilinear Dirichlet form associated with

$$(5.7) \quad A^\varepsilon = -\operatorname{div}(k(x/\varepsilon) \nabla).$$

As in the previous chapter, a strong solution does not exist in general, and neither does a free boundary; we are only interested in the weak solutions, for which some convergence to the weak solution of a similar "homogeneous" problem is of interest especially if one can obtain some "equivalent limit" constitutive laws for the limit problem (replacing (5.5) and (5.7)).

Existence and convergence of the weak solutions. Using methods similar to those applied in the case of a single medium (the scalar product on H_0^1 , being given by the form a^ε , depends upon ε but remains uniformly coercive, which allows us to keep the estimates uniform in ε), one can prove the following:

THEOREM. *Provided the α_i 's are bounded below away from zero, for each $\varepsilon > 0$, there is a unique solution $(u^\varepsilon, v^\varepsilon)$ to problem (5) (6), with homogeneous Dirichlet boundary conditions. Furthermore,*

u^ε is bounded in $W^{1,2}(0, T; H^{-1}(\Omega))$ and in $L^\infty(0, T; L^2(\Omega))$,

v^ε is bounded in $W^{1,2}(0, T; L^2(\Omega))$ and in $L^\infty(0, T; H'_0(\Omega))$.

The above bounds imply some compactness for u^ε and v^ε . We shall show that any converging sequence of $(u^\varepsilon, v^\varepsilon)$ converges to the solution of another Stefan-type problem for which there is uniqueness, so that the whole family $(u^\varepsilon, v^\varepsilon)$ will converge.

More precisely, for a suitable sequence of ε , u^ε converges to some u^0 in the proper weak topologies, whence in $C([0, T]; H^{-1}(\Omega))$. v^ε converges to a v^0 in the proper weak topologies, whence in $C([0, T]; L^2(\Omega))$.

The limit equation and constitutive laws. The problem now is to find what type of equations can be satisfied by (u^0, v^0) . Some ordinary common sense can help intuition here: first assume that neither medium undergoes a change of phase. Then one only deals with a (non-linear) diffusion problem, for which, in the linear case, the limit equation is known and should be recovered.

So, one should recall the linear result (see Bensoussan-Lions-Papanicolaou [1], or Tartar [1]) of elliptic homogenization, which applies to the linear heat equation.

The operators $A^\varepsilon = -\operatorname{div}(k(x/\varepsilon) \nabla)$ on Ω and $\partial/\partial t + A^\varepsilon$ on $(0, T) \times \Omega$, are homogenized into A^0 and $\partial/\partial t + A^0$, respectively, where

$$(5.8) \quad A^0 = -\sum_{j,l} q_{j,l} \frac{\partial^2}{\partial x_j \partial x_l}$$

with constant symmetric coefficients $q_{j,l}$ given by

$$q_{j,l} = \frac{1}{\operatorname{meas} Y} \int_Y k(y) \nabla(\chi^l - y_l) \cdot \nabla(\chi^j - y_j),$$

where χ^j is the solution (defined up to a constant) of

$$-\operatorname{div}(k(y) \nabla \chi^j) = -\operatorname{div}(k(y) e_j),$$

χ^j being periodic in Y , where e_j is the j th unit vector in R^N .

This means that, if f_ε converges strongly to f in $H^{-1}(\Omega)$ (resp. in $L^2(0, T; H^{-1}(\Omega))$) and u_ε is the solution of $A^\varepsilon u_\varepsilon = f_\varepsilon$ in Ω (resp. $\partial u_\varepsilon / \partial t + A^\varepsilon u_\varepsilon = f_\varepsilon$ in $Q = (0, T) \times \Omega$) in a variational form, i.e., using the bilinear form a^ε associated with A^ε , $a^\varepsilon(u_\varepsilon, w) = (f_\varepsilon, w)$ for every w in some closed subspace V of $H^1(\Omega)$, $u_\varepsilon \in V$ (respectively $u_\varepsilon \in L^2(0, T; V)$, $du_\varepsilon/dt \in L^2(0, T; V')$)

$$\int_0^T \left(\frac{du_\varepsilon}{dt}, w \right) + a^\varepsilon(u_\varepsilon, w) = \int_0^T (f_\varepsilon, w) \quad \forall w \in L^2(0, T; V),$$

then u_ε converges to the solution u_0 of $A^0 u_0 = f$ (similarly for the parabolic case).

Here V and a^ε incorporate the boundary conditions. We should note that any non-homogeneous variational type of boundary condition could be used here (just by getting a proper lifting of the boundary data in Ω ; in this case the lifting would depend upon ε via A^ε as a $g_\varepsilon(t)$, but the elliptic

homogenization theory then shows that $g_\varepsilon(t)$ converges to some $g(t)$, which is the proper lifting of the boundary conditions associated with A^0 .

