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SOME ADDITIVE PROBLEMS OF NUMBERS

AKIO FUJII

Department of Mathematics, Rikkyo University, Tokyo, Japan

1. Introduction

A well-known problem of Goldbach asks to prove that every even integer $N \geqslant 6$ can be written as

$$N = p_1 + p_2,$$

where p_1 and p_2 are odd primes. It is a well-known theorem of Vinogradov that every sufficiently large odd integer N can be written as

$$N = p_1 + p_2 + p_3,$$

where p_1, p_2 and p_3 are odd primes. It is also a well-known theorem of Hua [25] that every sufficiently large integer $N \equiv 5(24)$ can be written as

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2,$$

where p_j's are primes. The representations of integers by the sums of the higher powers of primes are also well known and can be seen in Hua's book [25]. Many investigations have been done concering these problems, some of which are discussed in a survey article by Bredihin [3].

The additive problems with which we are concerned here can be stated, in a less precise way, as follows:

PROBLEM 0. To represent an integer N using only the smaller primes.

More concretely, we have the following three problems in mind here. Let g be an integer $\geqslant 1$ and let $\delta_1, \delta_2, \ldots, \delta_g$ be positive numbers satisfying $\delta_1 + \ldots + \delta_g = 1$.

PROBLEM 1. Can every sufficiently large even integer N be written as

$$N = p_{1,1} \cdots p_{1,g} + p_{2,1} \cdots p_{2,g},$$

where $p_{i,j}$'s are odd primes satisfying $p_{i,j} \leqslant N^{\delta_j}$ for $i = 1, 2$ and $j = 1, \ldots, g$?

PROBLEM 2. Can every sufficiently large odd integer N be written as

$$N = p_{1,1} \cdots p_{1,g} + p_{2,1} \cdots p_{2,g} + p_{3,1} \cdots p_{3,g},$$

where $p_{i,j}$'s are odd primes satisfying $p_{i,j} \leqslant N^{\delta_j}$ for $i = 1, 2$ and $j = 1, \ldots, g$?

PROBLEM 3. Can every sufficiently large integer $N \equiv 5 (24)$ be written as

$$N = (p_{1,1} \cdots p_{1,g})^2 + \ldots + (p_{5,1} \cdots p_{5,g})^2,$$

where $p_{i,j}$'s are primes and satisfy $p_{i,j} \leqslant N^{\delta_j/2}$ for $i = 1, \ldots, 5$ and $j = 1, \ldots, g$?

The main purpose of the present article is to give a survey of the author's previous work concering these problems and add some new theorems with proofs. In fact, Problem 1 is still open, Problem 2 for $g \geqslant 2$ has been solved in [14] and Problem 3 for $g \geqslant 2$ will be solved in this article.

In Section 2 we shall deal with some binary additive problems which surround Problem 1. In Section 3 we shall give a survey on Problem 2 for $g \geqslant 2$. In Section 4 we shall state our theorems on Problem 3, whose proofs will be given in Section 5. In Section 6 we mention some related problems and results as supplementary remarks since we restrict ourselves to giving an exposition of the author's own work in the rest of the present article.

We remark that the results in Section 4 and the proofs in Section 5 are new.

2. The binary case – problems related to Problem 1

Although Problem 1 seems to be inaccessible, there are many problems surrounding it. In fact, this kind of problem was originated by Barban in [1]. It is an analogue of Titchmarsh's divisor problem. Titchmarsh [34] posed and solved under the generalized Riemann Hypothesis the problem of an asymptotic behaviour of the number of solutions of the equation

$$1 = p - n_1 n_2,$$

for a prime $p \leqslant X$ and natural numbers n_1 and n_2, namely an asymptotic formula for the sum

$$\sum_{p \leqslant X} \tau(p-1) \quad \text{as } X \to \infty,$$

where we put $\tau(n) = \sum_{d|n} 1$. Linnik [30] solved this unconditionally, using his dispersion method. Later it was proved without using the dispersion method by Rodriquez [33] and Elliott and Halberstam [9]. They used Bombieri's mean value theorem, which states that, for any large A, there exists a positive constant B such that

$$\sum_{q \leqslant Q} \max_{(a,q)=1} \max_{y \leqslant X} \left| \sum_{\substack{p \leqslant y \\ p \equiv a (\mathrm{mod}\, q)}} 1 - \frac{1}{\varphi(q)} \mathrm{Li}\, y \right| \ll X (\log X)^{-A},$$

where $\varphi(q)$ is the Euler function and we put $Q = X^{1/2}(\log X)^{-B}$. Barban [1] posed the problem of an asymptotic behaviour of the sum

$$\sum_{p_1 \leqslant X^\delta, p_2 \leqslant X^{1-\delta}} \tau(p_1 p_2 - 1) \quad \text{as } X \to \infty,$$

where $0 < \delta \leqslant \frac{1}{2}$ and p_1 and p_2 run over the primes. Linnik's dispersion method works for $0 < \delta < 1/6$, but it does not work for other values of δ. Barban [1] solved this for $\delta = \frac{1}{2}$. The present author [13] has shown the following

THEOREM 1. *Suppose that δ is in $0 < \delta \leqslant \frac{1}{2}$ and $\delta \log X$ tends to ∞ as $X \to \infty$. Then we have*

$$\sum_{p_1 \leqslant X^\delta, p_2 \leqslant X^{1-\delta}} \tau(p_1 p_2 - 1)$$
$$= \frac{315}{2\pi^4} \cdot \frac{\zeta(3)}{\delta(1-\delta)} \cdot \frac{X}{\log X} + O\left(X\delta^{-1} (\log X)^{-2} (\log\log X + \delta^{-1})\right)$$

uniformly for δ, where $\zeta(s)$ is the Riemann zeta function.

The main term on the left-hand side of the above theorem can be seen to be equal to

$$2 \sum_{q \leqslant Q} \sum_{p_1 \leqslant X^\delta, (p_1, q) = 1} \frac{\mathrm{Li}\, X^{1-\delta}}{\varphi(q)} + 2 \sum_{q \leqslant Q} \sum_{p_1 \leqslant X^\delta, (p_1, q) = 1} \left(\sum_{\substack{p_2 \leqslant X^{1-\delta} \\ p_2 \equiv p_1^* (\mathrm{mod}\, q)}} 1 - \frac{\mathrm{Li}\, X^{1-\delta}}{\varphi(q)} \right),$$

where p_1^* satisfies $p_1^* p_1 \equiv 1 (\mathrm{mod}\, q)$ and we put $Q = X^{1/2}(\log X)^{-B}$ with some positive constant B. To deal with the last sum we have to get an analogue of Bombieri's mean value theorem. This kind of sum was first treated by Chen [6], who proved that every sufficiently large even integer N can be written as

$$N = p_1 + p_2 \quad \text{or} \quad N = p_3 + p_4 p_5$$

with primes p_j's. In fact, we have proved and used the following lemma, where we put

$$E(y; a, q) = \sum_{\substack{p \leqslant y \\ p \equiv a (\mathrm{mod}\, q)}} 1 - \frac{\mathrm{Li}\, y}{\varphi(q)}.$$

LEMMA 1. *Suppose that* $\sum_{m \leqslant X} |b(m)|^2 \ll X (\log X)^C$ *with some positive* *constant C. Then, for any positive constants A and b* (<1), *there exists a positive constant B such that*

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\Big|\sum_{\substack{1\leqslant m\leqslant X^\delta\\(m,q)=1}} b_m E(X^{1-\delta};\, am^*,\, q)\Big| \ll X(\log X)^{-A}$$

uniformly for δ *in* $0\leqslant\delta<1-(\log X)^{-b}$, *where* $Q=X^{1/2}(\log X)^{-B}$ *and* $mm^*\equiv 1 \pmod q$.

We remark that the conclusion of Lemma 1 still holds even if we replace $E(X^{1-\delta};\, am^*,\, q)$ by $E(X/m;\, am^*,\, q)$. In a similar manner we get an asymptotic formula for the sum

$$\sum_{p_1\leqslant N^\delta,\, p_2\leqslant N^{1-\delta}} \tau(N-p_1p_2),$$

where $0<\delta\leqslant 1/2$ and N is an integer. Namely, we get an asymptotic formula for the number of solutions of the equation

$$N = p_1p_2+mn$$

for primes p_1 ($\leqslant N^\delta$) and p_2 ($\leqslant N^{1-\delta}$) and natural numbers m and n.

