THE CHARACTERIZATION OF EFFICIENT SEQUENTIAL PLANS FOR THE ORNSTEIN-UHLENBECK VELOCITY PROCESS

R. MAGIERA and R. RÓŻAŃSKI

*Institute of Mathematics, Wrocław Technical University,
 Wrocław, Poland*

1. Introduction

The paper considers the problem of sequential estimation for the family of stochastic processes $\{\xi(t) = \xi_{\vartheta}(t), t \in T = [0, \infty)\}$, $\vartheta \in \Theta$, satisfying the stochastic differential equation

$$d\xi_{\vartheta}(t) = [\alpha(\vartheta)\xi_{\vartheta}(t) + \beta(\vartheta)]dt + \gamma dW(t), \quad (1)$$

where $W(t)$, $t \in T$, denotes the standard Wiener process. The characterization of efficient sequential plans is presented and the class of efficiently estimable functions is investigated in the following two important cases of equation (1):

A. $\xi(0) = \text{const} = c$, $\vartheta \in (-\infty, \infty)$, $\alpha(\vartheta) = \text{const} = -a$, $\beta(\vartheta) = a\vartheta$, $\gamma = (2a)^{1/2}\sigma$ with $a > 0$, $\sigma > 0$;

B. $\xi(0)$ is normally distributed with zero mean and the variance $(2\vartheta)^{-1}$, $\vartheta \in (0, \infty)$, $\alpha(\vartheta) = -\vartheta$, $\beta(\vartheta) = 0$, $\gamma = 1$.

In both case A and case B, for every $\vartheta \in \Theta$, the unique solution $\xi_{\vartheta}(t) = \xi(t)$, $t \in T$, of equation (1) is the homogeneous Gaussian Markov process. In the first case the expected value and the covariance function of the process $\xi_{\vartheta}(t)$ are defined by

$$E\xi_{\vartheta}(t) = \vartheta + (c - \vartheta)\exp(-at)$$

and

$$\begin{aligned} K(s, t) &= E[\xi_{\vartheta}(t) - E\xi_{\vartheta}(t)][\xi_{\vartheta}(s) - E\xi_{\vartheta}(s)] \\ &= \sigma^2 \{\exp(-a|t-s|) - \exp[-a(t+s)]\}, \end{aligned}$$

respectively, and the problem is to estimate the drift parameter ϑ . In the second case the process $\xi_\vartheta(t)$ is stationary with $E\xi_\vartheta(t) = 0$ and

$$K(s, t) = (2\vartheta)^{-1} \exp(-\vartheta|t-s|)$$

and the problem is to estimate the parameter ϑ of the spectral density $[2\pi(\lambda^2 + \vartheta^2)]^{-1}$, $-\infty < \lambda < \infty$.

In both case A and case B one obtains two quite different classes of efficient sequential plans and efficiently estimable functions.

2. Preliminaries

An equivalent model of the processes $\{\xi_\vartheta(t), t \in T\}$, $\vartheta \in \Theta$, obtained by the canonical form, will be useful in the forthcoming considerations.

Let C be the space of all continuous real-valued functions $x = x(t)$, $t \in T$, and let $\mathcal{C} = \sigma\{x = x(t), t \in T\}$ denote the minimal σ -algebra consisting of all cylinder sets of C . By μ_ϑ let us denote the measure on $(C, \mathcal{C})$ corresponding to the process $\xi_\vartheta(t)$, $t \in T$: $\mu_\vartheta(B) = P(\xi_\vartheta(\cdot) \in B)$, $B \in \mathcal{C}$. Let $\mu_{\vartheta,t}$ be the truncation of the measure μ_ϑ on $\mathcal{C}_t = \sigma\{x = x(s), s \leq t, s, t \in T\}$.

Let us consider the sequential statistical space $(C, \mathcal{C}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$, corresponding to the family of processes $\{\xi_\vartheta(t), t \in T\}$, $\vartheta \in \Theta$. Let R be the real line and let $\mathcal{B}_R$ denote the σ -algebra of Borel subsets of R . A function $Z(t, x): T \times C \rightarrow R^k$ such that for every $t \in T$ the transformation $Z(t, \cdot)$ is $(\mathcal{C}_t, \mathcal{B}_{R^k})$ -measurable will be called a (k -dimensional) *statistic* on the space $(C, \mathcal{C}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$.

