Conspectus materiae tomi XLVII, fasciculi 4

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction et de l'échange

Address of the Editorial Board and of the exchange

Die Adresse der Schriftleitung und des Austausches

Адрес редакции и книгообмена

ACTA ARITHMETICA

ul. Śniadeckich 8, 00-950 Warszawa

Les auteurs sont priés d'envoyer leurs manuscrits en deux exemplaires
The authors are requested to submit papers in two copies
Die Autoren sind gebeten um Zusendung von 2 Exemplaren jeder Arbeit
Рукописи статей редакция просит предлагать в друх экземплярах

ISBN 83-01-07132-X ISSN 0065-1036

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

An asymptotic formula for *B*-twins

by

G. Bantle (Ulm)

1. Introduction and statement of the results. Let $r(n)$ be the number of representations of n as a sum of two squares and $r_1(n)$ be 1 or 0 according as to whether n be a sum of two squares or not.

The pairs $(n, n+1)$ are said to be *B-twins* if $r_1(n) = r_1(n+1) = 1$. An old result of T. Estermann [4] states that

$$\sum_{n\leqslant x} r(n)r(n+1) = 8x+O(x^{5/6+\varepsilon}). \tag{1.1}$$

A similar result for $r_1(n)r_1(n+1)$ is not known.

In 1965 Rieger [8] proved with the one-dimensional sieve the upper bound

$$\sum_{n\leqslant x} r_1(n)r_1(n+1) \ll x/\log x, \tag{1.2}$$

whereas the analoguous lower bound was independently proved in 1974 by C. Hooley [5] with the aid of formula (1.1) and by Indlekofer [6] with the sieve method.

P. J. Kelly [7] transformed in 1978 the method of Hooley to short intervals and showed that

$$\sum_{x-y<n\leqslant x} r_1(n)r_1(n+1) \gg y/\log x \quad \text{for} \quad y \geqslant x^{5/6+\varepsilon}.$$

In this paper we improve upon (1.1) and (1.2) by

Theorem I. *Let λ be composed entirely of prime factors $p \equiv 1(4)$ and let $\omega(\lambda)$ be the number of distinct prime factors of λ; then, for $\varepsilon > 0$,*

$$\sum_{\substack{n\leqslant x\\ n(n+1)\equiv 0(\lambda)}} r(n)r(n+1) = \frac{8\cdot 2^{\omega(\lambda)}\psi(\lambda)\bar{\omega}(\lambda)}{\lambda}x+O(\lambda^{5/4}x^{2/3+\varepsilon})$$

where

$$\bar{\omega}(\lambda) = \prod_{p|\lambda}\left(1+\frac{1}{p}\right)^{-1} \quad \textit{and} \quad \psi(\lambda) = \prod_{p^l\|\lambda}\left(l+1-\frac{l}{p}\right).$$

and

THEOREM II.

$$\sum_{x-y<n\leqslant x} r_1(n)r_1(n+1) \gg y/\log x \quad for \quad y \geqslant x^{2/3+\varepsilon}.$$

Theorem II is with the method of Hooley a direct consequence of Theorem I. The proof of Theorem I is, apart from technical complications, closely related to the method of J. M. Deshouillers and H. Iwaniec [1] to proving the analoguous asymptotic formula for the divisor function $\tau(n)$:

$$\sum_{n\leqslant x} \tau(n)\tau(n+1) = xP(\log x)+O(x^{2/3+\varepsilon}).$$

The dependence on λ of the error term in Theorem I may be improved, but this is not important for the application to Theorem II.

Proof of Theorem I

2. Basic decomposition. The left-hand side of the proposed formula may be written as

$$\sum_{\substack{\lambda_1\lambda_3=\lambda\\(\lambda_1,\lambda_3)=1}} \sum_{\substack{n\leqslant x\\ n\equiv 0(\lambda_1)\\ n+1\equiv 0(\lambda_3)}} r(n)r(n+1). \tag{2.1}$$

Thus, the proof of Theorem I reduces to showing that

$$\begin{aligned}B(x) &:= \sum_{\substack{n\leqslant x\\ n\equiv 0(\lambda_1)\\ n+1\equiv 0(\lambda_3)}} r(n)r(n+1)\\ &= \frac{8\psi(\lambda_1)\psi(\lambda_3)\bar{\omega}(\lambda_1)\bar{\omega}(\lambda_3)}{\lambda_1\lambda_3}x+O\big((\lambda_1\lambda_3)^{5/4}x^{2/3+\varepsilon}\big).\end{aligned} \tag{2.2}$$

