MATHEMATICAL CONTROL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 14
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

EIGENVALUE PROBLEMS IN CONVEX SETS

ERICH MIERSEMANN

Sektion Mathematik, Karl-Marx-Universität, Leipzig, DDR

1. Introduction

In this paper we present some results related to eigenvalue problems for variational inequalities. Without attempting to formulate any precise definitions, let us discuss some examples.

Consider a beam clamped at its ends and compressed by a force P. By $v(x)$ we denote the deflection of the beam from the x-axis. The *critical load of Euler* P_0 is given by

$$P_0^{-1} = \max_{\substack{v\in L\\ v\neq 0}} \frac{\int_0^l v'^2\,dx}{\mathrm{EJ}\int_0^l v''^2\,dx},$$

where $L = \{v \mid v(0) = v'(0) = v(l) = v'(l) = 0\}$, EJ is the bending stiffness. The critical load P_0 is the first eigenvalue of

$$\mathrm{EJ}\,u^{(4)} = -Pu'' \quad \text{in} \quad (0, l), \qquad u(0) = u'(0) = u(l) = u'(l) = 0.$$

Now we consider the case where the deflections of the beam are constrained by obstacles. Define

$$V = \{v \mid v \in L,\ \psi_1(x) \leqslant v(x) \leqslant \psi_2(x) \text{ on } (0, l)\},$$

a convex set of functions. ψ_1, ψ_2 are given functions on $(0, l)$ satisfying $\psi_1(x) \leqslant 0 \leqslant \psi_2(x)$ on $(0, l)$.

Also in this case it is possible to define a critical load. The "eigenfunctions" are solutions of a variational inequality. We shall consider the same problem for the thin elastic plate. Let $\Omega \subset R^2$ be a bounded domain with boundary $\partial\Omega$. Set

$$V = \left\{u \mid u = 0,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ \psi_1(x) \leqslant u(x) \leqslant \psi_2(x) \text{ in } \Omega\right\}$$

for the admissible deflections of the plate, perpendicular to the x-plane, $x = (x_1, x_2)$, where $\psi_1(x) \leqslant 0 \leqslant \psi_2(x)$ in Ω. The boundary $\partial\Omega$ is compressed by a force $P\mathfrak{n}$ where $\mathfrak{n}$ is the inner normal at $\partial\Omega$. In the case without constraints for the deflections the lowest critical value P_0 is given by

$$P_0^{-1} = \max \frac{\int_\Omega (v_{x_1}^2 + v_{x_2}^2)\,dx}{D \int_\Omega (\Delta v)^2\,dx},$$

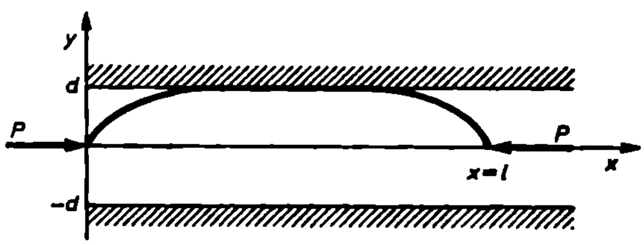
where D denotes the bending stiffness of the plate. The maximum is taken over all $v \neq 0$ with $v = \dfrac{\partial v}{\partial \mathfrak{n}} = 0$ on $\partial\Omega$.

Problems with constraints of such type we shall study in Section 2.

In Section 3 we deal with local minima in connection with variational inequalities. As an application of this theory, we consider the following problem for a compressed beam, Link [5]. The admissible deflections v are defined by

$$V = \{v \mid \; v(0) = v(l) = 0, \; |v(x)| \leqslant d \text{ in } (0, l)\},$$

where $0 < d = \text{const}$. For $P > P_0$ (here P_0 denotes the critical load of Euler) the beam leans for example at the line $y = d$ (see Fig.).



There exists a critical value P_{crit} for which a breakdown occurs.

I would like to thank Professor Klötzler for telling me this problem.

2. Eigenvalue problems for variational inequalities

Let H be a real Hilbert space with the inner product (u, v) and with the corresponding norm $\|u\|$. Denote by V a closed convex subset of H with $0 \in V$ and by $a(u, v)$, $b(u, v)$ real, symmetric, bounded bilinear forms defined on H. Suppose that the forms satisfy the following assumptions:

(2.1) $a(u, u) \geqslant 0$ for all $u \in H$.