As is well known, Hardy–Littlewood's problem belongs to the same category as Titchmarsh's divisor problem. It is to get an asymptotic formula for the number of representations of a natural number N as $N=p+m^2+n^2$, where p is a prime and m and n are natural numbers. Under the generalized Riemann Hypothesis, Hooley [24] derived such a formula. Linnik [30] succeeded in getting it without using any unproved hypothesis. He used his dispersion method. Later Elliott and Halberstam [9] proved it without using the dispersion method. They used instead Bombieri's mean value theorem. Linnik [30] and Poljanskii [31] derived an asymptotic formula for the number of representations of a natural number N as $N=p_1p_2+m^2+n^2$, where p_1 and p_2 are primes and m and n are natural numbers. They used the dispersion method. We remark here that we can derive that formula without using the dispersion method (cf. [13]). Instead, we have used Lemma 1 above and the following lemma:

LEMMA 1'. *For any positive constants A and b* (<1), *if* $\sum_{m\leqslant X}|b(m)|^2 \ll X(\log X)^C$, $b(m)\ll X^{1-\delta-\beta}$ *for* $m\leqslant X^\delta$, $\beta=(\log X)^{-f}$ *with some* f *in* $b<f<1$ *and some positive constant C, then there exists a positive constant B such that*

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\max_{1\leqslant y\leqslant X}\Bigg|\sum_{\substack{mp\leqslant y,\, m\leqslant X^\delta\\ mp\equiv a \pmod q}} b(m)-\frac{1}{\varphi(q)}\sum_{\substack{mp\leqslant y\\ m\leqslant X^\delta}} b(m)\Bigg| \ll X(\log X)^{-A}$$

uniformly for δ *in* $0\leqslant\delta<1-(\log X)^{-b}$, *where p runs over the primes and we put* $Q=X^{1/2}(\log X)^{-B}$.

3. The ternary case

Compared with the unsatisfactory results in the binary case, we can give a more satisfactory result in the ternary case. In fact, we have proved in [14] the following

THEOREM 2. *Let g be an integer $\geqslant 2$. Then every sufficiently large odd integer N can be written as is asked in Problem 2.*

We have proved our theorem using the Hardy–Littlewood–Vinogradov circle method. We have estimated the integral

$$\int_0^1 S_N^3(\alpha)\, e(-N\alpha)\, d\alpha,$$

where we put $e(\alpha) = \exp(2\pi i\alpha)$,

$$S_N(\alpha) = \sum_{p_j \leqslant N^{\delta_j}, j=1,\ldots,g} \log p_1 \ldots \log p_g\, e(p_1 \ldots p_g \alpha)$$

and p_j's run over the primes. The estimates of the minor arcs may be reduced to the estimates of the sum

$$\sum_\chi \prod_{j=1}^{g} \left| \sum_{p \leqslant N^{\delta_j}} \frac{\chi(p)\log p}{p^{1/2+it}} \right|,$$

where χ runs over all Dirichlet characters $\bmod q$, q belongs to $((\log N)^B, N(\log N)^{-B}]$, B is a sufficiently large constant and t is a real number. The last sum can be treated by Lemma 2 of Gallagher [17]. The major arcs can be treated by the Siegel–Walfisz theorem (cf. Satz 8.3 of [32]) on primes in arithmetic progressions. Thus we get the following quantitative result.

$$\sum_{N = p_{1,1}\cdots p_{1,g} + p_{2,1}\cdots p_{2,g} + p_{3,1}\cdots p_{3,g}} \prod_{i=1}^{3}\prod_{j=1}^{g} \log p_{i,j}$$
$$= \frac{1}{((g-1)!)^3}\,\mathfrak{S}(N)\,\tilde{r}_g(N) + O\left(N^2(\log N)^{-A}\right),$$

where

$$\mathfrak{S}(N) = \prod_{p \nmid N}\left(1 + \frac{1}{(p-1)^3}\right)\prod_{p \mid N}\left(1 - \frac{1}{(p-1)^2}\right),$$

$$\tilde{r}_g(N) = \sum_{N = h_1 + h_2 + h_3} \left(\log\frac{N}{h_1}\right)^{g-1}\left(\log\frac{N}{h_2}\right)^{g-1}\left(\log\frac{N}{h_3}\right)^{g-1},$$

p's run over the primes, h's are positive integers and A is a sufficiently large constant.

We may mention the following special case of Theorem 2.

COROLLARY 1. *Every sufficiently large odd integer N can be written as*

$$N = p_{1,1} \cdots p_{1,g} + p_{2,1} \cdots p_{2,g} + p_{3,1} \cdots p_{3,g},$$

where $p_{i,j}$'s are odd primes $\leqslant N^{1/g}$.

4. Waring's type of problem

Let k be an integer $\geqslant 2$. We put

$$s_0(k) = \begin{cases} 2^k & \text{for } 2 \leqslant k \leqslant 11, \\ 2k^2(2\log k + \log\log k + 2.5) - 2 & \text{for } k \geqslant 12. \end{cases}$$

Let p run over the primes. Let $\theta = \theta(p, k)$ be an integer such that $p^\theta \| k$. We put

$$\gamma = \gamma(p, k) = \begin{cases} \theta + 2 & \text{if } p = 2 \text{ and } p|k, \\ \theta + 1 & \text{otherwise} \end{cases}$$

and $K = \prod_{p-1|k} p^\gamma$. Our result concerning Problem 3 can be stated as follows.

THEOREM 3. *Let k be an integer $\geqslant 2$. Let s be an integer $\geqslant s_1(k)$, where we can take $s_1(k) = s_0(k) + 1$. Let g be an integer $\geqslant 1$ and let $\delta_1, \ldots, \delta_g$ be positive numbers satisfying $\delta_1 + \ldots + \delta_g = 1$. Then every sufficiently large integer $N \equiv s \pmod K$ can be written as*

$$N = n_1^k + n_2^k + \ldots + n_s^k,$$

where n's are of the form $p_1 p_2 \ldots p_g$ with primes p_j's satisfying $p_j \leqslant N^{\delta_j/k}$ for $j = 1, \ldots, g$.

This gives, in particular, an affirmative answer to our Problem 3. We mention the following corollaries as special cases of our Theorem 3.

COROLLARY 2. *For any integer $g \geqslant 1$, every sufficiently large integer $N \equiv 5 \pmod{24}$ can be written as*

$$N = (p_{1,1} \cdots p_{1,g})^2 + \ldots + (p_{5,1} \cdots p_{5,g})^2$$

with primes p's $\leqslant N^{1/2g}$.

COROLLARY 3. *For any integer $g \geqslant 1$, every sufficiently large odd integer N can be written as*

$$N = (p_{1,1} \cdots p_{1,g})^3 + \ldots + (p_{9,1} \cdots p_{9,g})^3$$

with primes p's $\leqslant N^{1/3g}$.

We remark here that if one uses the argument in Chapter IX of Hua [25] and our Lemma 2 below, then one can take the better $s_1(k)$ in Theorem 3 for $k \geqslant 5$. To prove our theorem we use the Hardy–Littlewood–Vinogradov circle method and the following lemma on the trigonometric sums over the primes:

LEMMA 2. *Let k be an integer $\geqslant 2$, let g be an integer $\geqslant 1$ and let $l_1, \ldots, l_g$ be integers $\geqslant 2$. Let N be an integer $> N_0$ and let $H, H_1, \ldots$ and H_g be integers $\geqslant 1$. Suppose that a real α satisfies $|\alpha - a/q| \leqslant q^{-2}$ with relatively prime positive integers a and q. Then for an arbitrarily small positive ν,*

$$\mathscr{S} \equiv \sum_{h \leqslant H} \sum_{h_1 \leqslant H_1} \cdots \sum_{h_g \leqslant H_g} \Big| \sum_{n \leqslant N} e(\alpha h h_1^{l_1} \ldots h_g^{l_g} n^k) \Lambda(n) \Big|$$

$$\ll HH_1 \ldots H_g N \mathscr{L}^C (\tilde{Q}^{-2^{-\eta}} + N^{-1/2})^{\frac{1-\nu}{2^{k-1} s_2(k)}},$$

where $s_2(2) = 2$ and $s_2(k) = s_0(k)$ for $k \geqslant 3$. $\mathscr{L} = \log(HH_1^{l_1} \ldots H_g^{l_g} N^k)$,

$$\tilde{Q} = \min(\hat{H}, q, HH_1^{l_1} \ldots H_g^{l_g} N^k/q),$$

$$\hat{H} = \begin{cases} \min\{H_i;\ H_i \gg 1,\ i = 1, \ldots, g\}, \\ \infty \quad \text{if } H_i \ll 1 \text{ for all } i = 1, \ldots, g, \end{cases}$$

$$\eta = \sum_{\substack{i=1 \\ H_i \gg 1}}^{g} (l_i - 1),$$

$\Lambda(n)$ is the von Mangoldt function, and C is some positive constant which may depend on $k, l_1, \ldots, l_g$ and ν.

