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STRUCTURAL PROPERTIES AND LIMIT BEHAVIOUR OF LINEAR STOCHASTIC SYSTEMS IN HILBERT SPACES

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1. INTRODUCTION

In this paper structural properties and asymptotic behaviour of solutions of the following linear stochastic infinite-dimensional equation:

$$dx = Axdt + Bdw, \qquad (1)$$

$$x(0) = x_0 \in H$$

are studied. In this equation A denotes the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geqslant 0$, acting on a Hilbert space H, w is a Wiener process with values in a Hilbert space U and with covariance operator R, and B is a bounded linear operator from U into H. By a solution of equation (1) we understand the so-called mild solution, see [3], given by the following explicit formula:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)Bdw(s), \qquad t \geqslant 0. \qquad (2)$$

Our aim is to give some answers to the following questions:

Under what conditions process (2) is non-degenerate in the sense that, for all initial conditions $x_0 \in H$ and any non-empty open set $V \subset H$, the probability of the fact that the process (2) will eventually hit V is positive?

Characterize those processes (2) for which all transition probabilities are absolutely continuous with respect to a fixed probability distribution.

Under what conditions there exists a stationary measure for process (2)?

Describe the set of all stationary measures and give conditions which imply uniqueness.

Characterize all recurrent processes (2).

Characterize all positive recurrent processes (2).

For all the above-formulated questions there are satisfactory answers if $\dim H < +\infty$, see papers [6], [8] and [19]. But to the best of our knowledge these questions have not been answered in the case of infinite dimensions and in the generality proposed here. As we shall see, results obtained in [6], [8] and [19] can be only partially extended to the case $\dim H = +\infty$. Moreover, if $\dim H = +\infty$, then some new interesting questions arise. For instance, an information that the closed support of a stationary measure is the whole H is not very satisfactory as there can also be some dense but not closed linear subspaces with the same property.

The connection of the results presented here with the control theory is twofold. First of all, several probabilistic properties of system (1) are closely related to controllability properties of the following deterministic system:

$$\dot{x} = Ax + BR^{1/2}u. \tag{3}$$

Secondly, results obtained in the paper can be applied to study stochastic controllability of the controlled system of the form:

$$dx = Ax\,dt + Cu\,dt + B\,dw.$$

In particular, an extension of the finite-dimensional results of papers [7] and [17] is possible. Such an extension will be treated in a subsequent paper.

There following basic assumption will be valid throughout the paper (although some results will be true in general):

There exists a right continuous version of the stochastic integral:

$$\int_0^t S(t-s)B\,dw(s), \quad t \geqslant 0.$$

It is still an open question whether the assumption is satisfied for all C_0-semigroups $S(t)$, $t \geqslant 0$. There are, however, several sufficient conditions under which this is true; see [2], [11], [14], [4] and survey [12].

The present paper is a rewritten version of the report [18].

2. NON-DEGENERACY AND EQUIVALENCE

By $\gamma(m, Q)$ we shall denote the Gaussian measure on the Hilbert space H determined uniquely by its mean value $m \in H$ and the covariance operator Q. The following fact is well known (see [16], p. 63, where a more general fact was proved):

PROPOSITION 1. *The smallest closed support of the measure* $\gamma(0, Q)$ *is identical with the closure of the range of the operator* Q (Range$Q$).

It follows from representation (2) that the transition probabilities of (1) are exactly:

(4) $$P(t, x_0, \cdot) = \gamma\big(S(t)x_0, Q_t\big),$$

where

(5) $$Q_t = \int_0^t S(r)BRB^*S^*(r)\,dr, \quad t \geqslant 0.$$

If $\dim H < +\infty$, $\dim U < +\infty$ and $R = I$, then the following characterization can be found in [19]:

PROPOSITION 2. *Supports of all measures* $\gamma(0, Q_t)$ *are identical with the* (A, B) *controllable subspace of* H. *Moreover, process* (2) *is non-degenerate if and only if*

(6) $$\operatorname{Rank}[B, AB, \ldots, A^{n-1}B] = n = \dim H.$$

To prove an analogous result in the case $\dim H = +\infty$ we need the following lemma. In its formulation L_t: $L^2[0, t; U] \to H$ is the controllability operator defined as

(7) $$L_t u = \int_0^t S(r)BR^{1/2}u(r)\,dr, \quad t \geqslant 0.$$

LEMMA 1. *For all* $t \geqslant 0$,

(8) $$\operatorname{Range} L_t = \operatorname{Range} Q_t^{1/2}.$$

Proof. We use the following well-known result (see [3], p. 55):

If X, Y, Z *are Hilbert spaces and* F: $X \to Z$, G: $Y \to Z$ *are linear bounded operators, then* $\operatorname{Range} F \subset \operatorname{Range} G$ *if and only if there exists* $c > 0$ *such that, for all* $z^* \in Z^*$,

$$\|F^*z^*\| \leqslant c\|G^*z^*\|.$$

It is easy to see that in our situation the element $L_t^* x$ is given by the formula

$$(L_t^* x)(s) = R^{1/2}B^*S^*(s)x, \quad s \in [0, t],$$

and therefore

$$\|L_t^* x\|^2 = \int_0^t \|R^{1/2}B^*S^*(s)x\|^2\,ds$$
$$= \|\sqrt{Q_t}x\|^2 = \|\sqrt{Q_t^*}x\|^2.$$

Consequently, (8) holds.

If the set composed of all elements of the form

$$S(t)x_0+\int_0^t S(t-s)BR^{1/2}u(s)\,ds,$$

where $t\geqslant 0$ and $u(\cdot)$ is any element from $L^2[0,t;U]$, is dense in H, then system (3) is called *approximately controllable from* x_0.