(2.2) There exists $c > 0$ such that $a(v, v) \geqslant c\|v\|^2$ for all $v \in V$.

(2.3) The form $b(u, v)$ is completely continuous on H.

We look for solutions (λ, u), $\lambda \in R$, $u \neq 0$, of the variational inequality

$$u \in V:\ a(u, v-u) \geqslant \lambda b(u, v-u) \quad \text{for all } v \in V. \tag{2.4}$$

We do not treat the more general problem

$$u \in V:\ \big(f'(u), v-u\big) \geqslant \lambda\big(g'(u), v-u\big) \quad \text{for all } v \in V \tag{2.5}$$

in this paper. Here f', g' denote the Fréchet derivatives of functionals, which are defined on H. For problem (2.5) and applications to buckling problems for the plate we refer to Miersemann [6]–[9] and Do [2], [3].

Let V be a cone with vertex at zero, i.e., a set such that $tu \in K$ for all $t > 0$ and for all $u \in K$. Furthermore, we assume that K is closed and convex. Under assumptions (2.1)–(2.3) we have

THEOREM 2.1 [6]. *Suppose there exists a $w \in K$ with $b(w, w) > 0$. Then the following maximum problem is solvable and λ_0 defined by*

$$\lambda_0^{-1} = \max_{\substack{v \in K \\ v \neq 0}} \frac{b(v, v)}{a(v, v)}$$

is the smallest positive eigenvalue of the variational inequality (2.4).

Remark. Under certain assumptions it was proved in Miersemann [6] that the number λ_0 is the smallest point of bifurcation for an associated nonlinear problem of type (2.5), where $V = K$. A different proof of this result was given by Do [2].

Now we consider the general case. Denote by $C(V)$ the tangential cone of V at zero, i.e., the closure of the set

$$\{w = tv \mid \text{for all } v \in V, \text{ for all } t > 0\}.$$

It is easy to see that $C(V)$ is a closed convex cone with vertex at zero.

DEFINITION. We say that λ is a *point of bifurcation* if there exists a sequence of solutions (λ_n, u_n) of (2.4) with $u_n \neq 0$, $\lambda_n \to \lambda$ and $u_n \to 0$ as $n \to \infty$.

THEOREM 2.2 [7]. *Assume the existence of a $w \in C(V)$ such that $b(w, w) > 0$. Then the positive number λ_0 defined by*

$$\lambda_0^{-1} = \max_{\substack{v \in C(V) \\ v \neq 0}} \frac{b(v, v)}{a(v, v)}$$

is the smallest positive point of bifurcation for the inequality (2.4).

Remark. For any eigenvalue λ of (2.4) we have $\lambda \geqslant \lambda_0$ since setting $v = 0$ in (2.4), we have the inequality $\lambda^{-1} \leqslant b(u, u)/a(u, u)$ for an eigensolution u. Since $u \in V \subset C(V)$, we have $b(u, u)/a(u, u) \leqslant \lambda_0^{-1}$.

THEOREM 2.3 [9]. *We assume that for every $w \in V$, $w \neq 0$, one can find a $v \in V$ with $b(w, v-w) > 0$. Then for every $0 < s < \infty$ there exists a solution u of the inequality (2.4) with $a(u, u) = s$.*

Sketched proof. We use a method due to Beckert [1], Krasnosel'skii [4], which we generalize to inequalities. Write

$$M_s = \{v \in V \mid a(v, v) \leqslant s\}, \quad \text{where} \quad 0 < s < \infty.$$

We seek the vectors $u \in M_s$ for which

$$b(u, u) = \max_{v \in M_s} b(v, v). \tag{2.6}$$

By using a lemma of Miersemann [9] it follows that for each solution of (2.6) we have $a(u, u) = s$ and that there exists a $v \in V$ such that $a(u, v-u) > 0$ and $b(u, v-u) > 0$. Let $v, z \in V$ be fixed with $a(u, z-u) \neq \neq 0$ and $0 < \varepsilon < \varepsilon_0$, ε_0 sufficiently small. We calculate $k(\varepsilon)$ such that we get $a(w, w) = s$ for $w = (1-k)\,[u+\varepsilon(v-u)]+kz$. We obtain

$$k(\varepsilon) = -\frac{a(u, v-u)}{b(u, z-u)}\,\varepsilon + o(\varepsilon).$$

Set

$$C_u^+ = \{v \in V \mid a(u, v-u) > 0\} \quad \text{and} \quad C_u^- = \{v \in V \mid a(u, v-u) < 0\}.$$