We shall prove our Lemma 2 and Theorem 3 in Section 5.

5. Proofs of Lemma 2 and Theorem 3

5.1. Some notation. Here we shall list some of the symbols which will be used in this section. We denote arbitrarily small positive numbers by ε or ν, arbitrarily large constants by A and some positive constants by C. $\mu(n)$ is the Möbius function, $\tau_j(n) = \sum_{n = d_1 \ldots d_j} 1$ and we denote $\tau_j(n)$ for some integer j by $\tau'(n)$. The following notation will be used except in 5.6. k is an integer $\geqslant 2$, $2b = 2^{k-1}$, g is an integer $\geqslant 1$, $l_1, \ldots, l_g$ are integers $\geqslant 2$, N is an integer $> N_0$, $H, H_1, \ldots, H_g$ are integers $\geqslant 1$.

α is a real number and $|\alpha - a/q| \leqslant q^{-2}$ with relatively prime positive integers a and q, $\tilde{Q}$ and η are the same as in Lemma 2 above, $\hat{Q} = \tilde{Q}^{2^{-\eta}}$, $\mathscr{L} = \log(N^k HH_1^{l_1} \ldots H_g^{l_g})$, $\tilde{\mathscr{L}} = \log(N^k HH_1^{l_1} \ldots H_g^{l_g} q)$, $\sum_h = \sum_{h \leqslant H}$, $\sum_{h_i} = \sum_{h_1 \leqslant H_1} \cdots \sum_{h_g \leqslant H_g}$, $\bar{H} = HH_1 \ldots H_g$, $\bar{h}_i = h_1^{l_1} \ldots h_g^{l_g}$, $2L_1 = 2^{l_1 - 1}, \ldots, 2L_g = 2^{l_g - 1}$, $2r$ is defined after Lemma 3 in 5.2 below.

5.2. Some lemmas. Here we shall list some of the lemmas which will be used for the proof of Lemma 2.

LEMMA 3. *We put $2r = 2^k$ for $2 \leqslant k \leqslant 11$ and let r be the least integer such that*

$$r > k^2(2\log k + \log\log k + 2.5) - 2 \quad \textit{for } k \geqslant 12.$$

Then

$$\int_0^1 \Big|\sum_{n \leqslant N} e(\alpha n^k)\Big|^{2r} d\alpha \ll N^{2r-k}(\log N)^C,$$

where C and the implied constant in $\ll$ may depend on k.

(Cf. Theorem 4 and Lemma 7.13 of Hua [25].)

Hereafter, except in 5.6, let $2r = 2$ for $k = 2$ and let $2r$ be as in Lemma 3 for $k \geqslant 3$.

Let a_m and b_m be any complex numbers which satisfy

$$\sum_{m \leqslant X} |a_m|^2 \ll X(\log X)^C \quad \text{and} \quad \sum_{m \leqslant X} |b_m|^E \ll X(\log X)^C,$$

where $X > X_0$, $E \geqslant 1$ and C may depend on E.

LEMMA 4. *Let M_1, M_2, N_1 and N_2 be positive integers such that $M_1 < M_2 \leqslant N$, $N_1 < N_2 \leqslant N$ and $M_1 N_1 \leqslant N$. If $N/N_2 \leqslant M_2$ then, for any positive $\varepsilon < 1$ and $\nu < 1$,*

$$R \equiv \sum_h \sum_{h_i} \Big| \sum_{M_1 < m \leqslant M_2} a_m \sum_{\substack{N_1 < n \leqslant N_2 \\ mn \leqslant N}} b_n e(\alpha h \tilde{h}_i m^k n^k)\Big|$$

$$\ll \bar{H} N^{1-\frac{k}{2r}(1-\varepsilon_0)} M_2^{\frac{k}{2r}(1-\varepsilon_0)} \mathscr{L}^C + \bar{H} N \mathscr{L}^C (\tilde{Q}^{-2^{-\eta}} + N_2/N)^{\frac{1-\nu}{4br}},$$

where $\varepsilon_0 = 1/2$ when $k = 2$ and $2r = 2$ and $\varepsilon_0 = \varepsilon$ when $k \geqslant 3$. If $N/N_2 \geqslant M_2$, then for any positive $\varepsilon < 1$ and $\nu < 1$,

$$R \ll \bar{H} N_2^{1-\frac{k}{2r}(1-\varepsilon_0)} M_2 \mathscr{L}^C +$$

$$+ \bar{H} M_2 N_2 \mathscr{L}^C \left(\frac{1}{M_2} + \left(\frac{N}{M_2 N_2}\right)^k \left(\tilde{Q}^{-2^{-\eta}} + \frac{N_2}{N}\right)^{1-\nu}\right)^{1/4br},$$

where ε_0 is the same as above.

If $b_n \ll D$, then ε may be taken to be 0 and we multiply the corresponding terms by D.

We shall prove Lemma 4 in 5.3. We need also the following

LEMMA 5. *Suppose that $b_n \equiv 1$ in Lemma 4. Then, under the same notations as Lemma 4, for any positive $\nu < 1$,*

$$R \ll \begin{cases} \bar{H} N \mathscr{L}^C M_2^{(k-1)/2b} (\tilde{Q}^{-2^{-\eta}} + M_2/N)^{(1-\nu)/2b} & \textit{if } N/N_2 \leqslant M_2, \\ \bar{H} M_2 N_2 \mathscr{L}^C \left(\dfrac{1}{N_2} + \dfrac{N^k}{N_2^k M_2}\left(\tilde{Q}^{-2^{-\eta}} + \dfrac{M_2}{N}\right)^{1-\nu}\right)^{1/2b} & \textit{if } N/N_2 \geqslant M_2. \end{cases}$$

We shall prove Lemma 5 in 5.4.

LEMMA 6. *Suppose that* $X \geqslant 1$, $Y \geqslant 1$, $(a, q) = 1$ *and* $|\alpha - a/q| \leqslant q^{-2}$. *Then*

$$\sum_{x \leqslant X} \min(Y, \|\alpha x\|^{-1}) \ll XYq^{-1} + (X+q)\log 2q,$$

$$\sum_{x \leqslant X} \min(XY/x, \|\alpha x\|^{-1}) \ll (XYq^{-1} + X + q)\log(2XYq).$$

(Cf. Vinogradov [40].)

5.3. *Proof of Lemma* 4. We may suppose that either $H_1 = H_2 = \ldots = H_g = 1$ or $H_1, \ldots, H_g \gg 1$. We decompose R into sums of the type

$$R(V, W) = \sum_h \sum_{h_i} \Big| \sum_{V < m \leqslant V'} a_m \sum_{\substack{W < n \leqslant W' \\ mn \leqslant N}} b_n e(\alpha h \bar{h}_i m^k n^k) \Big|,$$

where $1 \leqslant V < V' \leqslant 2V$, $1 \leqslant W < W' \leqslant 2W$ and $VW \leqslant N$.