THEOREM 1. *The stochastic system* (2) *is non-degenerate if and only if the controlled system* (3) *is approximately controllable from all initial states* $x_0\in H$.

Proof. It is clear that, for all $t\geqslant 0$,

$$\overline{\mathrm{Range}Q_t}=\overline{\mathrm{Range}Q_t^{1/2}}.$$

Moreover, if $x(t)$ is given by (2) and $P(t,x_0,V)>0$ for an open set $V\subset H$- then, by Proposition 1, the intersection of the set V with the affine hyper, plane $\mathrm{Range}Q_t^{1/2}+S(t)x_0$ is non-empty. By Lemma 1, $\mathrm{Range}L_t=\mathrm{Range}Q_t^{1/2}$, and therefore the intersection of the set V with $\mathrm{Range}L_t+S(t)x_0$ is non-empty as well, and (3) is approximately controllable from x_0. The converse implication follows in a similar way.

Remark 1. Explicit conditions for approximate controllability which generalize the rank condition (6) to infinite dimensions were studied by several authors, in particular by R. Triggiani [15], see also [3] and references there.

Although Theorem 1 is a natural extension of Proposition 2 to the case $\dim H=+\infty$, nevertheless if $\dim H=+\infty$, the following new and pathological situation can happen.

PROPOSITION 3. *If* $\dim H=+\infty$, *then there exists a stochastic process of the form* (2) *such that:*

(1) *For any non-empty open set* V *there exists* $t>0$ *such that*

$$P(t,0,V)>0;$$

(2) *There exists an open set* $V\neq\emptyset$ *and an initial condition* $x_0\in H$ *such that for all* $t\geqslant 0$,

$$P(t,x_0,V)=0.$$

Proof. An equivalent formulation of Proposition 3 is that there exists a controlled system of the form (3), with $R\geqslant 0$ a nuclear operator, which is approximately controllable from 0 but not from any other state. Let $Z=L^2[0,+\infty)$ and define on Z the "left shift" semigroup $S(t)$:

$$S(t)z(\theta)=\begin{cases}0 & \text{if}\quad \theta\leqslant t,\\ z(\theta-t) & \text{if}\quad \theta>t,\end{cases}$$

and an element $b \in Z$:

$$b(\theta) = \begin{cases} 1 & \text{if} \quad \theta \leqslant 1, \\ 0 & \text{if} \quad \theta > 1. \end{cases}$$

As the Hilbert space H we take

$$H = \overline{\text{lin}}\{S(t)b;\ t \geqslant 0\} \subset Z$$

and define controlled system as

$$x(t) = S(t)x_0 + \int_0^t S(t-s)bu(s)ds, \qquad t \geqslant 0. \tag{9}$$

In this case

$$\overline{\text{Range} L_t} = \overline{\text{lin}}\{S(r)b;\ r \leqslant t\}$$

and therefore, by the very definition of the space H, system (9) is approximately controllable from $x_0 = 0$. If, however, the initial condition x_0 equals $S(1)b$, then $S(t)x_0 = S(t+1)b$ and consequently, for any $t \geqslant 0$ and control $u \in L^2[0, t]$,

$$\left\| S(t)x_0 + \int_0^t S(t-s)bu(s)ds \right\| \geqslant \|S(t)x_0\| = \|b\| > 0.$$

Therefore system (9) is not approximately controllable from $x_0 = S(1)b$.

Remark 2. If there exists a universal time for the approximate controllability of (3) from 0, then system (3) is approximately controllable from any initial condition and the situation described in Proposition 3 cannot happen.

We proceed now to the main result of this section. Let us recall (see [3], p. 68) that system (3) is *exactly null controllable on the interval* $[0, t]$ if, for any $x_0 \in H$, there exists a control $u \in L^2[0, t;\ U]$ which steers x_0 to 0:

$$S(t)x_0 + \int_0^t S(t-s)B\sqrt{R}u(s)ds = 0.$$

THEOREM 2. *The transition probabilities* (4) *are equivalent for all* $t > t_0$ *and all* $x_0 \in H$ *if and only if the corresponding system* (3) *is exactly null controllable on arbitrary time interval* $[0, t]$, *with* $t > t_0$.

To prove Theorem 2 we shall need the following lemmas whose proofs are given, for instance, in [9].

LEMMA 2. *Let* R_1 *and* R_2 *be self-adjoint non-negative operators defined on* H *such that*:

$$\text{Range} R_1^{1/2} = \text{Range} R_2^{1/2} = H_0 \quad \textit{and} \quad \bar{H}_0 = H.$$

Then

(i) *The operator* $C_1 = R_1^{-1/2}R_2^{1/2}$ *is bounded;*

(ii) *The operator* $C_2 = R_2^{1/2}R_1^{-1/2}$ *has continuous extension* $\bar{C}_2$ *to* H;

(iii) $C_1^* = \bar{C}_2$ *and* $C_1 = \bar{C}_2^*$.

LEMMA 3. *Two non-degenerate Gaussian measures* $\gamma(m_1, R_1)$ *and* $\gamma(m_2, R_2)$ *are equivalent if and only if:*

(i) $\mathrm{Range}\, R_1^{1/2} = \mathrm{Range}\, R_2^{1/2}$;

(ii) $m_2 - m_1 \in \mathrm{Range}\, R_2^{1/2}$;

(iii) *The operator* $C_1 C_1^* - I$, *where* $C_1 = R_1^{-1/2}R_2^{1/2}$, *is Hilbert–Schmidt with all eigenvalues greater than* -1.