LEMMA 1. 1° The statistical space $(C, \mathcal{C}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$, is dominated, i.e., for every $t \in T$, all measures $\mu_{\vartheta,t}$, $\vartheta \in \Theta$, are absolutely continuous with respect to a measure $\mu_{\vartheta_0,t}$ ($\vartheta_0 \in \Theta$).

2° The densities $d\mu_{\vartheta,t}/d\mu_{\vartheta_0,t}$ are defined by

$$\begin{aligned} \frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(\xi_{\vartheta_0}(\cdot)) = p_{\vartheta_0}(\xi_{\vartheta_0}(0); \vartheta) \exp \left\{ -\frac{1}{2\gamma^2} [(\alpha^2(\vartheta) - \alpha^2(\vartheta_0)) \int_0^t \xi_{\vartheta_0}^2(s) ds + \right. \\ \left. + 2(\alpha(\vartheta)\beta(\vartheta) - \alpha(\vartheta_0)\beta(\vartheta_0)) \int_0^t \xi_{\vartheta_0}(s) ds - (\alpha(\vartheta) - \alpha(\vartheta_0))(\xi_{\vartheta_0}^2(t) - \xi_{\vartheta_0}^2(0)) - \right. \\ \left. - 2(\beta(\vartheta) - \beta(\vartheta_0))(\xi_{\vartheta_0}(t) - \xi_{\vartheta_0}(0)) + (\beta^2(\vartheta) - \beta^2(\vartheta_0) - \gamma^2(\alpha(\vartheta) - \alpha(\vartheta_0)))t \right\}, \quad (2) \end{aligned}$$

where $p_{\vartheta_0}(\cdot; \vartheta)$ denotes the distribution density of the values of the process $\xi_\vartheta(t)$, $t \in T$, at time $t = 0$ relative to that distribution for the process $\xi_{\vartheta_0}(t)$, $t \in T$.

Proof. Using the results on absolutely continuous substitution of measures ([1]) or Skorokhod theorems ([4]), we get assertion 1° of the lemma and the following formula for densities $d\mu_{\vartheta,t}/d\mu_{\vartheta_0,t}$:

$$\begin{aligned} \frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(\xi_{\vartheta_0}(\cdot)) &= p_{\vartheta_0}(\xi_{\vartheta_0}(\cdot); \vartheta) \exp \left\{ -\frac{1}{2\gamma^2} \left[\int_0^t ((\alpha(\vartheta) - \alpha(\vartheta_0)) \xi_{\vartheta_0}(s) + \right. \right. \\ &\quad \left. \left. + \beta(\vartheta) - \beta(\vartheta_0))^2 ds - 2\gamma \int_0^t ((\alpha(\vartheta) - \alpha(\vartheta_0)) \xi_{\vartheta_0}(s) + \beta(\vartheta) - \beta(\vartheta_0)) dW(s) \right] \right\}. \quad (3) \end{aligned}$$

From Ito's formula for processes satisfying equation (1) we obtain

$$\begin{aligned} &\int_0^t \xi_{\vartheta_0}(s) dW(s) \\ &= \frac{1}{2\gamma} \left[\xi_{\vartheta_0}^2(t) - \xi_{\vartheta_0}^2(0) - 2\alpha(\vartheta_0) \int_0^t \xi_{\vartheta_0}^2(s) ds - 2\beta(\vartheta_0) \int_0^t \xi_{\vartheta_0}(s) ds - \gamma^2 t \right]. \quad (4) \end{aligned}$$

By virtue of (1) we have

$$\int_0^t dW(s) = \frac{1}{\gamma} \left[\xi_{\vartheta_0}(t) - \xi_{\vartheta_0}(0) - \int_0^t (\alpha(\vartheta_0) \xi_{\vartheta_0}(s) + \beta(\vartheta_0)) ds \right]. \quad (5)$$

Substituting (4) and (5) in (3), we obtain formula (2).

Formula (2) has the form

$$(d\mu_{\vartheta,t}/d\mu_{\vartheta_0,t})(\xi_{\vartheta_0}(\cdot)) = h(t, Z(t, \xi_{\vartheta_0}(\cdot)); \vartheta, \vartheta_0).$$

It then follows from the Fisher-Neyman theorem on factorization that $Z(t, x)$ is a sufficient statistic on the space $(C, \mathcal{C}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$.