If $\chi(n)$ is the non-principal character mod 4, then

$$r(n) = 4\sum_{d|n}\chi(d).$$

Hence

$$\begin{aligned}B(x) &= 16\sum_{\substack{n\leqslant x\\ n\equiv 0(\lambda_1)\\ n+1\equiv 0(\lambda_3)}} \sum_{m_3|n}\chi(m_3)\sum_{m_1|n+1}\chi(m_1)\\ &= \sum_{\substack{0<m_{10},\dots,m_{40}\leqslant 4\\ m_{10}m_{20}-m_{30}m_{40}\equiv 1(4)}} \chi(m_{10}m_{30})\\ &\quad\times\sum_{j_1,\dots,j_4}\sum_{\substack{m_3m_4\leqslant x\\ m_3m_4\equiv 0(\lambda_3)}}\sum_{\substack{m_1m_2=m_3m_4+1\\ m_1m_2\equiv 0(\lambda_1)\\ m_i\equiv m_{i0}(4)}} b_{j_1}(m_1)\dots b_{j_4}(m_4),\end{aligned} \tag{2.3}$$

where $b_j(t)$ are functions of C^∞-class such that (cf. [1])

$$\begin{aligned}&\sum_j b_j(t) = 1 \quad for \quad t>0,\\ &\operatorname{supp} b_j(t) = [M_{j-1}, M_{j+1}], \quad where \quad M_j = 2^j,\\ &b_j^{(q)}(t) \ll M_j^{-q}, \quad q = 0, 1, 2, \dots\end{aligned} \tag{2.4}$$

Let (j) stand for $j_1,\dots,j_4$ and (m_0) for $m_{10},\dots,m_{40}$ and define

$$D^{\pm}_{(j)(m_0)}(\mu_1,\mu_3,z) = \sum_{\substack{m_3m_4\leqslant z\\ m_3m_4\equiv 0(\mu_3)}}\sum_{\substack{m_1m_2=m_3m_4\pm 1\\ m_1m_2\equiv 0(\mu_1)\\ m_i\equiv m_{i0}(4)}} b_{j_1}(m_1)\dots b_{j_4}(m_4), \tag{2.5}$$

where μ_1 and μ_3 are of the same type as λ_1 and λ_3, i.e. $(\mu_1,\mu_3) = 1$ and $p|\mu_i$ implies $p\equiv 1(4)$. Then the inner sum in (2.3) may be written as $D^+_{(j)(m_0)}(\lambda_1,\lambda_3,x)$.

For the estimation of the main term, we note that $M_{j_1} \ll \sqrt{x}$ and $M_{j_3} \ll \sqrt{x}$. Because of symmetry we write therefore

$$\begin{aligned}B(x) = 16\sum_{\substack{m_{10},\dots,m_{40}\bmod 4\\ m_{10}m_{20}-m_{30}m_{40}\equiv 1(4)}} &\Big\{\chi(m_{10}m_{30})\sum_{\substack{j_1\leqslant J+1\\ j_3\leqslant J+1\\ j_2,j_4}} +\chi(m_{10}m_{40})\sum_{\substack{j_1\leqslant J+1\\ j_4>J+1\\ j_2,j_3}}\\ &+\chi(m_{20}m_{30})\sum_{\substack{j_2>J+1\\ j_3\leqslant J+1\\ j_1,j_4}} +\chi(m_{20}m_{40})\sum_{\substack{j_2>J+1\\ j_4>J+1\\ j_1,j_3}}\Big\} D^+_{(j)(m_0)}(\lambda_1,\lambda_3,x),\end{aligned}$$

where J is the integer such that $2^J < \sqrt{x} \leqslant 2^{J+1}$.