Proof of Theorem 2. Since the distribution $P(t, x_0, \cdot)$ is exactly $\gamma\big(S(t)x_0, Q_t\big)$, therefore from Lemma 3 the equivalence

$$\gamma\big(S(t)x_0, Q_t\big) \simeq \gamma(0, Q_t)$$

holds if and only if $S(t)x_0 \in \mathrm{Range}\, Q_t^{1/2}$. Consequently, for a fixed $t > 0$ and all initial states $x_0 \in H$, the measures $\gamma\big(S(t)x_0, Q_t\big)$ are equivalent if and only if

$$\mathrm{Range}\, S(t) \subset \mathrm{Range}\, Q_t^{1/2}. \tag{10}$$

Taking into account Lemma 1 we see that inclusion (10) is equivalent to the exact null controllability of system (3) on the interval $[0, t]$. The theorem is thus proved in one direction. Let us now assume that condition (10) holds for all $t > t_0$ and let us remark that, for all $u \in L^2[0, t; U]$ and $t > s > t_0$,

$$L_t u = \int_0^t S(r)BR^{1/2}u(r)\,dr$$
$$= \int_0^s S(r)BR^{1/2}u(r)\,dr + S(s)\int_0^{t-s} S(r)BR^{1/2}u(s+r)\,dr.$$

Therefore

$$\mathrm{Range}\, L_t \subset \mathrm{Range}\, L_s \cup \mathrm{Range}\, S(s),$$

and thus from (10) and Lemma 1,

$$\mathrm{Range}\, L_t \subset \mathrm{Range}\, L_s.$$

Consequently, for all $t > s > t_0$,

$$\mathrm{Range}\, Q_t^{1/2} = \mathrm{Range}\, Q_s^{1/2}.$$

Let $H_1 = \overline{\mathrm{Range}\, Q_t^{1/2}}$ for some $t > t_0$ and therefore for all $t > t_0$. It follows from definition (5) that

$$Q_t - Q_s = S(s)Q_{t-s}S^*(s), \qquad t > s > t_0. \tag{11}$$

Applying the operator $Q_s^{-1/2}$ to both sides of (11) and using Lemma 2 with the space H replaced by H_1, one obtains, for all $x \in H_1$,

$$C_1 Q_t^{1/2} x - Q_s^{1/2} x = \big(Q_s^{-1/2} S(s)\big) Q_{t-s} S^*(s) x,$$

where $C_1 = Q_s^{-1/2} Q_t^{1/2}$. Take $x = Q^{-1/2} y$, where $y \in \operatorname{Range} Q_s^{1/2}$; then

$$C_1 C_1^* y - y = D Q_{t-s} D^* y. \tag{12}$$

In (12) $D = Q_s^{-1/2} S(s)$ is a bounded operator such that D^* is the unique extension of $S^*(s) Q_s^{-1/2}$. Consequently, (12) holds for all $y \in H_1$:

$$C_1 C_1^* - I = D Q_{t-s} D^*.$$

Since Q_{t-s} is a nuclear operator, $D Q_{t-s} D^*$ is also nuclear and hence Hilbert–Schmidt. Moreover, $D Q_{t-s} D^*$ is a non-negative operator, consequently all its eigenvalues are non-negative, thus greater than -1. In this way we have shown that all conditions of Lemma 3 are satisfied and the proof of Theorem 2 is complete.

Remark 3. Processes (2) for which the assumptions of Theorem 2 are satisfied will be called *regular* processes. Proposition 4 below shows that there are non-trivial regular processes also if $\dim H = +\infty$. On the other hand, the regularity assumption imposes severe restrictions on the semigroup $S(t)$. For instance, if process (2) is regular, then all operators $S(t)$, $t > t_0$, are necessarily Hilbert–Schmidt. To see this, let us notice that from (10), for some $\delta > 0$ and all $x \in H$,

$$\|S^*(t) x\| \leqslant \delta \|Q_t^{1/2} x\|. \tag{13}$$

If (λ_k, e_k) is the sequence of eigenvalues and eigenvectors of the nuclear operator Q_t, then, from (13),

$$\sum_{k=1}^{+\infty} \|S(t) e_k\|^2 = \sum_{k=1}^{+\infty} \|S^*(t) e_k\|^2$$

$$\leqslant \delta^2 \sum_{k=1}^{+\infty} \|Q_t^{1/2} e_k\|^2 \leqslant \delta^2 \sum_{k=1}^{+\infty} \lambda_k < +\infty.$$

Therefore $S(t)$ is a Hilbert–Schmidt operator.

PROPOSITION 4. *Let (a_k) and (λ_k) be positive sequences such that*:

(i) $\sum_{k=1}^{+\infty} \lambda_k < +\infty$;

(ii) $a_k \to +\infty$ *and there exists* $C > 0$ *such that*

$$\ln a_k \lambda_k^{-1} \leqslant a_k + C, \quad k = 1, 2, \ldots \tag{14}$$

If $H = l_2$ and the operators A, B and R are defined as

$$Ax = (-a_k \xi_k), \quad B = I, \quad Rx = (\lambda_k \xi_k), \quad x = (\xi_k) \in l_2,$$

then process (1) *is regular.*

Proof. In the present situation condition (10) is equivalent to the existence, for all $t > 0$, of constants $C(t) > 0$ such that

(15) $$S(2t) \leqslant C(t) Q_t.$$

Taking into account that the eigenvalues of the operators $S(2t)$ and Q_t are respectively $e^{-2a_k t}$ and $\frac{\lambda_k}{2a_k}(1 - e^{-2a_k t})$, $k = 1, 2, \ldots$, we see that (14) implies (15).