Let $\tau = \tau(x)$, be a finite Markov time with respect to the family $\mathcal{C}_t$, $t \in T$, i.e., $\tau: C \rightarrow [0, \infty]$ such that $\{x: \tau(x) \leq t\} \in \mathcal{C}_t$ for every $t \in T$ and $\mu_{\vartheta}(\{x: \tau(x) < \infty\}) = 1$ for all $\vartheta \in \Theta$. Let $U = T \times R^k$ and let $t = t(u)$ and $z = z(u)$ be the components of the point $u \in U$ such that $t(u) \in T$ and $z(u) \in R^k$. The pair $\mathcal{Z}(x) = (\tau(x), Z(\tau(x), x))$ of both $\mathcal{C}_\tau$ -measurable functions generates for every $\vartheta \in \Theta$ the measure m_{ϑ} on $(U, \mathcal{B}_U)$ in the standard way: for every $A \in \mathcal{B}_U$

$$m_{\vartheta}(A) = \mu_{\vartheta}(\mathcal{Z}^{-1}(A)) = \mu_{\vartheta}((\tau(x), Z(\tau(x), x)) \in A).$$

From the modification of the Sudakov lemma obtained in [3] for right-continuous functionals it follows that the measures m_{ϑ} , $\vartheta \in \Theta$, are absolutely continuous with respect to m_{ϑ_0} and

$$\frac{dm_{\vartheta}}{dm_{\vartheta_0}}(u) = h(t(u), z(u); \vartheta, \vartheta_0).$$

Let $g(\vartheta)$ be a real-valued function of the parameter $\vartheta \in \Theta$. We observe the process $\xi(t)$, $t \in T$, up to time τ and want to estimate the function $g(\vartheta)$. A function $f: U \rightarrow R$ and $(\mathcal{B}_U, \mathcal{B}_R)$ -measurable will be called an *estimator* for $g(\vartheta)$.

A *sequential estimation plan* for $g(\vartheta)$ is a pair $\delta = (\tau, f)$ consisting of a Markov time τ and an estimator f satisfying the following conditions:

$$\mu_{\vartheta}(0 < \tau(x) < \infty) = 1 \quad \text{for each } \vartheta \in \Theta, \quad (6)$$

$$E_{\vartheta} f(\mathcal{Z}(\xi)) = \int_U f(u) h(u; \vartheta, \vartheta_0) dm_{\vartheta_0}(u) = g(\vartheta) \quad (7)$$

and

$$E_{\vartheta} f^2(\mathcal{Z}(\xi)) = \int_U f^2(u) h(u; \vartheta, \vartheta_0) dm_{\vartheta_0}(u) < \infty, \quad \vartheta \in \Theta.$$

It follows from (6) that the observation of the process $\xi(t)$, $t \in T$, terminates in a finite time. Condition (7) means that estimator f is an unbiased estimator for $g(\vartheta)$.

By (6) we have

$$\int_U h(u; \vartheta, \vartheta_0) dm_{\vartheta_0}(u) = 1 \quad (8)$$

for every $\vartheta \in \Theta$.

In the sequel, for simplicity, we put $Z(\tau) = Z(\tau(\xi), \xi)$.

We shall make the following regularity conditions:

- (i) $g(\vartheta)$ is differentiable and not identically constant on Θ ;
- (ii) $0 < E_{\vartheta} \left[\frac{\partial \log h(\tau, Z(\tau); \vartheta, \vartheta_0)}{\partial \vartheta} \right]^2 < \infty$ for every $\vartheta \in \Theta$;
- (iii) the differentiation and repeated differentiation under the integral sign with respect to ϑ in identities (7) and (8), respectively, is allowed.

THEOREM 1. *For every sequential plan (τ, f) satisfying conditions (i)–(iii) the following inequality of the Cramér–Rao type holds:*

$$D_{\vartheta} f(\tau, Z(\tau)) \geq - \frac{[g'(\vartheta)]^2}{E_{\vartheta} \frac{\partial^2 \log h(\tau, Z(\tau); \vartheta, \vartheta_0)}{\partial \vartheta^2}}. \quad (9)$$

The equality holds at a particular value of ϑ if and only if

$$f(u) = k(\vartheta) \frac{\partial \log h(t(u), z(u); \vartheta, \vartheta_0)}{\partial \vartheta} + g(\vartheta) \quad \text{a.e. } m_{\vartheta_0},$$

where $k(\vartheta) \neq 0$.