In order to obtain in all sums the condition $(m_{10}, 2) = 1$, we rewrite this sum as

$$\begin{aligned}B(x) = 16\sum_{\substack{m_{10},\dots,m_{40}(4)\\ m_{10}m_{20}-m_{30}m_{40}\equiv 1(4)\\ m_{10}\equiv 1(2)}} &\Big\{\chi(m_{10}m_{30})\sum_{\substack{j_1\leqslant J+1\\ j_3\leqslant J+1\\ j_2,j_4}} +\chi(m_{10}m_{40})\sum_{\substack{j_1\leqslant J+1\\ j_4>J+1\\ j_2,j_3}}\\ &+\chi(m_{20}m_{40})\sum_{\substack{j_2>J+1\\ j_4>J+1\\ j_1,j_3}}\Big\} D^+_{(j)(m_0)}(\lambda_1,\lambda_3,x)\\ +16\sum_{\substack{m_{10},\dots,m_{40}(4)\\ m_{10}m_{20}-m_{30}m_{40}\equiv -1(4)\\ m_{10}\equiv 1(2)}} &\Big\{\chi(m_{10}m_{40})\sum_{\substack{j_1\leqslant J+1\\ j_4>J+1\\ j_2,j_3}}\\ &+\chi(m_{20}m_{40})\sum_{\substack{j_2>J+1\\ j_4>J+1\\ j_1,j_3}}\Big\} D^-_{(j)(m_0)}(\lambda_3,\lambda_1,x+1).\end{aligned} \tag{2.6}$$

Next we have to transform $D^{\pm}_{(j)(m_0)}(\mu_1, \mu_3, z)$ with Poisson's summation formula in a manner similar to [1].

First we omit the summation on m_2; by (2.5) it follows that

$$(2.7)\quad D^{\pm}_{(j)(m_0)}(\mu_1, \mu_3, z) = \sum_{\substack{m_1, m_3 \\ ([m_1,\mu_1],[m_3,\mu_3])=1 \\ m_i \equiv m_{i0}(4)}} b_{j_1}(m_1) b_{j_3}(m_3) \sum_{\substack{m_4 \leqslant z/m_3 \\ m_4 \equiv m_{40}(4),\, m_4 \equiv 0(\mu_3/(m_3,\mu_3)) \\ m_3 m_4 \equiv \mp 1([m_1,\mu_1])}} b_{j_2}\left(\frac{m_3 m_4}{m_1}\right) b_{j_4}(m_4) + O(\log^2 x).$$

Let $\overline{\left(\frac{\mu_3}{(m_3, \mu_3)}\right)}$, $\bar{4}$, $\overline{[m_1, \mu_1]}$, $\overline{[m_3, \mu_3]}$ be solutions of the congruences

$$(2.8)\quad \begin{aligned} &\frac{\mu_3}{(m_3, \mu_3)} x \equiv 1(4), \\ &4x \equiv 1([m_1, \mu_1]), \\ &[m_1, \mu_1]\, x \equiv 1(4), \\ &[m_3, \mu_3]\, x \equiv 1([m_1, \mu_1]), \quad \text{respectively,} \end{aligned}$$

then the congruence conditions for m_4 in (2.7) are equivalent to the congruence

$$m_4 \equiv q \bmod \left(4 \frac{\mu_3}{(m_3, \mu_3)} [m_1, \mu_1]\right),$$

where

$$q = m_{40} \frac{\mu_3}{(m_3, \mu_3)} \overline{\left(\frac{\mu_3}{(m_3, \mu_3)}\right)} [m_1, \mu_1] \overline{[m_1, \mu_1]} \mp 4\bar{4} \frac{\mu_3}{(m_3, \mu_3)} \overline{[m_3, \mu_3]}.$$

We are now in a position to apply Poisson's summation formula (cf. [1], Lemma 1) to the innermost sum in (2.7), obtaining:

LEMMA 1.

$$D^{\pm}_{(j)(m_0)}(\mu_1, \mu_3, z) = \sum_{\substack{m_1, m_3 \\ ([m_1,\mu_1],[m_3,\mu_3])=1 \\ m_i \equiv m_{i0}(4)}} b_{j_1}(m_1) b_{j_3}(m_3)$$
$$\times \Bigg[\frac{1}{4[m_1, \mu_1][m_3, \mu_3]} \int_0^z b_{j_2}\left(\frac{t}{m_1}\right) b_{j_4}\left(\frac{t}{m_3}\right) dt$$
$$+ \sum_{h \neq 0} \frac{1}{2\pi i h} e\left(\pm h \frac{\bar{4}\,\overline{[m_3, \mu_3]}}{[m_1, \mu_1]} - h \frac{m_1 m_{40}}{4}\right)$$
$$\times \Bigg\{ b_{j_2}\left(\frac{z}{m_1}\right) b_{j_4}\left(\frac{z}{m_3}\right) e\left(\frac{hz}{4[m_1, \mu_1][m_3, \mu_3]}\right)$$

$$- \int_0^z \left(b_{j_2}\left(\frac{t}{m_1}\right) b_{j_4}\left(\frac{t}{m_3}\right)\right)' e\left(\frac{ht}{4[m_1, \mu_1][m_3, \mu_3]}\right) dt \Bigg\} \Bigg]$$
$$+ O(\log^2 x).$$