3. INVARIANT MEASURES

3.1. Existence

The following theorem extends to infinite dimensions a similar result, valid for finite-dimensional spaces, contained in paper [19].

THEOREM 3. *The following conditions are equivalent:*

(i) *There exists a stationary distribution for process* (2);

(ii) *There exists a non-negative nuclear operator Q satisfying the equation*

(16) $$2\langle QA^* x, x\rangle + \langle RB^* x, B^* x\rangle = 0 \quad \text{for} \quad x \in D(A^*);$$

(iii) $\sup_{t \geqslant 0} \operatorname{Trace} Q_t < +\infty$.

If one of the conditions (i)–(iii) *holds, then any invariant probability measure μ for* (2) *is of the form*

(17) $$\mu = \nu * \gamma(0, Q),$$

where ν is an invariant measure for the semigroup $S(t)$: $\nu S(t) = \nu$, $t \geqslant 0$, and Q is the smallest non-negative solution of (16).

Proof. Let us assume that μ is an invariant measure for process (2) and let $\hat{\mu}$ denote its characteristic functional:

$$\hat{\mu}(\lambda) = \int_H \exp i\langle\lambda, x\rangle \mu(dx), \quad \lambda \in H.$$

Stationarity implies that

$$\hat{\mu}(\lambda) = \hat{\mu}(S^*(t)\lambda)\exp(-1/2\langle Q_t\lambda, \lambda\rangle), \quad t > 0, \ \lambda \in H.$$

Let us fix $\lambda \in H$; then, for an $\varepsilon > 0$, $\hat{\mu}(\varepsilon\lambda) \neq 0$ and therefore $\exp(-1/2\varepsilon^2$

$\cdot\langle Q_t\lambda, \lambda\rangle)\to 0$ as $t\to+\infty$. Consequently, $\sup_{t\geqslant 0}\langle Q_t\lambda, \lambda\rangle < +\infty$ for all λ, and the limit $\lim_{t\to+\infty} Q_t = Q$ is a bounded, non-negative operator. It is a standard procedure to show that Q satisfies equation (16), see [3] and [5]. Since

$$\hat{\mu}\big(S^*(t)\lambda\big) = \hat{\mu}(\lambda)\exp(1/2\langle Q_t\lambda, \lambda\rangle),$$

therefore

$$\hat{\nu}(\lambda) = \lim_{t\to+\infty}\big(S^*(t)\lambda\big) = \hat{\mu}(\lambda)\exp(1/2\langle Q\lambda, \lambda\rangle). \tag{18}$$

Also, since characteristic functional $\hat{\mu}$ is an S-continuous function (see [13], p. 160), there exists a nuclear non-negative operator S such that if $\langle S\lambda, \lambda\rangle \leqslant 1$, then $|\hat{\mu}(\lambda)| \geqslant 1/2$. Consequently, if $\langle S\lambda, \lambda\rangle \leqslant 1$, then

$$\langle Q\lambda, \lambda\rangle \leqslant 2\ln 2$$

and $\langle Q\lambda, \lambda\rangle \leqslant 2\ln 2\langle S\lambda, \lambda\rangle$ for all $\lambda\in H$. We see that Q is also a nuclear operator. In this way we have shown that (i) implies (ii) and (iii). It follows from (18) that $\hat{\nu}$ is necessarily an S-continuous function and since it is also positive definite, $\hat{\nu}$ is a characteristic functional of a probability measure ν. But $\hat{\nu}\big(S^*(t)\lambda\big) = \hat{\nu}(\lambda)$ for all $t\geqslant 0$ and $\lambda\in H$, and so ν is an invariant measure for the semigroup $S(t)$, $t\geqslant 0$. Thus necessarily, any invariant measure for (2) is of the form (17). It is easy to show that, conversely, if one of the conditions (i)–(iii) is satisfied, ν is an invariant measure for $S(t)$, $t\geqslant 0$ and $Q = \lim_{t\to+\infty} Q_t$, then the measure $\mu = \nu * \gamma(0, Q)$ is invariant for (2).

Let us consider, as an illustration, the equation

$$dx = Ax\,dt + b\,d\beta, \tag{19}$$

where $b\in H$ and β is a 1-dimensional Brownian motion. The following proposition is a simple consequence of Theorem 3.

PROPOSITION 5. *A stationary probability distribution for* (19) *exists if and only if*

$$\int_0^{+\infty} \|S(t)b\|^2\,dt < +\infty. \tag{20}$$

If, in addition to (20),

$$\overline{\mathrm{lin}}\{S(t)b;\ t\geqslant 0\} = H, \tag{21}$$

then H is the smallest closed support of any invariant measure for (19).

Proof. Let us remark that in the present situation

$$\langle Q_t x, x\rangle = \int_0^t |\langle S(r)b, x\rangle|^2\,dr.$$

Therefore if (e_k) is a fixed orthonormal basis in H, then

$$\operatorname{Trace} Q_t = \sum_{k=1}^{+\infty} \langle Q_t e_k, e_k\rangle = \int_0^t \Big(\sum_{k=1}^{+\infty} |\langle S(r)b, e_k\rangle|^2\Big)\, dr$$
$$= \int_0^t \|S(r)b\|^2 dr.$$

Consequently, Theorem 3 (iii) implies (20). The second part of the theorem follows from an easy to check identity

$$\overline{\operatorname{Range} L_t} = \overline{\operatorname{lin}}\{S(r)b;\ r\leqslant t\} = \overline{\operatorname{Range} Q_t^{1/2}}.$$

3.2. Uniqueness

As for the uniqueness of stationary measure we have the following result.