A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient at ϑ* if (9) becomes an equality at ϑ . The estimator f is then called efficient at this value ϑ , and the function $g(\vartheta)$ is efficiently estimable at the point ϑ .

A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient* if it is efficient at each $\vartheta \in \Theta$. The estimator f is then called efficient and the function $g(\vartheta)$ is efficiently estimable.

3. Description of efficient sequential plans

Let us now consider in detail the problem of finding sequential estimation plans for processes satisfying the stochastic differential equation (1) in the above-mentioned two cases A and B.

A ([3]). In this case equation (1) takes the form

$$d\xi_{\vartheta}(t) = -a[\xi_{\vartheta}(t) - \vartheta]dt + \sqrt{2a}\sigma dW(t). \quad (1')$$

If we substitute $\alpha(\vartheta) = -a$, $\beta(\vartheta) = a\vartheta$, $\gamma = (2a)^{1/2}\sigma$, $\vartheta_0 = 0$ in formula (2) and take into account that $\xi(0) = \text{const} = c$, we obtain from (2)

$$h(t, Z(t, \xi_{\vartheta_0}(\cdot)); \vartheta, \vartheta_0) = \exp \left\{ \frac{\vartheta}{2\sigma^2} \left[a \int_0^t \xi_0(s) ds + \xi_0(t) - \frac{a\vartheta t}{2} - c \right] \right\}. \quad (10)$$

Thus the sufficient statistic $Z(t, x)$ on the space $(C, \mathcal{C}_t, \{\mu_{\vartheta, t}, \vartheta \in \Theta\})$, $t \in T$, has the form

$$Z(t, x) = x(t) + a \int_0^t x(s) ds. \quad (11)$$

In view of (10) we have

$$\begin{aligned} \frac{dm_{\vartheta}}{dm_{\vartheta_0}}(u) &= h(t(u), z(u); \vartheta, \vartheta_0) = \exp \left\{ \frac{\vartheta}{2\sigma^2} \left[z(u) - \frac{a\vartheta}{2} t(u) - c \right] \right\}, \\ \frac{\partial \log h(t(u), z(u); \vartheta, \vartheta_0)}{\partial \vartheta} &= \frac{1}{2\sigma^2} [z(u) - a\vartheta t(u) - c] \end{aligned}$$

and

$$\frac{\partial^2 \log h(t(u), z(u); \vartheta, \vartheta_0)}{\partial \vartheta^2} = -\frac{a}{2\sigma^2} t(u).$$

We have the following lemma:

LEMMA 2. For a sequential plan (τ, f) let the regularity conditions (i)–(iii) be satisfied.

1° Suppose, moreover, that

(iv') $E_{\vartheta}\tau$ is a differentiable function of the variable $\vartheta \in \Theta$. Then the following identities hold:

$$\begin{aligned} E_{\vartheta} Z(\tau) &= a\vartheta E_{\vartheta}\tau + c, \\ E_{\vartheta} [Z(\tau) - a\vartheta\tau - c]^2 &= 2a\sigma^2 E_{\vartheta}\tau, \\ E_{\vartheta} \{f(\tau, Z(\tau)) [Z(\tau) - a\vartheta\tau - c]\} &= 2\sigma^2 g'(\vartheta), \\ E_{\vartheta} \{\tau [Z(\tau) - a\vartheta\tau - c]\} &= 2\sigma^2 E'_{\vartheta}\tau, \\ D_{\vartheta} Z(\tau) &= a^2 \vartheta^2 D_{\vartheta}\tau + 2a\sigma^2 E_{\vartheta}\tau + 4a\vartheta\sigma^2 E'_{\vartheta}\tau \end{aligned}$$

($D_{\vartheta}(\cdot)$ denotes the variance evaluated at ϑ).