For the insertion of this formula in (2.6) we need the following:

LEMMA 2. *Let $(m_1, 2) = 1$ and let $G(1, \chi)$ be the Gaussian sum to the non-principal character $\chi \bmod 4$, then*

(i)
$$\sum_{\substack{m_{20}, m_{40} \bmod 4 \\ m_1 m_{20} - m_3 m_{40} \equiv \pm 1(4)}} \chi(m_{40}) = \sum_{\substack{m_{20}, m_{40} \bmod 4 \\ m_1 m_{20} - m_3 m_{40} \equiv \pm 1(4)}} \chi(m_{20} m_{40})$$
$$= \sum_{\substack{m_{20}, m_{40} \bmod 4 \\ m_1 m_{20} - m_3 m_{40} \equiv \pm 1(4)}} e\left(-h \frac{m_1 m_{40}}{4}\right) = 0,$$

(ii)
$$\sum_{\substack{m_{20}, m_{40} \bmod 4 \\ m_1 m_{20} - m_3 m_{40} \equiv \pm 1(4)}} \chi(m_1 m_{40}) e\left(-h \frac{m_1 m_{40}}{4}\right) = \chi(-h) G(1, \chi),$$

(iii)
$$\sum_{\substack{m_{20}, m_{40} \bmod 4 \\ m_1 m_{20} - m_3 m_{40} \equiv \pm 1(4)}} \chi(m_{20} m_{40}) e\left(-h \frac{m_1 m_{40}}{4}\right)$$
$$= \begin{cases} \pm \chi(-h) G(1, \chi) & \text{if } m_3 \equiv 0(4), \\ \mp \chi(-h) G(1, \chi) & \text{if } m_3 \equiv 2(4), \\ 0 & \text{otherwise.} \end{cases}$$

(2.6) together with Lemmas 1 and 2 now gives the following basic decomposition:

LEMMA 3.

$$B(x) = \mathrm{MT} + O\Big(x^{\varepsilon} \max_{\substack{j_1 \leqslant J+3 \\ j_3 \leqslant J+3 \\ j_2, j_4}} \max_{\gamma = 0, 1, \emptyset} \Big\{ |R^{+}_{(j)}(\lambda_1, \lambda_3, x, \gamma)| + \Big| \int_0^x r^{+}_{(j)}(\lambda_1, \lambda_3, t, \gamma)\, dt \Big|$$
$$+ |R^{-}_{(j)}(\lambda_3, \lambda_1, x+1, \gamma)| + \Big| \int_0^{x+1} r^{-}_{(j)}(\lambda_3, \lambda_1, t, \gamma)\, dt \Big| \Big\}\Big) + O(\log^2 x)$$

where

$$\mathrm{MT} = 16 \sum_{\substack{j_1 \leqslant J+1 \\ j_3 \leqslant J+1 \\ j_2, j_4}} \sum_{\substack{m_1, m_3 \\ ([m_1 \lambda_1],[m_3, \lambda_3])=1}} \chi(m_1 m_3) \frac{b_{j_1}(m_1) b_{j_3}(m_3)}{[m_1, \lambda_1][m_3, \lambda_3]}$$
$$\times \int_0^x b_{j_2}\left(\frac{t}{m_1}\right) b_{j_4}\left(\frac{t}{m_3}\right) dt,$$

$$R^{\pm}_{(j)}(\mu_1, \mu_3, z, \gamma) = \sum_{\substack{m_1, m_3 \\ ([m_1,\mu_1],[m_3,\mu_3])=1 \\ m_3 \equiv 2\gamma(4), m_1 \equiv 1(2)}} b_{j_1}(m_1) b_{j_2}\left(\frac{z}{m_1}\right) b_{j_3}(m_3) b_{j_4}\left(\frac{z}{m_3}\right) \times \sum_{h \neq 0} \frac{\chi(-h)}{2\pi i h} e\left(\pm h \overline{4} \frac{\overline{[m_3, \mu_3]}}{[m_1, \mu_1]} + \frac{hz}{4[m_1, \mu_1][m_3, \mu_3]}\right),$$