PROPOSITION 6. *If for any $x\in H$ either $\|S(t)x\|\to 0$ or $\|S(t)x\|\to +\infty$ as $t\to +\infty$, then there exists at most one stationary measure for* (2).

Proof. It is sufficient to deduce from the assumption of the proposition that if ν is an invariant probability measure for the semigroup $(S(t))$, then ν is concentrated at 0. Let, for instance, $K\subset H$ be a compact set such that $0\notin K$, $\nu(K)>0$, and

$$K\subset \{x\colon\ 0<|S(t)x|\to 0 \ \text{as}\ t\to +\infty\}.$$

There exists a sequence $t_n\to +\infty$ such that the sets $K_n = S(t_n)K$ are mutually disjoint. But

$$\nu(K_n) = \nu S(t_n)(K_n) = \nu\{x\colon\ S(t_n)x\in S(t_n)K\}\geqslant \nu(K),\qquad n=1,2,\ldots$$

and $\nu(H)\geqslant \sum_{n=1}^{+\infty}\nu(K_n) = +\infty$, a contradiction.

In a similar way one can consider the case of a compact set

$$K\subset\{x\colon\ \|S(t)x\|\to +\infty \ \text{as}\ t\to+\infty\},\qquad \nu(K)>0.$$

COROLLARY 1. *If, for all $x\in H$, $S(t)x\to 0$ as $t\to +\infty$ and $\int_0^{+\infty}\|S(t)b\|^2dt < +\infty$, then there exists exactly one stationary measure for* (19).

If $\dim H< +\infty$ and process (2) *is non-degenerate, then there exists at most* 1 *stationary measure. This is no longer true if $\dim H = +\infty$.*

PROPOSITION 7. *Take $H = L^2[0, +\infty)$ and define for $\lambda\geqslant 0$ and $\varkappa>0$ a semigroup $S(t)$, $t\geqslant 0$ and an element $b\in H$ as follows:*

$$\begin{aligned} S(t)x(\theta) &= \exp\lambda t\, x(\theta+t), && x\in H,\\ b(\theta) &= \exp(-\varkappa\theta^2), && \theta\geqslant 0,\ t\geqslant 0. \end{aligned} \tag{22}$$

Then, for any $\lambda \geqslant 0$, system (19) *is non-degenerate and there exists at least one stationary distribution for* (19). *There exists exactly one stationary measure for* (19) *if and only if $\lambda = 0$.*

Proof. The proof easily follows from Theorem 1, Theorem 3, Proposition 5 and the following lemma:

LEMMA 4. *Any periodic trajectory of the equation*

$$\dot{x} = Ax,$$

where A is the infinitesimal generator of the semigroup (22) *with $\lambda > 0$, corresponds to the following initial conditions x_0:*

(i) $x_0(\theta) = e^{-\lambda\theta}$, $\theta \geqslant 0$;

or

(ii) $x_0(\theta) = e^{-\lambda rk} y(\theta - r)$ *for* $\theta \in [kr, (k+1)r)$, $k = 0, 1, 2, \ldots$, *where r is an arbitrary, positive number and y arbitrary element from $L^2[0, r]$.*

From Lemma 4 it follows that if $\lambda > 0$, then there are many periodic trajectories of the form $t \to S(t)x_0$ and therefore there are many invariant measures for the semigroup $S(t)$.

3.3. Stationary distributions and stability

If $\dim H < +\infty$ and process (2) is non-degenerate, then the existence of a stationary distribution (2) implies the stability of the semigroup $S(t)$:

$$\text{for all } x \in H,\ S(t)x \to 0 \text{ as } t \to +\infty \tag{23}$$

or, equivalently,

$$\sup\{\mathrm{Re}\lambda\colon \lambda \in \sigma(A)\} < +\infty. \tag{24}$$

Proposition 7 shows that this is no longer true if $\dim H = +\infty$, because if $\lambda > 0$, $\|S(t)\| = e^{\lambda t} \to +\infty$. Even if the stationary measure is unique, (24) may fail as Proposition 7 shows for $\lambda = 0$. If $\lambda = 0$, then the spectrum of the operator A is exactly the imaginary axis.

3.4. Non-closed supports of stationary distributions

If the distribution $\mu = \gamma(0, Q)$ is invariant for (2) and $\dim H < +\infty$, then the distribution μ is concentrated on the subspace $H_0 = \{x\colon S(t)x \to 0$ as $t \to +\infty\}$. Although we conjecture that this is true in general if $\dim H = +\infty$, we are able to prove only the following proposition.

PROPOSITION 8. *If measure $\mu = \gamma(0, Q)$ is a stationary distribution for* (2), *then there exists a sequence $t_n \to +\infty$ such that*

$$\mu\{x\colon S(t_n)x \to 0\} = 1.$$

If, in addition, $\int_0^{+\infty} \operatorname{Trace}(Q - Q_t)\,dt < +\infty$, *then*

$$\mu H_0 = 1.$$

Proof. Let ζ be a random variable with distribution $\gamma(0, Q)$, then $E(\|S(t)\zeta\|^2) = \operatorname{Trace} S(t)QS^*(t)$. But $S(t)QS^*(t) = Q - Q_t$ and since $Q_t \uparrow Q$, therefore $E\|S(t)\zeta\|^2 \to 0$ as $t \to +\infty$. Consequently, there exists a sequence $t_n \to +\infty$ such that $\|S(t_n)\zeta\| \to 0$ almost everywhere. In a similar way we check that

$$E\left(\int_0^{+\infty} \|S(t)\zeta\|^2 dt\right) = \int_0^{+\infty} \operatorname{Trace}(Q - Q_t)\,dt.$$

Therefore if $\int_0^{+\infty} \operatorname{Trace}(Q - Q_t)\,dt < +\infty$, then the measure μ is concentrated on the set H_1:

$$H_1 = \left\{x: \int_0^{+\infty} \|S(t)x\|^2 dt < +\infty\right\}.$$

Since $H_1 \subset H_0$ (see [5]), $\mu(H_0) = 1$.