2° The following inequality of the Cramér–Rao type holds:

$$D_{\vartheta} f(\tau, Z(\tau)) \geq \frac{2\sigma^2 [g'(\vartheta)]^2}{aE_{\vartheta}\tau}. \quad (9')$$

3° A sequential estimation plan (τ, f) for $g(\vartheta)$ is efficient at a point ϑ if and only if the estimator f is of the form

$$f(u) = k(\vartheta) [z(u) - a\vartheta t(u) - c] + g(\vartheta) \quad \text{a.e. } m_0,$$

where $k(\vartheta) \neq 0$.

THEOREM 2. If (τ, f) is an efficient sequential estimation plan for $g(\vartheta)$, then there exist constants α_1, α_2 not both zero and a constant α_3 such that

$$\alpha_1 z(u) + \alpha_2 t(u) + \alpha_3 = 0 \quad \text{a.e. } m_0. \quad (12)$$

From Lemma 2, 1° and 3°, it follows that in a given sequential plan (τ, f) the function $g(\vartheta)$ is efficiently estimable at a point $\vartheta = \vartheta^0$ if and only if it is of the form

$$g(\vartheta) = k(\vartheta^0) a(\vartheta - \vartheta^0) E_{\vartheta^0}\tau + g(\vartheta^0).$$

Moreover, if $g(\vartheta)$ is an efficiently estimable function, then it must be of the form

$$g(\vartheta) = \frac{k_1 \vartheta + k_0}{l_1 \vartheta + l_0}.$$

It follows from Theorem 2 that looking for efficient plans one should consider those Markov stopping times for which equation (12) holds. Let us consider the following Markov times:

$$\begin{aligned} \tau^{(1)}(x) &= t_0 > 0, \\ \tau^{(2)}(x) &= \inf \{t: Z(t, x) = z_0\}, \\ \tau^{(3)}(x) &= \inf \{t: Z(t, x) = \lambda_1 t + \lambda_2\}, \end{aligned}$$

where $t_0, z_0, \lambda_1, \lambda_2$ are boundaries given in advance and $Z(t, x)$ is defined by (11). A sequential plan determined by $\tau^{(1)}$ is called a *fixed-time plan*, a sequential plan determined by $\tau^{(2)}$ is called an *inverse plan* and a sequential plan determined by $\tau^{(3)}$ will be called an *oblique plan*.

THEOREM 3. The fixed-time plan $\delta^{(1)} = (\tau^{(1)}, f^{(1)})$ with

$$f^{(1)} = c_1 Z(t_0) + c_2$$

is efficient for

$$g(\vartheta) = c_1 a t_0 \vartheta + c c_1 + c_2$$

where $c_1 \neq 0$ and c_2 are arbitrary constants. The variance of estimator $f^{(1)}$ is equal to $D_{\vartheta} f^{(1)} = c_1^2 \sqrt{2a} \sigma t_0$.

For example, $f^{(1)} = [Z(t_0) - c]/a t_0$ is an efficient estimator for $g(\vartheta) = \vartheta$.

LEMMA 3. If $z_0 > c$, $\vartheta > 0$ or $z_0 < c$, $\vartheta < 0$ or $\vartheta = 0$, then

$$P(\tau^{(2)}(\xi_{\vartheta}) < \infty) = 1$$

and, moreover, for $\vartheta \neq 0$, $E_{\vartheta}[\tau^{(2)}]^2 < \infty$.

Proof. We have

$$\tau^{(2)}(\xi) = \inf \{t: Z(t, \xi) = z_0\} = \inf \{t: \xi(t) + a \int_0^t \xi(s) ds = z_0\}.$$

Taking into account (1'), we obtain

$$\xi(t) + a \int_0^t \xi(s) ds = W_1(t) + c,$$

where

$$W_1(t) = \sqrt{2a} \sigma W(t) + a \vartheta t$$

with

$$E_{\vartheta} W_1(t) = a \vartheta t \quad \text{and} \quad D_{\vartheta} W_1(t) = \sqrt{2a} \sigma t.$$

Thus

$$\tau^{(2)} = \inf \{t: W_1(t) = z_0 - c\},$$

i.e., $\tau^{(2)}$ is equal to the time of first attaining the line $w_1 = z_0 - c$ for the Wiener process with the mean $a \vartheta t$. If $-a \vartheta(z_0 - c) < 0$, i.e., if $\vartheta(z_0 - c) > 0$ for $a > 0$, it then follows from Trybula's results ([5], p. 36) that $P(\tau^{(2)} < \infty) = 1$ and $E_{\vartheta}[\tau^{(2)}]^2 < \infty$.