$$r^{\pm}_{(j)}(\mu_1, \mu_3, t, \gamma) = \sum_{\substack{m_1, m_3 \\ ([m_1,\mu_1],[m_3,\mu_3])=1 \\ m_3 \equiv 2\gamma(4), m_1 \equiv 1(2)}} b_{j_1}(m_1) b_{j_3}(m_3) \left(b_{j_2}\left(\frac{t}{m_1}\right) b_{j_4}\left(\frac{t}{m_3}\right)\right)' \times \sum_{h \neq 0} \frac{\chi(-h)}{2\pi i h} e\left(\pm h \overline{4} \frac{\overline{[m_3, \mu_3]}}{[m_1, \mu_1]} + \frac{ht}{4[m_1, \mu_1][m_3, \mu_3]}\right).$$

$\gamma = \emptyset$ means that the congruence condition $m_3 \equiv 2\gamma(4)$ is omitted.

The main term MT will be evaluated in the next section. From the remainder terms we consider in detail $R^{+}_{(j)}(\lambda_1, \lambda_3, x, 1)$, denoted for simplicity by $R_{(j)}(\lambda_1, \lambda_3, x)$. The other terms behave similarly and can be treated in much the same way.

We write henceforth $b_1, \ldots, b_4$ and $M_1, \ldots, M_4$ for $b_{j_1}, \ldots, b_{j_4}$ and $M_{j_1}, \ldots, M_{j_4}$, respectively.

By the choice of J, we know that $M_1 \ll \sqrt{x}$ and $M_3 \ll \sqrt{x}$.

3. The main term. Let $\alpha(t) = \sum_{0 \leq j \leq J+1} b_j(t)$; then, because of (2.4),

$$\mathrm{MT} = \frac{16x}{\lambda_1 \lambda_3} \sum_v \mu(v) \sum_{\substack{m_1, m_3 \\ [m_1, \lambda_1] \equiv 0(v) \\ [m_3, \lambda_3] \equiv 0(v)}} \frac{\chi(m_1)\alpha(m_1)(m_1, \lambda_1)}{m_1} \frac{\chi(m_3)\alpha(m_3)(m_3, \lambda_3)}{m_3}$$

$$= \frac{16x}{\lambda_1 \lambda_3} \sum_v \mu(v) \sum_{\substack{m_1 \\ \frac{m_1}{(m_1, \lambda_1)} \equiv 0\left(\frac{v}{(\lambda_1, v)}\right)}} \frac{\chi(m_1)\alpha(m_1)(m_1, \lambda_1)}{m_1} \times \sum_{\substack{m_3 \\ \frac{m_3}{(m_3, \lambda_3)} \equiv 0\left(\frac{v}{(\lambda_3, v)}\right)}} \frac{\chi(m_3)\alpha(m_3)(m_3, \lambda_3)}{m_3}.$$

For the summation over m_3 we obtain with squarefree v:

$$\sum_{\substack{m_3 \\ \frac{m_3}{(m_3, \lambda_3)} \equiv 0\left(\frac{v}{(\lambda_3, v)}\right)}} \frac{\chi(m_3)\alpha(m_3)(m_3, \lambda_3)}{m_3} = \sum_{k | \lambda_3} k \sum_{\substack{(m_3, \lambda_3) = k \\ m_3 \equiv 0\left(\frac{vk}{(\lambda_3, v)}\right)}} \frac{\chi(m_3)\alpha(m_3)}{m_3}$$

$$= \sum_{\substack{k, l \\ kl | \lambda_3}} k\mu(l) \sum_{m_3 \equiv 0\left(\frac{vkl}{(\lambda_3, v)}\right)} \frac{\chi(m_3)\alpha(m_3)}{m_3}$$

$$= \frac{\chi(v)(\lambda_3, v)}{v} \sum_{kl | \lambda_3} \frac{\mu(l)}{l} \sum_m \frac{\chi(m)}{m} \alpha\left(m \frac{vkl}{(\lambda_3, v)}\right).$$