By Hölder's inequality,

$$R(V, W) \ll \bar{H}^{1-(1/2r)} \Big(\sum_m |a_m|^{2r/(2r-1)}\Big)^{1-(1/2r)} \times$$
$$\times \Big(\sum_h \sum_{h_i} \sum_m \Big| \sum_{\substack{n \\ mn \leqslant N}} b_n e(\alpha h \bar{h}_i m^k n^k)\Big|^{2r}\Big)^{1/2r}$$
$$= \bar{H}^{1-(1/2r)} R_1^{1-(1/2r)} R_2^{1/2r}, \text{ say},$$

where m runs over $V < m \leqslant V'$ and n runs over $W < n \leqslant W'$. We have

$$R_1 \ll V\mathscr{L}^c,$$
$$R_2 = \sum_h \sum_{h_i} \sum_m \sum'_{n_i, mn_i \leqslant N} b_{n_1} \ldots \bar{b}_{n_{2r}} + \sum_h \sum_{h_i} \sum_m \sum''_{n_i, mn_i \leqslant N} b_{n_1} \ldots \bar{b}_{n_{2r}} e(\alpha h \bar{h}_i m^k F)$$
$$= R_3 + R_4, \text{ say},$$

where the dash indicates that we sum over all $n_1, \ldots, n_{2r}$ such that $W < n_i \leqslant W'$ and $n_1^k + \ldots + n_r^k = n_{r+1}^k + \ldots + n_{2r}^k$, the double dash indicates that we sum over all $n_1, \ldots, n_{2r}$ such that $W < n_i \leqslant W'$ and $n_1^k + \ldots + n_r^k \neq n_{r+1}^k + \ldots + n_{2r}^k$ and we denote $n_1^k + \ldots + n_r^k - n_{r+1}^k - \ldots - n_{2r}^k$ by F.

If $2r = 2$, then

$$R_3 \ll \bar{H} VW \mathscr{L}^c.$$

If $2r > 2$ and $b_n \ll D$, then by Lemma 3

$$R_3 \ll D^{2r} \bar{H} V W^{2r-k} \mathscr{L}^c.$$

Generally, for $2r > 2$ and for any positive $\varepsilon < 1$,

$$R_3 \ll \bar{H} V W^{2r-(1-\varepsilon)k} \mathscr{L}^C,$$

$$R_4 \ll \sum_h \sum_{h_i} \sum_{n_i \leqslant N/V}'' |b_{n_1}| \dots |b_{n_{2r}}| \,\Big| \sum_{m \leqslant N/W} e(\alpha h \bar{h}_i m^k F) \Big|$$

$$\ll \bar{H}^{1-(1/2b)} \Big(\sum_n |b_n|^{2b/(2b-1)}\Big)^{2r(1-(1/2b))} \times$$

$$\times \Big(\sum_h \sum_{h_i} \sum_{n_i \leqslant N/V}'' \Big| \sum_{m \leqslant N/W} e(\alpha h \bar{h}_i m^k F) \Big|^{2b}\Big)^{1/2b}$$

$$= \bar{H}^{1-(1/2b)} R_5^{2r(1-(1/2b))} R_6^{1/2b}, \text{ say,}$$

$$R_5 \ll W \mathscr{L}^C.$$

By Weyl's method (cf. pp. 10–12 of Davenport [8]),

$$R_6 \ll V^{2b-1} \bar{H} W^{2r} +$$

$$+ V^{2b-k} \sum_h \sum_{h_i} \sum_{n_i}'' \sum_{y_1, \dots, y_{k-1} \leqslant N/W} \min(N/W, \|\alpha h \bar{h}_i k!\, y_1 \dots y_{k-1} F\|^{-1})$$

$$= V^{2b-1} H W^{2r} + R_7, \text{ say.}$$

$$R_7 = V^{2b-k} \sum_{h_i} \sum_{1 \leqslant |d| \ll HN^{k-1} W} c(d) \min(N/W, \|\alpha d \bar{h}_i\|^{-1}),$$

where $d = hk!\, y_1 \dots y_{k-1} F \ll HN^{k-1} W$, $N/W \ll N^k H/|d|$ and

$$c(d) \ll \sum_{d = d_1 d_2, d_1 > 0} \tau'(d_1) \Big(\sum_{\substack{n_i \\ d_2 = n_1^k + \dots + n_r^k - n_{r+1}^k - \dots - n_{2r}^k}} 1 \Big) \ll W^{2r-k} \tau'(|d|) \mathscr{L}^C.$$

We first treat the case $H_1 = \dots = H_g = 1$. Then by Hölder's inequality, Lemma 6 and Lemma 1.1.2 of Linnik [30], for any positive $\nu < 1$,

$$R_7 \ll V^{2b-k} W^{2r-k} \mathscr{L}^C \sum_d \tau'(|d|) \min(HN^k/|d|, \|\alpha d\|^{-1})$$

$$\ll V^{2b-k} W^{2r-k} \mathscr{L}^C (N^k H)^\nu \Big(\sum_d \tau'(|d|)^{1/\nu}/|d|\Big)^\nu \Big(\sum_d \min(HN^k/|d|, \|\alpha d\|^{-1})\Big)^{1-\nu}$$

$$\ll V^{2b-k} W^{2r-k} \mathscr{L}^C (N^k H)^\nu (HN^k/q + HN^{k-1} W + q)^{1-\nu}$$

$$\ll V^{2b-k} W^{2r-k} N^k \bar{H} \mathscr{L}^C (\hat{Q}^{-1} + W/N)^{1-\nu},$$

where d runs over $1 \leqslant |d| \ll HN^{k-1} W$.

Now suppose that $H_1, \dots, H_g \gg 1$. Then for any positive $\nu < 1$,

$$R_7 \ll V^{2b-k} W^{2r-k} \mathscr{L}^C (\bar{H} N^k)^\nu \Big(\sum_{h_i} \sum_d \min(N/W, \|\alpha d \bar{h}_i\|^{-1})\Big)^{1-\nu}$$

$$= V^{2b-k} W^{2r-k} \mathscr{L}^C (\bar{H} N^k)^\nu R_8^{1-\nu}, \text{ say.}$$

We suppose first that $g = 1$. Then

$$\sum_{h_1 \leqslant H_1} \min(N/W, \|\alpha d h_1^{l_1}\|^{-1})$$

$$\ll \tilde{\mathscr{L}} \frac{N}{W} \sum_{1 \leqslant |j| \leqslant (N/W)^2} \min((N/W)^{-1}, |j|^{-1}) \Big| \sum_{h_1 \leqslant H_1} e(j\alpha d h_1^{l_1}) \Big| + H_1 \tilde{\mathscr{L}}$$

$$\ll \tilde{\mathscr{L}} \Phi_d((N/W)^2) W/N - \tilde{\mathscr{L}} \Phi_d(1) + (N/W) \tilde{\mathscr{L}} \int_{N/W}^{(N/W)^2} \Phi_d(y) y^{-2} dy + H_1 \tilde{\mathscr{L}},$$

where we put

$$\Phi_d(y) = \sum_{1 \leqslant |j| \leqslant y} \Big| \sum_{h_1 \leqslant H_1} e(j\alpha d h_1^{l_1}) \Big|$$

and have used the Fourier expansion of $\min(N/W, \|\alpha\|^{-1})$ as in pp. 265–266 of Ghosh [19].

Now

$$\sum_d \Phi_d(y) \ll (HN^{k-1} Wy)^{1-(1/2L_1)} \Big(\sum_d \sum_j \Big| \sum_{h_1 \leqslant H_1} e(j\alpha d h_1^{l_1}) \Big|^{2L_1}\Big)^{1/2L_1}$$
$$= (HN^{k-1} Wy)^{1-(1/2L_1)} R_9^{1/2L_1}, \text{ say,}$$

where $1 \leqslant |j| \leqslant y$. Then as before for any positive $v < 1$,

$$R_9 \ll H_1^{2L_1-1} HN^{k-1} Wy + H_1^{2L_1 - l_1} \times$$
$$\times \sum_d \sum_j \sum_{y_1, \ldots, y_{l_1-1} \leqslant H_1} \min(H_1, \|j\alpha d l_1! \, y_1 \ldots y_{l_1-1}\|^{-1})$$
$$\ll H_1^{2L_1-1} HN^{k-1} Wy + H^{2L_1-l_1} (HN^{k-1} W H_1^{l_1} y)^{v} \tilde{\mathscr{L}}^C \times$$
$$\times \Big(\sum_{1 \leqslant |t| \ll HN^{k-1} W H_1^{l_1-1} y} \min(H_1, \|\alpha t\|^{-1}) \Big)^{1-v}$$
$$\ll H_1^{2L_1} HN^{k-1} Wy \tilde{\mathscr{L}}^C \left(\frac{1}{q} + \frac{1}{H_1} + \frac{q}{HN^{k-1} W H_1^{l_1} y} \right)^{1-v}.$$