THEOREM 4. *Let the assumptions of Lemma 3 be satisfied and $\vartheta \neq 0$. Then the inverse plan $\delta^{(2)} = (\tau^{(2)}, f^{(2)})$ with*

$$f^{(2)} = c_1 \tau^{(2)} + c_2$$

is efficient for

$$g(\vartheta) = c_1 \frac{z_0 - c}{a\vartheta} + c_2.$$

The variance of estimator $f^{(2)}$ is equal to $D_{\vartheta} f^{(2)} = \frac{2c_1^2 \sigma^2 (z_0 - c)}{a^2 \vartheta^3}$.

For example, the estimator $f^{(2)} = \frac{a}{z_0 - c} \tau^{(2)}$ is efficient for $g(\vartheta) = 1/\vartheta$.

LEMMA 4. *If $\lambda_2 > c$, $\vartheta > \lambda_1/a$ or $\lambda_2 < c$, $\vartheta < \lambda_1/a$ or $\vartheta = 0$, $\lambda_1(\lambda_2 - c) < 0$, then*

$$P(\tau^{(3)}(\xi_{\vartheta}) < \infty) = 1$$

and $E_{\vartheta}[\tau^{(3)}]^2 < \infty$.

The proof, which is analogous to that of Lemma 3, makes use of Trybuła's results concerning the oblique plans for the Wiener process ([5], p. 36).

THEOREM 5. *If the assumptions of Lemma 4 are satisfied, then the oblique plan $\delta^{(3)} = (\tau^{(3)}, f^{(3)})$ with*

$$f^{(3)} = c_1 \tau^{(3)} + c_2$$

is efficient for

$$g(\vartheta) = c_1 \frac{\lambda_2 - c}{a\vartheta - \lambda_1} + c_2.$$

The variance of estimator $f^{(3)}$ is equal to $D_{\vartheta} f^{(3)} = \frac{2c_1^2 \sigma^2 a(\lambda_2 - c)}{(a\vartheta - \lambda_1)^3}$.

For example the estimator $f^{(3)} = \frac{\tau^{(3)}}{\lambda_2 - c}$ is efficient for $g(\vartheta) = \frac{1}{a\vartheta - \lambda_1}$.

B ([2]). In this case equation (1) takes the form

$$d\xi_{\vartheta}(t) = -\vartheta \xi_{\vartheta}(t) dt + dW(t). \quad (1'')$$

If we substitute $\alpha(\vartheta) = -\vartheta$, $\beta(\vartheta) = 0$, $\gamma = 1$ in formula (2) and take into account that $\xi(0)$ is normally distributed with mean zero and variance

$(2\vartheta)^{-1}$, we then obtain from (2)

$$\begin{aligned} & h(t, Z(t, \xi_{\vartheta_0}(\cdot)); \vartheta, \vartheta_0) \\ &= \exp \left\{ \frac{1}{2} \left[\log \vartheta - \log \vartheta_0 - (\vartheta - \vartheta_0) (\xi_{\vartheta_0}^2(0) + \xi_{\vartheta_0}^2(t) - t) - (\vartheta^2 - \vartheta_0^2) \int_0^t \xi_{\vartheta_0}^2(s) ds \right] \right\} \\ &= \exp \left\{ \frac{1}{2} \left[\log \vartheta - \log \vartheta_0 - (\vartheta - \vartheta_0) Z_1(t, \xi_{\vartheta_0}(\cdot)) - (\vartheta^2 - \vartheta_0^2) Z_2(t, \xi_{\vartheta_0}(\cdot)) \right] \right\} \\ &= h_1(Z(t, \xi_{\vartheta_0}(\cdot)); \vartheta, \vartheta_0), \end{aligned} \quad (13)$$

where

$$\begin{aligned} Z(t, \xi(\cdot)) &= (Z_1(t, \xi(\cdot)), Z_2(t, \xi(\cdot))) \\ Z_1(t, \xi(\cdot)) &= \xi^2(0) + \xi^2(t) - t, \end{aligned} \quad (14)$$