If u is a solution of (2.6), then we have $C_u^+ \neq \emptyset$, $C_u^- \neq \emptyset$. From $z \in C_u^+$ and $v \in C_u^-$ we conclude $0 < k(\varepsilon) < 1$ provided $\varepsilon_0 > 0$ is small enough. Hence we have $w \in V$. Since $b(w, w) \leqslant b(u, u)$, we deduce the inequality

$$\frac{b(u, z-u)}{a(u, z-u)}\,a(u, v-u) \geqslant b(u, v-u)$$

for all $z \in C_u^+$ and for all $v \in C_u^-$, or

$$\frac{b(u, z-u)}{a(u, z-u)} \leqslant \frac{b(u, v-u)}{a(u, v-u)}.$$

Set

$$\alpha = \sup_{z \in C_u^+} \frac{b(u, z-u)}{a(u, z-u)} \quad \text{and} \quad \beta = \inf_{v \in C_u^-} \frac{b(u, v-u)}{a(u, v-u)}.$$

Then u is a solution of the variational inequality (2.4) for all λ with $\lambda^{-1} \in [\alpha, \beta]$ and for all $v \in C_u^+ \cup C_u^-$. In the case $a(u, v-u) = 0$ we set $v_n = (1-1/n)v$ in the variational inequality (2.4). Since $v_n \in C_u^-$, the inequality follows for such v by letting $n \to \infty$. ∎

EXAMPLE 1. Set $H = \mathring{H}_{1,2}(0, l) \cap H_{2,2}(0, l)$ — the usual Sobolev space over $(0, l)$ with zero boundary conditions. Let

$$V = \{v \in H \mid \psi_1(x) \leqslant v(x) \leqslant \psi_2(x) \text{ on } (0, l)\},$$

where

$$\psi_1(x) \leqslant 0 \leqslant \psi_2(x) \quad \text{and} \quad \psi_1, \psi_2 \in H_{2,2}(0, l).$$

The variational inequality which describes the buckling problem for the simply supported beam is given by

$$u \in V: \int_0^l u''(v-u)''dx \geqslant \lambda \int_0^l u'(v-u)'dx \quad \text{for all } v \in V.$$

The assumption of Theorem 2.3 is fulfilled if we have $V \neq \{0\}$, $\psi_1'' \leqslant 0$ and $\psi_2'' \geqslant 0$ a.e. on $(0, l)$. For if not, we conclude from

$$\int_0^l u'(v-u)'dx \leqslant 0 \quad \text{for all } v \in V$$

that

$$-u'' = \begin{cases} 0 & \text{if} \quad \psi_1(x) < u(x) < \psi_2(x), \\ -\psi_1'' & \text{if} \quad u(x) = \psi_1(x), \\ -\psi_2'' & \text{if} \quad u(x) = \psi_2(x). \end{cases}$$

Therefore we obtain

$$\int_0^l u'^2 dx = \int_{u=\psi_1} -\psi_1''\psi_1 dx + \int_{u=\psi_2} -\psi_2''\psi_2 dx \leqslant 0,$$

which is impossible because $u \in V$ and $u \neq 0$.

EXAMPLE 2. Buckling problems for the clamped plate are described by the inequality, Miersemann [6], [7]:

$$u \in V: \int_\Omega \Delta u \Delta(v-u)dx \geqslant \lambda \int_\Omega a_{ij}(x)u_{x_i}(v-u)_{x_j}dx \quad \text{for all } v \in V,$$

where

$$V = \{v \in \mathring{H}_{2,2}(\Omega) \mid \psi_1(x) \leqslant v(x) \leqslant \psi_2(x) \text{ in } \Omega\}$$

with

$$\psi_1(x) \leqslant 0 \leqslant \psi_2(x) \quad \text{in} \quad \Omega, \quad \psi_1, \psi_2 \in H_{2,2}(\Omega).$$

Here Ω is a bounded open subset of R^2 with sufficiently regular boundary $\partial\Omega$. For $a_{ij} = a_{ji}$ we assume $a_{ij} \in C^1(\bar{\Omega})$. Set $L = -\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)$. Then the hypothesis of Theorem 2.3 is fulfilled if $V \neq \{0\}$, $L\psi_1 \leqslant 0$ and $L\psi_2 \geqslant 0$ a.e. in Ω, i.e., ψ_1 is a subsolution and ψ_2 a supersolution with respect to L.