Thus we get

$$\sum_d \Phi_d(y) \ll HN^{k-1} Wy H_1 \tilde{\mathscr{L}}^C \left(\frac{1}{q} + \frac{1}{H_1} + \frac{q}{HN^{k-1} W H_1^{l_1} y} \right)^{(1-v)/2L_1},$$

$$R_8 \ll HN^k H_1 \tilde{\mathscr{L}}^C \left(\left(\frac{1}{q} + \frac{1}{H_1} + \frac{q}{HN^k H_1^{l_1}} \right)^{(1-v)/2L_1} + \frac{W}{N} \right)$$

and

$$R_7 \ll V^{2h-k} W^{2r-k} \tilde{\mathscr{L}}^C HH_1 N^k \left(\frac{W}{N} + \left(\frac{1}{q} + \frac{1}{H_1} + \frac{q}{HN^k H_1^{l_1}} \right)^{1/2L_1} \right)^{1-v}.$$

For $g \geqslant 2$, we repeat the above procedure and get

$$R_7 \ll V^{2b-k} W^{2r-k} \tilde{\mathscr{L}}^C \bar{H} N^k \Big(\frac{W}{N} + H_1^{-1/2L_1} + H_2^{-1/4L_1L_2} + \ldots +$$

$$+ H_g^{-1/2^g L_1 \ldots L_g} + \Big(\frac{1}{q} + \frac{q}{HN^k H_1^{l_1} \ldots H_g^{l_g}}\Big)^{-1/2^g L_1 \ldots L_g}\Big)^{1-\nu}$$

$$\ll V^{2b-k} W^{2r-k} \tilde{\mathscr{L}}^C \bar{H} N^k (\hat{Q}^{-1} + W/N)^{1-\nu}.$$

Combining all these estimates, we get the following

LEMMA 7.

$$R(V, W) \ll \bar{H} V W \tilde{\mathscr{L}}^C \big(W^{-2bk(1-\varepsilon_0)} + V^{-1} + (N/VW)^k (W/N + \hat{Q}^{-1})^{1-\nu}\big)^{1/4br},$$

where ε_0 is defined in Lemma 4.

From this we immediately deduce Lemma 4.

5.4. *Proof of Lemma* 5. We decompose R into sums of the type

$$R'(V, W) = \sum_h \sum_{h_i} \Big| \sum_{V < m \leqslant V'} a_m \sum_{\substack{W < n \leqslant W' \\ mn \leqslant N}} e(\alpha h \tilde{h}_i m^k n^k) \Big|,$$

where $1 \leqslant V < V' \leqslant 2V$, $1 \leqslant W < W' \leqslant 2W$ and $VW \leqslant N$. Then

$$R'(V, W) \ll (\bar{H} V)^{1-(1/2b)} \mathscr{L}^C \Big(\sum_h \sum_{h_i} \sum_m \Big| \sum_{n, nm \leqslant N} e(\alpha h \tilde{h}_i m^k n^k)\Big|^{2b}\Big)^{1/2b}$$

$$= (\bar{H} V)^{1-(1/2b)} \mathscr{L}^C R_{10}^{1/2b}, \text{ say.}$$

$$R_{10} \ll W^{2b-1} \bar{H} V +$$

$$+ W^{2b-k} \sum_h \sum_{h_i} \sum_m \sum_{y_1, \ldots, y_{k-1} \leqslant N/V} \min(N/V, \|\alpha h \tilde{h}_i m^k y_1 \ldots y_{k-1} k!\|^{-1})$$

$$\ll W^{2b-1} \bar{H} V + W^{2b-k} \sum_{h_i} \sum_{1 \leqslant d \ll HN^{k-1}V} \tau'(d) \min(N/V, \|\alpha \tilde{h}_i d\|^{-1}).$$

The last sum can be treated as R_7 and we get

$$R_{10} \ll W^{2b-1} \bar{H} V + W^{2b-k} \tilde{\mathscr{L}}^C \bar{H} N^k (V/N + \hat{Q}^{-1})^{1-\nu}.$$

Thus we get the following

LEMMA 8.

$$R(V, W) \ll \bar{H} V W \tilde{\mathscr{L}}^C \Big(W^{-1} + \frac{N^k}{VW^k} (V/N + \hat{Q}^{-1})^{1-\nu}\Big)^{1/2b}.$$

This immediately gives our Lemma 5.

5.5. *Proof of Lemma* 2. We may suppose that $q \leqslant N^k \bar{H}$. Let Z be a positive number $\leqslant N^{1/3}$, which will be chosen later. By Vaughan's lemma (cf.

Vaughan [35] and Ghosh [19]), we get

$$\sum_{n\leqslant N}\Lambda(n)e(\alpha h\tilde{h}_i n^k)=\int_1^N\sum_{d\leqslant \min(Z,N/y)}\mu(d)\sum_{y<l\leqslant N/d}e\big(\alpha h\tilde{h}_i(dl)^k\big)\frac{dy}{y}-$$

$$-\Big(\sum_{l\leqslant Z}\sum_{rl\leqslant N}+\sum_{Z<l\leqslant\sqrt{N}}\sum_{r\leqslant Z}+\sum_{Z<l\leqslant\sqrt{N}}\sum_{Z\leqslant r\leqslant\sqrt{N}}+$$

$$+\sum_{\sqrt{N}<l\leqslant Z^2}\sum_{rl\leqslant N}+\sum_{Z<l\leqslant\sqrt{N}}\sum_{\sqrt{N}\leqslant r\leqslant N/l}\Big)g(l)\,e\big(\alpha h\tilde{h}_i(rl)^k\big)-$$

$$-\Big(\sum_{Z<m\leqslant\sqrt{N}}\sum_{Z<n\leqslant\sqrt{N}}+\sum_{Z<m\leqslant\sqrt{N}}\sum_{N<n\leqslant N/m}+$$

$$+\sum_{\sqrt{N}<m\leqslant N/Z}\sum_{Z<n\leqslant N/m}\Big)t(m)\Lambda(n)\,e(\alpha h\tilde{h}_i m^k n^k)+O(N^{1/3})$$

$$=S_1-\sum_{j=1}^{5}S_2^{(j)}-\sum_{j=1}^{3}S_3^{(j)}+O(N^{1/3}),\ \text{say},$$

where

$$g(l)=\sum_{dn=l,d\leqslant Z}\mu(d)\Lambda(n)\qquad(\ll\log l)$$

and

$$t(m)=\sum_{d|m,d\leqslant Z}\mu(d)\qquad(\ll\tau'(m)).$$

We observe that, if $Z^2\leqslant N$, then we suppose that $g(l)=0$ for $l>Z^2$ and the estimates below hold also in this case. We denote $\sum_h\sum_{h_i}|S_1|$ by $\mathscr{S}_1$ and $\sum_h\sum_{h_i}|S_n^{(j)}|$ by $\mathscr{S}_n^{(j)}$.

By Lemma 5, we get

$$\mathscr{S}_1,\ \mathscr{S}_2^{(1)}\ll\bar{H}N\mathscr{L}^C Z^{(k-1)/2b}(\hat{Q}^{-1}+Z/N)^{(1-\nu)/2b}.$$

By Lemma 4, we get

$$\mathscr{S}_2^{(3)}\ll\bar{H}N^{1-\frac{k}{4r}(1-\varepsilon_1)}\mathscr{L}^C+\bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+Z/N)^{\frac{1-\nu}{2b}},$$

$$\mathscr{S}_2^{(4)}\ll\bar{H}N^{1-\frac{k}{2r}(1-\varepsilon_1)}Z^{\frac{2k}{2r}(1-\varepsilon_1)}\mathscr{L}^C+\bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+1/\sqrt{N})^{\frac{1-\nu}{4br}},$$

$$\mathscr{S}_2^{(5)}\ll\bar{H}NZ^{-\frac{k}{2r}(1-\varepsilon_1)}\mathscr{L}^C+\bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+1/\sqrt{N})^{\frac{1-\nu}{4br}},$$

$$\mathscr{S}_3^{(1)}\ll\bar{H}N^{1-\frac{k}{4r}(1-\varepsilon_1)}\mathscr{L}^C+\bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+1/\sqrt{N})^{\frac{1-\nu}{4br}},$$

$$\mathscr{S}_3^{(2)}\ll\bar{H}NZ^{-(1-\varepsilon_0)\frac{k}{2r}}\mathscr{L}^C+\bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+1/\sqrt{N})^{\frac{1-\nu}{4br}},$$

$$\mathscr{S}_3^{(3)}\ll\bar{H}NZ^{-(1-\varepsilon_1)\frac{k}{2r}}\mathscr{L}^C+\bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+1/\sqrt{N})^{\frac{1-\nu}{4br}},$$

where $\varepsilon_1 = 1/2$ if $k = 2$ and $2r = 2$ and $\varepsilon_1 = 0$ if $k \geqslant 3$ and ε_0 is the same as before.