$$Z_2(t, \xi(\cdot)) = \int_0^t \xi^2(s) ds. \quad (15)$$

In view of (13) we obtain

$$\begin{aligned} \frac{dm_{\vartheta}}{dm_{\vartheta_0}}(u) &= h_1(z(u); \vartheta, \vartheta_0) \\ &= \exp \left\{ \frac{1}{2} \left[\log \vartheta - \log \vartheta_0 - (\vartheta - \vartheta_0) z_1(u) - (\vartheta^2 - \vartheta_0^2) z_2(u) \right] \right\} \\ &= \exp \left\{ \frac{1}{2} \left[\log \vartheta - \vartheta z_1(u) - \vartheta^2 z_2(u) \right] \right\} \frac{dm_{\tau}}{dm_{\vartheta_0}}(u), \end{aligned}$$

where

$$\frac{dm_{\tau}}{dm_{\vartheta_0}}(u) = \exp \left\{ \frac{1}{2} \left[-\log \vartheta_0 + \vartheta_0 z_1(u) + \vartheta_0^2 z_2(u) \right] \right\},$$

and

$$\begin{aligned} \frac{\partial \log h_1(z(u); \vartheta, \vartheta_0)}{\partial \vartheta} &= \frac{1}{\vartheta} - z_1(u) - 2\vartheta z_2(u), \\ \frac{\partial^2 \log h_1(z(u); \vartheta, \vartheta_0)}{\partial \vartheta^2} &= -\frac{1}{\vartheta^2} - 2z_2(u). \end{aligned}$$

We have the following lemma.

LEMMA 5. For a sequential plan (τ, f) let the regularity conditions (i)–(iii) be satisfied.

1° Suppose, moreover, that

(iv'') $E_{\vartheta} Z_2(\tau)$ is a differentiable function of the variable $\vartheta \in \Theta$. Then the following identities hold:

$$\vartheta E_{\vartheta} Z_1(\tau) = 1 - 2\vartheta^2 E_{\vartheta} Z_2(\tau),$$

$$\vartheta^2 E_{\vartheta} \left[\frac{1}{\vartheta} - Z_1(\tau) - 2\vartheta Z_2(\tau) \right]^2 = 4\vartheta^2 E_{\vartheta} Z_2(\tau) + 2,$$

$$E_{\vartheta} \left\{ f(\tau, Z(\tau)) \left[\frac{1}{\vartheta} - Z_1(\tau) - 2\vartheta Z_2(\tau) \right] \right\} = 2g'(\vartheta),$$

$$E_{\vartheta} \left\{ Z_2(\tau) \left[\frac{1}{\vartheta} - Z_1(\tau) - 2\vartheta Z_2(\tau) \right] \right\} = 2E'_{\vartheta} Z_2(\tau),$$

$$\vartheta^2 D_{\vartheta} Z_1(\tau) = 4\vartheta^4 D_{\vartheta} Z_2(\tau) + 4\vartheta^2 E_{\vartheta} Z_2(\tau) + 8\vartheta^3 E'_{\vartheta} Z_2(\tau) + 2.$$

2° The inequality of the Cramér-Rao type takes the form

$$D_{\vartheta} f(\tau, Z(\tau)) \geq \frac{2\vartheta^2 [g'(\vartheta)]^2}{1 + 2\vartheta^2 E_{\vartheta} Z_2(\tau)}. \quad (9'')$$

3° A sequential estimation plan (τ, f) for $g(\vartheta)$ is efficient at a point ϑ if and only if the estimator f is of the form

$$f(u) = k(\vartheta) \left[\frac{1}{\vartheta} - z_1(u) - 2\vartheta z_2(u) \right] + g(\vartheta) \quad \text{a.e. } m_{\tau},$$

where $k(\vartheta) \neq 0$.

THEOREM 6. If (τ, f) is an efficient sequential estimation plan for $g(\vartheta)$, then there exist constants α_1, α_2 not both zero and a constant α_3 such that

$$\alpha_1 z_1(u) + \alpha_2 z_2(u) + \alpha_3 = 0 \quad \text{a.e. } m_{\tau}. \quad (16)$$

THEOREM 7. If in a given sequential plan (τ, f) the function $g(\vartheta)$ is efficiently estimable, then it must be of the form

$$g(\vartheta) = \frac{k_2 \vartheta^2 + k_1 \vartheta + k_0}{l_2 \vartheta^2 + l_1 \vartheta}$$

with $k_0 \neq 0$.