COROLLARY 2. *If* $\int_0^{+\infty} t\|S(t)b\|^2 dt < +\infty$, *then the stationary measure* $\gamma(0, Q)$ *corresponding to* (19) *is concentrated on* H_0.

4. RECURRENCE

A process defined by (2) is called *recurrent* if and only if for any initial condition $x_0 \in H$ and any non-empty open set $V \subset H$ the probability that the process will eventually hit V is 1. It is therefore clear that recurrence implies non-degeneracy. If $\dim H < +\infty$, then a non-degenerate process (2) is recurrent if and only if (see [6] and [8]) there exists a decomposition of H into the direct product of two $S(t)$-invariant subspaces H_0, H_1 such that:

(1) For all $x \in H_0$, $\|S(t)x\| \to 0$ as $t \to +\infty$;

(2) $\dim H_1 \leqslant 2$ and for all $x \in H_1$, $\sup_{t \geqslant 0} \|S(t)x\| < +\infty$.

From the above characterization one can deduce the following sufficient conditions for recurrence:

If $\dim H < +\infty$, process (2) is non-degenerate and

(1) For all $x \in H$, $S(t)x \to 0$ as $t \to +\infty$; or

(2) There exists a stationary measure for (2).

Then process (2) is recurrent.

As we shall see later, if $\dim H = +\infty$, neither of those conditions implies recurrence. The following theorem gives some sufficient and some necessary conditions for recurrence.

THEOREM 4. *Let us assume that there exists a stationary measure for a non-degenerate process $x(\cdot)$ of the form* (2). *If*

(i) *For all $x \in H$, $S(t)x \to 0$ as $t \to +\infty$;*

or

(ii) *Process is regular,*

then the process $x(\cdot)$ is recurrent. On the other hand, if

(iii) *For some $x_0 \in H$, $\|S(t)x_0\|\sqrt{\ln t} \to \infty$ as $t \to \infty$,*

then the process $x(\cdot)$ is not recurrent.

Proof. Let us assume first that (i) holds and let $\mu = \gamma(0, Q)$ be the unique invariant measure for the process under consideration. If $K = \{x \in H\colon |x-y| < r\}$, then $\mu\{\partial K\} = 0$ and consequently it follows from the weak convergence: $\gamma\big(S(t)x, Q_t\big) \to \mu$ (see [13], p. 40) that, for every $x \in H$,

$$P(t, x, K) = \gamma\big(S(t)x, K\big) \to \mu(K) = \mu_0 > 0.$$

We shall need the following lemma.

LEMMA 5. *For arbitrary sequence of positive numbers (μ_k), $\mu_k < \mu_0$, $k = 1, 2, \ldots$, there exists an increasing sequence $t_1 < t_2 < \ldots$ such that, for every $k = 1, 2, \ldots$,*

$$P^{x_0}\big(x(t_1) \notin K \text{ and } \ldots \text{ and } x(t_k) \notin K \text{ and } x(t_{k+1}) \in K\big) \geqslant P^{x_0}\big(x(t_1) \notin K \text{ and } \ldots \text{ and } x(t_k) \notin K\big)\mu_{k+1}.$$

Proof. Let μ_k be a measure concentrated on the complement K^c of the set K and defined, for Borel sets $\Gamma \subset K^c$, by the formula

$$\mu_k(\Gamma) = P^{x_0}\big(x(t_1) \notin K \text{ and } \ldots \text{ and } x(t_{k-1}) \notin K \text{ and } x(t_k) \in \Gamma\big).$$

Since for all $x \in H$, $P(t, x, K) \to \mu_0$, therefore, for arbitrary $\delta > 0$, one can find a compact set $K_\delta \subset K^c$ and a positive number t_{k+1} such that

$$\mu_k(K_\delta) \geqslant \mu(K^c) - \delta \quad \text{and} \quad P(t_{k+1} - t_k, x, K) \geqslant \mu_0 - \delta \quad \text{for all} \quad x \in K_\delta.$$

Consequently,

$$P^{x_0}\big(x(t_1) \notin K \text{ and } \ldots \text{ and } x(t_k) \notin K \text{ and } x(t_{k+1}) \in K\big) \geqslant \int_{K_\delta} P(t_{k+1} - t_k, y, K)\mu_k(dy) \geqslant (\mu_0 - \delta)\big(\mu(K^c) - \delta\big)$$

and therefore the proof of the lemma is complete.

It follows from Lemma 5 and from a simple induction argument that

$$P^{x_0}\big(x(t_1)\in K \text{ or } x(t_2)\in K \text{ or } \ldots \text{ or } x(t_k)\in K\big) \geqslant 1-(1-\mu_k)\cdot\ldots\cdot(1-\mu_2)P^{x_0}\big(x(t_1)\in K\big).$$

Taking $\mu_k = \frac{1}{2}\mu_0$, $k = 1, 2, \ldots$, we see that our process is recurrent.