Trivially, we get

$$\mathscr{S}_2^{(2)} \ll Z\sqrt{N}\bar{H}\mathscr{L}.$$

Thus we get

$$\mathscr{S} \ll \bar{H}N\mathscr{L}^C Z^{\frac{k-1}{2b}}(\hat{Q}^{-1}+Z/N)^{\frac{1-\nu}{2b}} + \bar{H}NZ^{-(1-\varepsilon_0)\frac{k}{2r}}\mathscr{L}^C + \\ + \bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+N^{-1/2})^{\frac{1-\nu}{4br}}.$$

Here we take

$$Z = \left(\min(\hat{Q},\, N^{K/K'})\right)^{\frac{2r}{K}(1-\nu)},$$

where $K = 2r(k-1)+k(1-\varepsilon_0)2b$ and $K' = 2r(k-a)+k(1-\varepsilon_0)2b$. Then $Z \leqslant N^{1/3}$ for $k \geqslant 2$.

Hence, we get

$$\mathscr{S} \ll \bar{H}N\mathscr{L}^C(\hat{Q}^{-1}+N^{-1/2})^{\frac{1-\nu}{4br}}.$$

By our choice of $2r$ we get our Lemma 2.

5.6. *Proof of Theorem* 3. Let A be an arbitrarily large constant, let B be a sufficiently large constant and $Q = N(\log N)^{-B}$. We put $X_j = N^{\delta_j/k}$ for $j = 1, \ldots, g$ and $\mathscr{L} = \log N$. Let

$$I_1 = \sum_{\substack{1 \leqslant q \leqslant \mathscr{L}^B \\ (a,q)=1,\, 1 \leqslant a \leqslant q}} \left[\frac{a}{q}-\frac{1}{Q},\, \frac{a}{q}+\frac{1}{Q}\right] \quad \text{and} \quad I_2 = \left[-\frac{1}{Q},\, 1-\frac{1}{Q}\right) - I_1.$$

We put

$$T(\alpha) = \sum_{p_j \leqslant X_j} \log p_1 \ldots \log p_g\, e\left((p_1 \ldots p_g)^k \alpha\right).$$

We shall estimate

$$r(N) = \int_0^1 T^s(\alpha)e(-N\alpha)\,d\alpha = \Big(\int_{I_1} + \int_{I_2}\Big) T^s(\alpha)e(-N\alpha)\,d\alpha = r_1(N)+r_2(N), \text{ say.}$$

We shall estimate $r_2(N)$ first. Let $2r$ be defined as in Lemma 3 and suppose that $s \geqslant 2r+1$. We have

$$r_2(N) \ll \left(\max_{\alpha \in I_2} T(\alpha)\right)^{s-2r} \int_0^1 |T(\alpha)|^{2r}\,d\alpha.$$

If $\alpha \in I_2$, then there exist integers a and q such that

$$|\alpha - a/q| \leqslant 1/qQ, \quad \mathscr{L}^B < q \leqslant Q, \quad (a, q) = 1 \quad \text{and} \quad 0 < a \leqslant q.$$

Hence, by Lemma 2, we get

$$\max_{\alpha\in I_2} |T(\alpha)| \ll N^{1/k}(\log N)^{-A}.$$

The last integral is

$$\ll \mathscr{L}^C \sum_{\substack{n_i\leqslant N^{1/k}\\ n_1^k+\ldots+n_r^k=n_{r+1}^k+\ldots+n_{2r}^k}} c(n_1)\ldots c(n_{2r}) \ll \mathscr{L}^C \sum_{\substack{n_i\leqslant N^{1/k}\\ n_1^k+\ldots+n_r^k=n_{r+1}^k+\ldots+n_{2r}^k}} 1$$

$$= \mathscr{L}^C \int_0^1 \Big|\sum_{n\leqslant N^{1/k}} e(\alpha n^k)\Big|^{2r} d\alpha \ll N^{(2r-k)/k}\mathscr{L}^C,$$

where

$$c(n) \equiv \sum_{p_j\leqslant X_j,\, n=p_1\ldots p_g} 1 \ll 1.$$

Hence, we get

$$r_2(N) \ll N^{\frac{s-2r}{k}}(\log N)^{-A} N^{\frac{2r}{k}-1}(\log N)^C \ll N^{\frac{s}{k}-1}(\log N)^{-A}.$$

Next we shall treat $r_1(N)$. Suppose that $\alpha\in I_1$, $\alpha = a/q+\beta$, $1\leqslant q\leqslant \mathscr{L}^B$, $(a, q) = 1$ and $|\beta| \leqslant 1/Q$.

We shall prove the following

LEMMA 9. *Let k and g be integers $\geqslant 1$. Then we have*

$$T(\alpha) = \frac{S_{a,q}}{\varphi(q)}\cdot\frac{1}{k^g(g-1)!}\sum_{h\leqslant N} h^{1/k-1}\left(\log\frac{N}{h}\right)^{g-1} + O\left(N^{1/k}(\log N)^{-A}\right),$$

where $(\log N/h)^{g-1}$ is replaced by 1 for $g=1$ and we put

$$S_{a,q} = \sum_{b=1,(b,q)=1}^{q} e\left(b^k\frac{a}{q}\right).$$

Proof. We suppose that $g\geqslant 2$.

$$T(\alpha) = \sum_{h\leqslant N-1} A(h)\big(e(h\beta)-e((h+1)\beta)\big)+A(N)\,e(N\beta),$$

where we put

$$A(h) = \sum_{p_1\ldots p_g\leqslant h^{1/k}}' \log p_1\ldots\log p_g\, e\left((p_1\ldots p_g)^k\frac{a}{q}\right),$$

and the dash indicates that $p_j\leqslant X_j$ for $j = 1, \ldots, g$.

Now,

$$A(h) = \sum_{b=1,(b,q)=1}^{q} e\left(b^k \frac{a}{q}\right) \sum_{\substack{p_1 \dots p_g \leqslant h^{1/k} \\ p_1 \dots p_g \equiv b(q)}}' \log p_1 \dots \log p_g + O\left(N^{1/k} (\log N)^{-A}\right)$$

$$= \frac{1}{\varphi(q)} \sum_{b=1,(b,q)=1}^{q} e\left(b^k \frac{a}{q}\right) \bar{\chi}(b) \sum_{p_1 \dots p_g \leqslant h^{1/k}}' \log p_1 \dots \log p_g \chi(p_1 \dots p_g) + $$

$$+ O\left(N^{1/k} (\log N)^{-A}\right)$$

$$= \frac{S_{a,q}}{\varphi(q)} \sum_{p_1 \dots p_g \leqslant h^{1/k}}' \log p_1 \dots \log p_g +$$

$$+ O\left(\frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \left| \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(b^k \frac{a}{q}\right) \bar{\chi}(b) \right| \left| \sum_{p_1 \dots p_g \leqslant h^{1/k}}' \log p_1 \dots \right.\right.$$

$$\left.\left. \dots \log p_g \chi(p_1 \dots p_g) \right| \right) + O\left(N^{1/k} (\log N)^{-A}\right)$$

$$= M_1 + M_2 + O\left(N^{1/k} (\log N)^{-A}\right), \text{ say,}$$

where χ runs over Dirichlet characters $\bmod q$ and χ_0 is the principal character $\bmod q$.