It follows from Theorem 6 that one should seek the efficient sequential plans for a given function $g(\vartheta)$ from the class described in Theorem 7 among the plans determined by Markov stopping times for which equation (16) holds.

Let us consider the following Markov times:

$$\tau^{(1)}(x) = \inf \{t: Z_2(t, x) = z_{2,0}\}, \quad 0 < z_{2,0} < \infty,$$

$$\tau^{(2)}(x) = \inf \{t: Z_1(t, x) = z_{1,0}\},$$

$$\tau^{(3)}(x) = \inf \{t: Z_1(t, x) = \lambda_1 Z_2(t, x) + \lambda_2\},$$

where $z_{1,0}$, $z_{2,0}$, λ_1 , λ_2 are boundaries given in advance and $Z_1(t, x)$ and $Z_2(t, x)$ are defined by (14) and (15), respectively. A sequential plan determined by $\tau^{(1)}$ will be called a *fixed-energy plan*.

From the ergodic theorem we obtain the following lemma:

LEMMA 6.

$$P(\tau^{(1)}(\xi_{\vartheta}) < \infty) = 1 \quad \text{for all } \vartheta \in \Theta.$$

THEOREM 8. The fixed-energy plan $\delta^{(1)} = (\tau^{(1)}, f^{(1)})$ with

$$f^{(1)} = c_1 Z_1(\tau^{(1)}) + c_2$$

is efficient for

$$g(\vartheta) = \frac{c_1(1 - 2z_{2,0}\vartheta^2) + c_2\vartheta}{\vartheta}.$$

The variance of estimator $f^{(1)}$ is given by

$$D_{\vartheta}f^{(1)} = \frac{2c_1^2}{\vartheta^2}(1 + 2z_{2,0}\vartheta^2).$$

LEMMA 7. If $-\infty < z_{1,0} < 0$, then $E_{\vartheta}[\tau^{(2)}]^n < \infty$ for every $n = 1, 2, \dots$ and all $\vartheta \in \Theta$.

In particular, it follows from Lemma 7 that, for $-\infty < z_{1,0} < 0$, the condition (6) of the closedness of the sequential plan $(\tau^{(2)}, f)$ is satisfied. Henceforth we shall suppose that $-\infty < z_{1,0} < 0$.

THEOREM 9. A sequential plan $\delta^{(2)} = (\tau^{(2)}, f^{(2)})$ with

$$f^{(2)} = c_1 Z_2(\tau^{(2)}) + c_2$$

is efficient for

$$g(\vartheta) = \frac{c_1(1 - z_{1,0}\vartheta) + 2c_2\vartheta^2}{2\vartheta^2}.$$

The variance of estimator $f^{(2)}$ is equal to

$$D_{\vartheta}f^{(2)} = \frac{c_1^2(2 - z_{1,0}\vartheta)}{2\vartheta^4}.$$

In particular, it follows from Theorem 9 that

$$Z_2(\tau^{(2)}) = \int_0^{\tau^{(2)}} \xi^2(s) ds$$

is an efficient sequential estimator for

$$g(\vartheta) = \frac{1 - z_{1,0} \vartheta}{2\vartheta^2}.$$

LEMMA 8. *If $\lambda_1 > 0$ and $\lambda_2 < 0$, then*

$$P(\tau^{(3)}(\xi_\vartheta) < \infty) = 1 \quad \text{for all } \vartheta \in \Theta.$$

References

- [1] И.В. Гирсанов, *О преобразованиях одного класса случайных процессов с помощью абсолютно непрерывной замены меры*, Теория вероятн. и ее примен. **5** (1960), 314-330.
- [2] R. Magiera, *Efficient sequential plans for the spectral density parameter of a stationary Gaussian process*, Institute of Mathematics, Wrocław Technical University, Preprints, Report Nr 75, 1980.
- [3] R. Róžański, *A modification of Sudakov's lemma and efficient sequential plans for the Ornstein-Uhlenbeck process*, Zastos. Mat. **17** (1980) 73-86.
- [4] А.В. Скороход, *О дифференцируемости мер соответствующих случайным процессам, II. Марковские процессы*, Теория вероятн. и ее примен. **5** (1960), 45-53.
- [5] S. Trybuła, *Sequential estimation in processes with independent increments*, Dissertationes Math. (Rozprawy Mat.) **60** (1968).

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