Let us now assume that the regularity assumption (ii) holds. Let us fix $t > 0$ and consider the Markov chain $Z_n = x(tn)$, $n = 0, 1, \ldots$ For arbitrary $\delta \in (0, 1)$ and a Borel set $\Gamma \subset H$ such that $\mu(\Gamma) > 0$ let

$$\Gamma_\delta = \{x \in \Gamma: \varphi_\Gamma(x) \leqslant \delta\},$$

where

$$\varphi_\Gamma(x) = P^x\{Z_n \in \Gamma \text{ for some } n = 1, 2, \ldots\}.$$

A simple geometric estimate shows that the potential VI_{Γ_δ}

$$VI_{\Gamma_\delta}(x) = E^x\Big(\sum_{k=0}^{+\infty} I_{\Gamma_\delta}(Z_k)\Big) = \sum_{k=0}^{+\infty} P_{kt} I_{\Gamma_\delta}(x)$$

is a bounded function on Γ_δ and therefore uniformly bounded on the whole H. But

$$+\infty > \int VI_{\Gamma_\delta}(x)\mu(dx) = \sum_{k=0}^{+\infty} \mu(\Gamma_\delta).$$

Therefore $\mu(\Gamma_\delta) = 0$ and, for μ-almost all $x \in \Gamma$, $\varphi_\Gamma(x) = 1$. If $D = \{x \in \Gamma^c: \varphi_\Gamma(x) < 1\}$, then $\mu(D) = 0$ for if $\mu(D) > 0$ and x is an element such that $\varphi_\Gamma(x) = 1$, then $P^x(Z_1 \in D) > 0$ and the Markov property implies $\varphi_\Gamma(x) < 1$, a contradiction. In an analogical way, if $x_0 \notin C = \{x: \varphi_\Gamma(x) = 1\}$, then $1 = \mu(c) = P(x_0, t, C) = P^{x_0}(Z_1 \in C)$ and by the Markov property, $x_0 \in C$.

Before passing to the last part of the theorem we formulate the following lemma whose proof requires a standard application of the strong Markov property and therefore will be omitted.

LEMMA 6. *If a process $x(\cdot)$ of the form* (2) *satisfies the basic assumption from Section* 1 *and, for some $t_1 > 0$ and $R > 0$,*

$$\int_{t_1}^{+\infty} P(t, 0, \{x: \|x\| \leqslant R\})\,dt < +\infty,$$

then $x(\cdot)$ is not recurrent.

Assume now that (iii) holds. If $\lambda > 0$ is an arbitrary number larger than the largest eigenvalue of the covariance matrix Q, then (see [16], p. 87) there exist positive constants C, t_1 such that, for all $t \geqslant t_1$ and all $r > r_0 > 0$,

$$P(t, 0, \{x: r \geqslant \|x\| \geqslant r_0\}) \leqslant c \int_{r_0}^{r} \exp\left(-\frac{s}{2\lambda}\right) ds. \tag{25}$$

If $\|S(t)x_0\| > R$ and

(26) $$r_0 = R - \|S(t)x_0\|, \quad r = R + \|S(t)x_0\|,$$

then

(27) $$P(t, x_0, \{x: \|x\| \leqslant R\}) = P(t, 0, \{x: \|x - S(t)x_0\| \leqslant R\}) \leqslant P(t, 0, \{x: r_0 \leqslant \|x\| \leqslant r_1\}).$$

Taking into account (iii), (25), (26) and (27) one easily obtains that

$$\int_{t_1}^{+\infty} P(t, x_0, \{x: \|x\| \leqslant R\})\,dt < +\infty.$$

COROLLARY 3. *The stochastic system defined in Proposition 7 is recurrent if and only if $\lambda = 0$, although it has stationary non-degenerate distributions for all $\lambda \geqslant 0$.*

Proposition 9 below shows that non-degeneracy and stability property: $S(t)x \to 0$ as $t \to +\infty$ for all x, do not imply recurrence (although the exponential convergence does, see Theorem 4).

PROPOSITION 9. *Let (λ_k), (t_k) and (a_k) by positive sequences with the following properties:*

(1) $\sum_{k=1}^{+\infty} \lambda_k < +\infty$, $\lambda_{3k-2} = \lambda_{3k-1} = \lambda_{3k}$;

(2) $\sum_{k=1}^{+\infty} (t_k)^{-1/2}(\lambda_{3k})^{-3/2} < +\infty$, $t_k < t_{k+1} < \ldots$;

(3) $\int_0^t \exp(-2a_k r)\,dr \geqslant \frac{1}{2}t$ *for* $t \in [t_k, t_{k+1})$, $k = 1, 2, \ldots$

Moreover, let $H = l_2$, $S(t)x = (\exp(-a_k t)\xi_k)$, $w(t) = (\sqrt{\lambda_k}\beta^k(t))$, *where* $x = (\xi_k) \in H$ *and* $\beta^1, \beta^2, \ldots$ *are independent normalized Brownian motions. Then the process* $x(\cdot)$ *given by* (2) *is non-degenerate,*

$$S(t)x \to 0 \quad \textit{as} \quad t \to +\infty \quad \textit{for all } x \in H,$$

but nevertheless the process $x(\cdot)$ is not recurrent.