By p. 46 of [14], we get for $h \leqslant N-1$

$$M_1 = \frac{S_{a,q}}{\varphi(q)} h^{1/k} \sum_{\mu=0}^{k-1} \frac{1}{\mu!} \left(\frac{1}{k} \log \frac{N}{h}\right)^{\mu} + O\left(N^{1/k} (\log N)^{-A}\right).$$

For $h = N$,

$$M_1 = \frac{S_{a,q}}{\varphi(q)} N^{1/k} + O\left(N^{1/k} (\log N)^{-A}\right).$$

As in p. 45 of [14], we get for $h \leqslant N$

$$M_2 \ll q h^{1/2k} (X_1 \dots X_g)^{1/2} (\log N)^{-A} \ll N^{1/k} (\log N)^{-A}.$$

Thus we get

$$A(h) = \frac{S_{a,q}}{\varphi(q)} h^{1/k} \sum_{\mu=0}^{g-1} \frac{1}{\mu!} \left(\frac{1}{k} \log \frac{N}{h}\right)^{\mu} + O\left(N^{1/k} (\log N)^{-A}\right) \quad \text{for} \quad h \leqslant N-1$$

and

$$A(h) = \frac{S_{a,q}}{\varphi(q)} N^{1/k} + O\left(N^{1/k} (\log N)^{-A}\right).$$

Thus we get

$$T(\alpha) = \frac{S_{a,q}}{\varphi(q)} \sum_{h \leqslant N-1} \left(h^{1/k} \sum_{\mu=0}^{g-1} \frac{1}{\mu!} \left(\frac{1}{k} \log \frac{N}{h} \right)^{\mu} \right) \left(e(h\beta) - e((h+1)\beta) \right) +$$

$$+ O\left(N^{1/k} (\log N)^{-A} N |1 - e(\beta)| \right) + \frac{S_{a,q}}{\varphi(q)} N^{1/k} e(N\beta) + O\left(N^{1/k} (\log N)^{-A} \right)$$

$$= \frac{S_{a,q}}{\varphi(q)} \sum_{h \leqslant N} \left(h^{1/k} \sum_{\mu=1}^{g-1} \frac{1}{\mu!} \left(\frac{1}{k} \log \frac{N}{h} \right)^{\mu} - \right.$$

$$\left. - (h-1)^{1/k} \sum_{\mu=1}^{g-1} \frac{1}{\mu!} \left(\frac{1}{k} \log \frac{N}{h-1} \right)^{\mu} \right) e(h\beta) +$$

$$+ \frac{S_{a,q}}{\varphi(q)} \sum_{h \leqslant N} \left(h^{1/k} - (h-1)^{1/k} \right) e(h\beta) + O\left(N^{1/k} (\log N)^{-A} \right)$$

$$= \frac{S_{a,q}}{\varphi(q)} \cdot \frac{1}{k^{g}(g-1)!} \sum_{h \leqslant N} h^{1/k-1} \left(\log \frac{N}{h} \right)^{g-1} e(h\beta) + O\left(N^{1/k} (\log N)^{-A} \right).$$

Thus we get our conclusion for $g \geqslant 2$. With obvious modifications we get our conclusion for $g = 1$. ∎

Now, with obvious modification for $g = 1$,

$$r_1(N) = \sum_{q \leqslant \mathscr{L}^B} \sum_{a=1,(a,q)=1}^{q} e\left(-\frac{a}{q} N \right) \left(\frac{S_{a,q}}{\varphi(q)} \right)^{s} \left(\frac{1}{k^{g}(g-1)!} \right)^{s} \times$$

$$\times \int_{-1/Q}^{1/Q} \left(\sum_{h \leqslant N} h^{1/k-1} \left(\log \frac{N}{h} \right)^{g-1} e(h\beta) \right)^{s} e(-N\beta)\, d\beta +$$

$$+ O\Big(\sum_{q \leqslant \mathscr{L}^B} \sum_{a=1,(a,q)=1}^{q} N^{s/k} (\log N)^{-A} Q^{-1} \Big)$$

$$= \left(\frac{1}{k^{g}(g-1)!} \right)^{s} \sum_{q \leqslant \mathscr{L}^B} \frac{1}{\varphi(q)^{s}} \sum_{a=1,(a,q)=1}^{q} S_{a,q}^{s} e\left(-\frac{a}{q} N \right) \times$$

$$\times \sum_{h_1 + \ldots + h_s = N} (h_1 \ldots h_s)^{1/k-1} \left(\log \frac{N}{h_1} \ldots \log \frac{N}{h_s} \right)^{g-1} + O\left(N^{s/k-1} (\log N)^{-A} \right)$$

$$= \left(\frac{1}{k^{g}(g-1)!} \right)^{s} \mathfrak{S}(N) M(N) + O\left(N^{s/k-1} (\log N)^{-A} \right),$$

where we put

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{1}{\varphi(q)^{s}} \sum_{a=1,(a,q)=1}^{q} S_{a,q}^{s} e\left(-\frac{a}{q} N \right)$$

and

$$M(N) = \sum_{h_1+\ldots+h_s=N} (h_1\ldots h_s)^{1/k-1}\left(\log\frac{N}{h_1}\ldots\log\frac{N}{h_s}\right)^{g-1}.$$

Thus we get

$$r(N) = \left(\frac{1}{k^g(g-1)!}\right)^s \mathfrak{S}(N)M(N)+O\left(N^{s/k-1}(\log N)^{-A}\right).$$

Here we remark, by Lemma 8.12 of Hua [25], that if $s \geqslant 3k$,

$$\mathfrak{S}(N) \geqslant C > 0$$

and that, if $k \neq p^\theta\frac{p-1}{2}$, this is valid for $s \geqslant 2k$ and, if $k = p^\theta\frac{p-1}{2}$, this is valid with $N \equiv \pm s, \pm(s-2), \ldots, \pm(s-2[\frac{1}{2}s])(\bmod p^\gamma)$ for any s, where θ and γ are the same as in the introduction. Since $M(N) \gg N^{s/k-1}$, we get $r(N) > CN^{s/k-1}$ under the above circumstances and get our Theorem 3.

6. Some other problems related to our problems

We shall mention as supplements to our problems and results, some related results, which may suggest future studies in the additive theory of numbers. We only state the results without mentioning the methods (e.g., the Rosser–Iwaniec sieve method [26] for 6.6 below) or conjectures (e.g., Car's problem for 6.1 below (cf. [23])). We remark again that there are many other results and problems, some of which can be seen in Bredihin's article [3].

6.1. Heath-Brown [23] has shown that, if $\theta > 3/4$, then there exist $n_0 = n_0(\theta)$ and $\delta = \delta(\theta) > 0$ such that every $n > n_0$ is representable as $n = p+ab$ with p prime and integers a and b satisfying $1 \leqslant a, b \leqslant n^{1/2-\delta}$ and $ab \leqslant n^\theta$.

6.2. Fouvry (cf. Cor. 1 and Cor. 3 of [10-I] and Cor. 1 of [10-II]) has shown that if N is sufficiently large, the equation $2N = p_1p_2+P_2$ with the conditions $p_1 \sim N^{10/21}$, $p_2 \sim N^{11/21}$ and $p_1 \equiv 1 (\bmod 100)$ is solvable, where p_1 and p_2 are primes and P_2 has at most two prime factors; that, if N is sufficiently large, the equation $2N = p_1p_2p_3+P_2$ with the conditions $p_1 \sim N^{1/3}$, $p_2 \sim N^{7/15}$ and $p_3 \sim N^{1/5}$ is solvable, where p_j's are primes; and also that, if N is sufficiently large, the equation $2N = p_1p_2+P_2$ with the conditions $p_1 \sim N^{1/19}$, $p_2 \sim N^{18/19}$ and $p_2 \equiv 1 (\bmod 1000)$ is solvable, where p_1 and p_2 are primes.

6.3. The author has shown in [12] that, if N is a sufficiently large even integer, then

$$|\{p \leqslant N;\, N-p = p_1 p_2,\, N^{1/\alpha} < p_1 \leqslant p_2 \leqslant N^{1/2}\}| \leqslant (16+\varepsilon) NC_N (\log N)^{-3} \quad \text{for any } \alpha > 2$$

and

$$|\{p \leqslant N;\, N-p = p_1 p_2,\, N^{1/\alpha} < p_1 \leqslant N^{1/\beta} < p_2\}| \leqslant 8(1+\varepsilon)\log((\alpha-1)/(\beta-1)) NC_N (\log N)^{-2} \quad \text{for any } \alpha \geqslant \beta \geqslant 2$$

where p's are primes, ε is any small positive number and

$$C_N = \prod_{\substack{p|N\\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

He has also shown that, if we assume the Halberstam–Richert's type of conjecture, then 8 in the last inequality may be replaced by 4 and

$$|\{p \leqslant N;\, N-p \text{ is a prime or } N-p = p_1 p_2 \text{ with primes } p_1 \text{ and } p_2 \text{ satisfying } p_1 \notin (1, N^{1/2-\varepsilon}] \text{ and } p_1 \leqslant p_2\}| \geqslant ANC_N (\log N)^{-2},$$

where p runs over the primes, ε is any positive small number and A is some positive constant.