The argument is the same as in Example 1 and will be omitted. In this example we must assume that there exists a $w \in V$ such that

$$\int_\Omega a_{ij} w_{x_i} w_{x_j} dx > 0.$$

3. Stability problems

Now we suppose that the bilinear form $a(v, v)$ is *coercive* on H, i.e., there exists $c > 0$ such that $a(v, v) \geqslant c\|v\|^2$ for all $v \in H$. Set

$$I_\lambda(v) = \tfrac{1}{2}a(v, v) - \tfrac{1}{2}\lambda b(v, v). \tag{3.1}$$

DEFINITION. A vector (λ_0, u_0), $u_0 \in V$, $\lambda_0 \in R$, is a ***strong local minimum*** of (3.1) if there exist positive numbers ϱ, c such that

$$I_{\lambda_0}(v) - I_{\lambda_0}(u_0) \geqslant c\|v-u_0\|^2 \quad \text{for all } v \in V, \quad \text{where} \quad \|v-u_0\| \leqslant \varrho.$$

The constant c does not depend on v.

Remark. Local extrema in connection with nonlinear variational equations were studied in Beckert [1].

Let (λ, u) be a solution of the variational inequality (2.4). We shall give a criterion for (λ, u) to define a strong local minimum of functional (3.1). For $t > 0$ set

$$V_t(u) = \{w \in H \mid a(w, w) = 1, u + tw \in V\}.$$

Denote by $K_{\lambda,u}$ the closure of the set

$$\{h = t(v-u) \mid t > 0, F_{\lambda,u}(v-u) = 0, v \in V\},$$

where $F_{\lambda,u}(w) = a(u, w) - \lambda b(u, w)$. We assume that $K_{\lambda,u} \neq \{0\}$ and for

$$\mu_{\lambda,u}^{-1} = \max_{\substack{h \in K_{\lambda,u} \\ h \neq 0}} b(h, h)/a(h, h)$$

we have the inequality

$$\mu_{\lambda,u}^{-1} > 0. \tag{3.2}$$

HYPOTHESIS H_0. For every sequence $t_n \to 0$, $t_n > 0$, and for every weakly convergent sequence $w_n \rightharpoonup w$, $w_n \in V_{t_n}(u)$, from $\overline{\lim\limits_{n\to\infty}} \dfrac{F_{\lambda,u}(w_n)}{t_n} < \infty$ follows the inequality $1 - \lambda b(w, w) > 0$.

THEOREM 3.1 [10]. *Under hypothesis* H_0 *a solution* (λ, u) *of the variational inequality defines a strong local minimum of* (3.1).

Now let (λ, u_λ), $\lambda_1 < \lambda < \lambda_2$, $\lambda_1 < \lambda_2$, be a continuous branch $\mathfrak{L}$ of solutions (2.4). (We call a branch *continuous* if $u_\lambda \to u_{\lambda_0}$ as $\lambda \to \lambda_0$, where $\lambda, \lambda_0 \in (\lambda_1, \lambda_2)$.)

HYPOTHESIS H_1. *Let $(\lambda_n, u_n) \in \mathfrak{L}$ be a sequence, where $\lambda_n \to \lambda_0$, $\lambda_n, \lambda_0 \in (\lambda_1, \lambda_2)$. For every sequence $t_n \to 0$, $t_n > 0$, and for every weakly convergent sequence $w_n \rightharpoonup w$, $w_n \in V_{t_n}(u_n)$ from $\overline{\lim_{n\to\infty}} \frac{F_{\lambda_n, u_n}(w_n)}{t_n} < \infty$ follows the inequality $1 - \lambda_0 b(w, w) > 0$.*