Proof. Let $B = \{x \in H: \|x\| \leqslant 1\}$ and

$$B_k = \{x \in H: |\xi_{3k-2}| \leqslant 1 \text{ and } |\xi_{3k-1}| \leqslant 1 \text{ and } |\xi_{3k}| \leqslant 1\}.$$

Then $B \subset B_k$ for all $k = 1, 2, \ldots$ and

$$\int_{t_1}^{+\infty} P(t, 0, B)\,dt \leqslant \sum_{k=1}^{+\infty} \int_{t_k}^{t_{k+1}} P(t, 0, B_k)\,dt.$$

Moreover,

$$P(t, 0, B_k) \leqslant 2^3 \big(\lambda_{3k-2}(t)\lambda_{3k-1}(t)\lambda_{3k}(t)\big)^{-1/2},$$

where

$$\lambda_m(t) = \lambda_m \int_0^t \exp(-2a_m r)\,dr.$$

Therefore, taking into account the properties of the sequences (λ_k), (t_k), (a_k), we see that, for $t \in [t_k, t_{k+1})$,

$$P(t, 0, B_k) \leqslant 2^{3/2} t^{-3/2} \lambda_{3k}^{-3/2}.$$

Consequently,

$$\int_{t_1}^{+\infty} P(t, 0, B)\,dt \leqslant 2^{9/2} \sum_{k=1}^{+\infty} \int_{t_k}^{t_{k+1}} t^{-3/2}\,dt\,\lambda_{3k}^{-3/2} < +\infty.$$

Lemma 6 now completes the proof.

COROLLARY 4. *It follows from Theorem 4, Proposition 5 and Proposition 9 that there exists an infinitesimal generator A, and bounded operators B_1 and B_2 such that the solutions of both the equations*

$$dx = Ax\,dt + B_1\,dw,$$

$$dx = Ax\,dt + B_2\,dw$$

are non-degenerate and the former equation defines a recurrent process whereas the latter equation defines a non-recurrent one.

Such situation is impossible in the finite-dimensional case.

5. POSITIVE RECURRENCE

A recurrent process $x(\cdot)$ is called *positive recurrent* if and only if, for any initial condition $x_0 \in H$ and any non-empty set $V \subset H$, we have

$$E^{x_0}(T_V) < +\infty,$$

where

$$T_V = \inf\{t > 0:\ x(t) \in V\}.$$

We conjecture that if the semigroup $S(t)$ is exponentially stable: $S(t) \to 0$ exponentially as $t \to +\infty$, and the process $x(\cdot)$ given by (2) is non-degenerate, then this process is positive recurrent. However, at present we can prove this statement under some additional conditions.

THEOREM 5. *If the process $x(\cdot)$ given by (2) is non-degenerate in finite time and the semigroup $S(t)$ has the following two properties:*

(1) $S(t)x \to 0$ exponentially as $t \to +\infty$ for all $x \in H$;

(2) There exists $t > 0$ such that $S(t)$ is a compact operator,

then the process $x(\cdot)$ is positive recurrent.

Proof. The proof given here is a modification of a proof of positive recurrence for non-degenerate diffusion which can be found in [10]. Let t_0 be a positive number such that the operator $S_0 = S(t_0)$ is a compact contraction: $\|S_0\| < 1$, and, moreover, the random variable $\zeta = \int\limits_0^{t_0} S(t_0 - s) B\,dw$ has non-degenerate Gaussian distribution $\mu = \gamma(0, Q)$. To prove the theorem it is enough to show that the Markov chain (Z_n):

$$Z_n = S_0 Z_{n-1} + \zeta_n,$$
$$Z_0 = x_0,$$

where $\zeta_0, \zeta_1, \zeta_2, \ldots$ are independent random variables with the distribution μ, is positive recurrent. We divide the proof into several steps.

Firstly, it is easy to see that there exists a number $R > 0$ such that, for all $x_0 \in H$, $\|x_0\| \geqslant R$, we have

$$E^{x_0}(T) \leqslant C_1 + C_2 \|x_0\|^2, \tag{28}$$

where

$$T = \inf\{n \geqslant 1:\ \|Z_n\| \leqslant R\}$$

and C_1, C_2 are some constants. Estimate (28) follows, for example, from the observation that there exists a non-negative solution $W \geqslant 0$ of the equation

$$S_0^* W S_0 + I = W$$

and from the martingale property of the sequence (Y_n):

$$Y_n = \langle W Z_n, Z_n \rangle + \sum_{k=0}^{n-1} (\|Z_k\|^2 - \delta);$$

here $\delta = E\|W^{1/2}\zeta_0\|^2$ and $n = 0, 1, 2, \ldots$

Secondly, if $R > 0$, $r > 0$ and $y \in H$, then

$$\inf\{P^{x_0}(\|Z_1 - y\| \leqslant r):\ \|x_0\| \leqslant R\} = a > 0. \tag{29}$$

To prove (29), let us remark that

$$Z_1 - y = (S_0 x_0 - y) + \zeta_0.$$

Since ζ_0 has non-degenerate distribution, for any fixed x_0 (see Proposition 1),

$$P^{x_0}(\|Z_1 - y\| \leqslant r) > 0. \tag{30}$$

But S_0 transforms the set $\{x_0:\ \|x_0\| \leqslant R\}$ into a compact set and therefore (30) easily follows.

Thirdly, one shows easily by induction that, for a constant $c > 0$ and $k = 1, 2, \ldots$,

$$\sup\{E^{x_0}(T_k):\ \|x_0\| \leqslant R\} \leqslant kc, \tag{31}$$

where

$$T_1 = \inf\{n \geqslant 1: \|Z_n\| \leqslant R\}$$

and

$$T_{k+1} = \inf\{n \geqslant T_k+1: \|Z_n\| \leqslant R\}.$$

The final step is the observation that the probability of the fact that the ball $\{x: \|x-y\| \leqslant r\}$ will not be hit by the process $x(\cdot)$ at one of the moments $T_1+1, T_2+1, \ldots, T_k+1$ is not greater than $(1-a)^k$. Strong Markov property and estimate (31) imply positive recurrence.

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