6.4. After Statlevicius (cf. [230] of References of Bredihin [3]), Haselgrove and Pan Cheng-tung, Chen [7] has shown that, if N is a large odd integer, then $N = p_1 + p_2 + p_3$ with primes p_i's satisfying $p_i = \frac{1}{3}N + O(N^{(2/3)+\varepsilon})$, where ε is any positive small number.

6.5. In [5], Bredihin and Linnik have proposed to express N as $N = p + m^k p_1 + n^k p_2$, where k is a given integer $\geqslant 1$ p's are primes, m and n are natural numbers and $m^k \leqslant N^\delta$, $p_1 \leqslant N^{1-\delta}$, $n^k \leqslant N^{\delta'}$ and $p_2 \leqslant N^{1-\delta'}$ with given positive numbers δ and $\delta' < 1$.

Bredihin [4] has solved this with primes m and n if $0 < \delta = \delta' < 1/4$ and $k > 1$.

Our method of the proof of Theorem 2 in Section 3 solves this with primes m and n if $k = 1$ and $0 < \delta, \delta' < 1$. Our method of the proof of Theorem 3 in Section 4 solves the equation

$$N = p_{1,1} \cdots p_{1,g} + p_{2,1} \cdots p_{2,g'} + p_3^k p_4$$

if k is an integer $\geqslant 1$, g and g' are integers $\geqslant 1$, p's are primes $p_3^k \leqslant N^\delta$, $p_4 \leqslant N^{1-\delta}$, $p_{1,j} \leqslant N^{\delta_{1,j}}$ for $j = 1, \ldots, g$ and $p_{2,j} \leqslant N^{\delta_{2,j}}$ for $j = 1, \ldots, g'$ with given positive δ's which satisfy $\delta < 1$, $\delta_{1,1} + \ldots + \delta_{1,g} = 1$ and $\delta_{2,1} + \ldots + \delta_{2,g'} = 1$.

6.6. Greaves [20] has shown that every sufficiently large integer N which is $\not\equiv 0, 1, 5 \bmod 8$ is representable as

$$N = p_1^2 + p_2^2 + m^2 + n^2,$$

where p_1 and p_2 are odd primes and m and n are natural numbers.

References

[1] M. B. Barban, *The large sieve method and its applications in the theory of numbers*, Uspekhi Mat. Nauk 21 (1) (127) (1966), 51–102 = Russian Math. Surveys 21 (1) (1966), 49–103.

[2] E. Bombieri, *On the large sieve*, Mathematika 12 (1965), 201–225.

[3] B. M. Bredihin, *Partitions into terms with prime numbers*, Itogi Tekhniki, Algebra, Topologya, Geometrya 16 (1978), 5–33.

[4] –, *A refinement of a new method in ternary and semiternary problems with prime numbers*, Soviet Math. Dokl. 15 (4) (1974), 989–993.

[5] – and Yu. V. Linnik, *A new method in analytic number theory, Current problems of Analytic Number Theory* (in Russian), Nauka i Tekhnika, Minsk (1974), 5–22.

[6] J. Chen, *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Sc. Sinica 16 (1973), 157–176.

[7] –, *On large odd numbers as sum of three almost equal primes*, ibid. 14 (1965), 1113–1117.

[8] H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Ann Arbor Publications, Ann Arbor 1962.

[9] P. D. T. A. Elliott and H. Halberstam, *Some applications of Bombieri's theorem*, Mathematika 13 (1966), 196–203.

[10] E. Fouvry, *Repartition des suites dans les progressions arithmetiques*, I, II, to appear.

[11] E. Fouvry and H. Iwaniec, *On a theorem of Bombieri–Vinogradov type*, Mathematika 27 (1980), 135–152.

[12] A. Fujii, *A local study of some additive problems in the theory of number*, Proc. Japan Academy 52 (3) (1976), 113–115.

[13] –, *On some analogues of Titchmarsh divisor problem*, Nagoya Math. J. 64 (1976), 149–158.

[14] –, *An additive problem in the theory of numbers*, Acta Arith. 40 (1981), 41–49.

[15] –, *Some remarks on Goldbach's problem*, ibid. 32 (1976), 27–35.

[16] P. X. Gallagher, *Bombieri's meanvalue theorem*, Mathematika 15 (1968), 1–6.

[17] –, *A large sieve density estimate near $\sigma = 1$*, Invent. Math. 11 (1970), 329–339.

[18] –, *Primes and powers of* 2, ibid. 29 (1975), 125–142.

[19] A. Ghosh, *The distribution of αp^2 modulo* 1, Proc. London Math. Soc. (3) 42 (1981), 252–269.

[20] G. Greaves, *On the representation of a number in the form $x^2+y^2+p^2+q^2$ where p, q are odd primes*, Acta Arith. 29 (1976), 257–274.

[21] H. Halberstam and H. E. Richert, *Sieve Methods*, Academic Press, London, New York and San Francisco 1974.

[22] G. H. Hardy and J. E. Littlewood, *Some problems of partitio numerorum III*, Acta Math. 44 (1923), 1–70.

[23] D. R. Heath-Brown, *Representation of an integer as a prime plus a product of two small factors*, Math. Proc. Cambridge Philos. Soc. 89 (1981), 29–33.

[24] C. Hooley, *On the representation of a number as the sum of two squares and a prime*, Acta Math. 97 (1957), 189–210.

[25] L. K. Hua, *Additive theory of prime numbers*, Translations of Mathematical Monographs 13, Providence R. I. 1965.

[26] H. Iwaniec, *Primes of the type $\varphi(x, y)+A$ where φ is a quadratic form*, Acta Arith. 21 (1972), 203–234.

[27] —, *The generalized Hardy–Littlewood's problem involving a quadratic polynomial with coprime dicscriminants*, ibid. 27 (1975), 421–446.

[28] A. K. Karsiev, *The generalized problems of Titchmarsh divisors*, Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk 13 (1) (1969), 69–70.

[29] A. K. Karsiev and A. B. Sokolovskii, *The generalized problem of Titchmarsh divisors*, Mat. Zametki 3 (2) (1968), 187–194.

[30] Yu. V. Linnik, *The dispersion method in binary additive problems*, Leningrad 1961, 208 pp.

[31] A. A. Poljanskii, *An analogue of the Hardy–Littlewood equation*, Soviet Math. Dokl. 9 (3) (1968), 587–590.

[32] K. Prachar, *Primzahlverteilung*, Springer Verlag, Berlin, Göttingen, Heidelberg 1957.

[33] G. Rodriquez, *Sul problema dei divisori di Titchmarsh*, Boll. Uni. Mat. Ital. 3,20 (1965), 358–366.

[34] E. C. Titchmarsh, *A divisor problem*, Rend. Circ. Mat. Palermo, 54 (1930), 414–429.

[35] R. C. Vaughan, *Sommes trigonometriques sur les nombres premiers*, C. R. Acad. Sci. Paris, Ser. A 285 (1977), 981–983.

[36] —, *On the distribution of αp mod 1*, Mathematika 24 (1977), 136–141.

[37] —, *On the number of solutions of the equation $p = a+n_1+\ldots+n_k$ with $a < p \leqslant x$*, J. London Math. Soc. (2) 6 (1972), 43–55.

[38] I. M. Vinogradov, *A new estimate of a trigonometric sum containing primes*, Bull. Acad. Sci. USSR Ser. Mat. 2 (1938), 1–13.

[39] —, *A general distribution law for the fractional parts of a polynomial with variable running over primes*, Doklady Acad. Nauk SSSR (NS) 51 (1946), 491–492.

[40] —, *The method of trigonometrical sums in the theory of numbers*, Interscience, New York 1954.

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