THEOREM 3.2. *Under the assumption $\mu_{\lambda,u}^{-1} \geqslant c_0 > 0$, where c_0 does not depend on $\lambda \in [\lambda_1, \lambda_2]$, and under hypothesis H_1, there is no bifurcation from $\mathfrak{L}$. This means that there is no sequence (λ_n, u_n) of solutions of (2.4) such that $\lambda_n \to \lambda$, $\lambda, \lambda_n \in (\lambda_1, \lambda_2)$, and $(\lambda_n, u_n) \notin \mathfrak{L}$.*

An application to the beam [10]. The energy of the compressed beam according to the linear theory, is given by

$$I_\lambda(v) = \tfrac{1}{2}\mathrm{EJ}\left(\int_0^l v''^2 dx - \lambda \int_0^l v'^2 dx\right), \tag{3.3}$$

where $\lambda = P/\mathrm{EJ}$, l is the length of the beam, EJ is the bending stiffness. Suppose that the beam is simply supported at the ends, i.e., the boundary conditions $v(0) = v(l) = 0$ are prescribed. Set

$$H = \dot{H}_{1,2}(0, l) \cap H_{2,2}(0, l) \quad \text{and} \quad V = \{v \in H \mid |v(x)| \leqslant d \text{ on } (0, l)\},$$

where $0 < d = \text{const}$.

The family of functions

$$u_\lambda = \begin{cases} \dfrac{d}{\pi}\left(\dfrac{\pi}{k}x + \sin\dfrac{\pi}{k}x\right) & \text{if} \quad 0 \leqslant x \leqslant k, \\ d & \text{if} \quad k < x < l-k, \\ \dfrac{d}{\pi}\left(\dfrac{\pi}{k}(l-x) + \sin\dfrac{\pi}{k}(l-x)\right) & \text{if} \quad l-k \leqslant x \leqslant l, \end{cases}$$

where $0 < k \leqslant \frac{1}{2}$ and $\lambda = \left(\dfrac{\pi}{k}\right)^2$, defines solutions of the variational inequality

$$u \in V: \int_0^l u''(v-u)'' dx \geqslant \lambda_0 \int_0^l u'(v-u)' dx \quad \text{for all } v \in V. \tag{3.4}$$

In Miersemann [10] it was proved:

(a) The solution (λ, u_λ) of inequality (3.4) is a strong local minimum of (3.3) if λ satisfies the inequalities $(2\pi/l)^2 \leqslant \lambda < (4\pi/l)^2$.

(b) There is no bifurcation from the branch

$$\mathfrak{L} = \{(\lambda, u_\lambda) \mid (2\pi/l)^2 \leqslant \lambda < (4\pi/l)^2\}.$$

(c) The solution $(\lambda_0, u_{\lambda_0})$, where $\lambda_0 = (4\pi/l)^2$ is a point of bifurcation.

References

[1] H. Beckert, *Variations- und Eigenwertprobleme zu nichtlinearen Differentialgleichungssystemen höherer Ordnung*, Math. Nachr. 49 (1971), 311–341.

[2] Cl. Do, *Problèmes de valeurs propres pour une inéquation variationnelle sur un cône et application au flambement unilatéral d'une plaque mince*, C. R. Acad. Sci. Paris Sér. A 280 (1975), 45–48.

[3] —, *Bifurcation theory for elastic plate subjected to unilateral conditions*, J. Math. Anal. Appl. 60 (1977), 435–448.

[4] M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, Moscow 1956 [in Russian].

[5] H. Link, *Über den geraden Knickstab mit begrenzter Durchbiegung*, Ing. Arch. 22 (1954), 237–250.

[6] E. Miersemann, *Verzweigungsprobleme für Variationsungleichungen*, Math. Nachr. 65 (1975), 187–209.

[7] —, *Über nichtlineare Eigenwertaufgaben in konvexen Mengen*, ibid. 88 (1979), 191–205.

[8] —, *Über höhere Verzweigungspunkte nichtlinearer Variationsungleichungen*, ibid. 85 (1978), 195–213.

[9] —, *Eigenwertaufgaben für Variationsungleichungen*, ibid. 100 (1981), 221–228.

[10] —, *Zur Lösungsverzweigung bei Variationsungleichungen mit einer Anwendung auf den geraden Knickstab*, ibid. 102 (1981), 7–15.