We now introduce a weak formulation of the Baiocchi type (see also Duvaut), which is equivalent to the weak formulation we have used so far, because we know that u^ε , as well as u^0 , have a derivative with respect to time in $L^2(0, T; H^{-1}(\Omega))$. In order to do so we write

$$V^\varepsilon(t) = \int_0^t v^\varepsilon(s) ds, \quad F(t) = \int_0^t f(s) ds,$$

and we note that the operators A^ε are linear (equivalently a^ε is bilinear) due to the hypotheses of temperature independence of heat conductivities. Now (5.6) can be replaced by (and is equivalent to)

$$(5.9) \quad a^\varepsilon(V^\varepsilon(t), \varphi) = \int_\Omega (F(t) + u_0 - u^\varepsilon(t)) \varphi \quad \text{a.e. in } t,$$

$$V^\varepsilon(0) = 0,$$

$$V^\varepsilon(t) = 0 \quad \text{on } \Gamma \text{ a.e. in } t,$$

or

$$A^\varepsilon V^\varepsilon(t) = F(t) + u_0 - u^\varepsilon(t),$$

$$V^\varepsilon(t) = 0 \quad \text{on } \Gamma \text{ a.e. in } t,$$

which therefore converges, since $u^\varepsilon(t)$ converges strongly in $H^{-1}(\Omega)$ for almost every t and has for the limit the solution $V^0(t)$ of

$$(5.10) \quad A^0 V^0(t) = F(t) + u_0 - u^0(t), \quad V^0(t) = 0 \quad \text{on } \Gamma \text{ a.e. in } t.$$

Here also, reversing the previous procedure, we find that v^0 and u^0 satisfy $v^0 = 0$ on $\Gamma \times (0, T)$ and

$$(5.11) \quad \int_Q -\varphi' u^0 + \int_0^T a^0(v^0, \varphi) = \int_Q f\varphi + \int_\Omega a_0 \varphi(0)$$

for every φ in $C^1(Q)$, $\varphi(T) = 0$, $\varphi = 0$ on $\partial\Omega$.

We now turn to the task of finding the limit for (5.5), that is, to find a relationship, if any, between u^0 and v^0 . Let c be a real number differing from the θ_i 's and $w_\varepsilon^c(x) = \gamma(x/\varepsilon, c)$. Using the fact that, for any φ in $L^p(Y)$, the sequence $\varphi(x/\varepsilon) = \varphi^\varepsilon$ is in $L^p(\Omega)$ and converges weakly in that space to its mean value on Y

$$\hat{\varphi} = \frac{1}{|Y|} \int_Y \varphi(y) dy,$$

we find that w_ε^c converges weakly in L^2 to the constant

$$(5.12) \quad \bar{\gamma}(c) = \frac{1}{|Y|} \int_Y \gamma(y, c) dy.$$

Using the monotonicity of $\gamma(x/\varepsilon, c)$, we have

$$\mu_\varepsilon(t, x) = (u^\varepsilon(t, x) - w_c^\varepsilon(x))(v^\varepsilon(t, x) - c) \geq 0 \quad \text{a.e. in } \Omega,$$

and, making use of the convergence properties we have established, we see that $\mu_\varepsilon(t, x)$ converges weakly in the sense of measures on Ω for all t to $(u^0(t, x) - \bar{\gamma}(c))(v^0 - c)$, which has to be non-negative on Ω . Therefore, by the maximality of the unique extension $\tilde{\gamma}$ of the monotone graph $\bar{\gamma}$, we get

$$(5.13) \quad u^0(t, x) \in \tilde{\gamma}(V^0(t, x)) \quad \text{a.e. in } x \text{ for all } t.$$

So (5.13) and (5.12) give the relationship between the enthalpy and the temperature for the limit problem.

For this problem the constitutive laws are given as follows:

- the diffusion operator is homogeneous but not necessarily (and not usually isotropic (this depends on the geometry of Y_1 and Y_2 in Y)).
- the enthalpy, as a function of the temperature, is obtained by the following method:

$\tilde{\gamma}(v)$ is the average enthalpy of the unit cell at uniform temperature v .

It is of great importance to remark that the discontinuities in γ_1 and γ_2 , which correspond to changes of phase, still appear for $\tilde{\gamma}$, so that the limit medium undergoes changes of phase at all the change-of phase temperatures of both initial media.

Similar equations arise in several problems in electromagnetism where composite ferromagnetic media are studied. The same result can be applied (although the Cauchy problem is replaced by a periodicity condition) and allows us to define an “equivalent magnetic permeability” for the limit medium. This notion is already being used in electrical engineering (see Bossavit–Damlamian [1]).

Many questions concerning similar problems are still unanswered, among them:

- what can be said for temperature-dependent heat conductivity?
- what can be said in the case of heat and transport equation e.g. in the case of hot air flowing through a mesh-like structure partially filled with water?

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