Since

$$\sum_m \frac{\chi(m)}{m} \alpha\left(m \frac{vkl}{(\lambda_3, v)}\right) = \frac{\pi}{4} + O\left(\frac{vkl}{(\lambda_3, v)} \frac{1}{\sqrt{x}}\right)$$

and

$$\sum_{l | \lambda_3} \frac{\mu(l)}{l} \tau\left(\frac{\lambda_3}{l}\right) = \prod_{p^\alpha \| \lambda_3} \left(\tau(p^\alpha) - \frac{1}{p} \tau(p^{\alpha-1})\right) = \psi(\lambda_3)$$

we finally obtain

$$\sum_{\substack{m_3 \\ \cdots}} = \frac{\pi}{4} \frac{\chi(v)(\lambda_3, v)}{v} \psi(\lambda_3) + O\left(\frac{\lambda_3 x^\varepsilon}{\sqrt{x}}\right).$$

Hence

$$\mathrm{MT} = \pi^2 x \frac{\psi(\lambda_1)}{\lambda_1} \frac{\psi(\lambda_3)}{\lambda_3} \sum_v \mu(v) \frac{\chi^2(v)(\lambda_1, v)(\lambda_3, v)}{v^2} + O(\lambda_1 \lambda_3 x^{1/2 + 2\varepsilon}).$$

Since $\mu(v)\frac{(\lambda_1, v)(\lambda_3, v)}{v^2}$ is multiplicative, we have

$$\sum_{v=1}^{\infty} \mu(v) \frac{\chi^2(v)(\lambda_1, v)(\lambda_3, v)}{v^2} = \prod_{p > 2} \left(1 - \frac{(\lambda_1, p)(\lambda_3, p)}{p^2}\right)$$

$$= \prod_{p | \lambda_1 \lambda_3} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)^{-1} \frac{4}{3} \prod_p \left(1 - \frac{1}{p^2}\right)$$

$$= \frac{8}{\pi^2} \bar{\omega}(\lambda_1) \bar{\omega}(\lambda_3)$$

and therefore

$$\mathrm{MT} = 8x \frac{\psi(\lambda_1)\bar{\omega}(\lambda_1)}{\lambda_1} \frac{\psi(\lambda_3)\bar{\omega}(\lambda_3)}{\lambda_3} + O(\lambda_1 \lambda_3 x^{1/2 + 2\varepsilon}). \tag{3.1}$$

4. Transformation of $R_{(j)}(\lambda_1, \lambda_3, x)$. Like Deshouillers and Iwaniec, we split up the sum in question into two parts

$$R_{(j)}(\lambda_1, \lambda_3, x) = 2 \operatorname{Re} R_{(j)}(\lambda_1, \lambda_3, x, H) + R^{\infty}_{(j)}(\lambda_1, \lambda_3, x, H), \tag{4.1}$$

where $R_{(j)}(\lambda_1, \lambda_3, x, H)$ is the partial sum with $0 < h \leqslant H$ and $R^{\infty}_{(j)}(\lambda_1, \lambda_3, x, h)$ is the sum with $|h| > H$.

For the last sum we obtain immediately

$$R^{\infty}_{(j)}(\lambda_1, \lambda_3, x, H)$$
$$\leqslant \sum_{\substack{m_1, m_3 \\ 0.5M_i \leqslant m_i \leqslant 2M_i \\ ([m_1,\lambda_1],[m_3,\lambda_3])=1}} \left| \sum_{|h|>H} \frac{\chi(-h)}{2\pi i h} e\left(h\bar{4}\frac{\overline{[m_3, \lambda_3]}}{[m_1, \lambda_1]} + \frac{hx}{4[m_1, \lambda_1][m_3, \lambda_3]}\right) \right|$$
$$\leqslant \sum_{\substack{m_1, m_3 \\ 0.5M_i \leqslant m_i \leqslant 2M_i \\ ([m_1,\lambda_1],[m_3,\lambda_3])=1}} \left| \sum_{|l|>H/4} \frac{1}{2\pi i l} e\left(l\left(\frac{\overline{[m_3, \lambda_3]}}{[m_1, \lambda_1]} + \frac{x}{[m_1, \lambda_1][m_3, \lambda_3]}\right)\right) \right|$$
$$+ O\left(\frac{M_1 M_3}{H}\right).$$

Defining $\tilde{m}_i = [m_i, \lambda_i]$, $l_i = (m_i, \lambda_i)$, $\tilde{M}_i = M_i \cdot (\lambda_i / l_i)$ and $\tilde{H} = H/4$, gives

$$R^{\infty}_{(j)}(\lambda_1, \lambda_3, x, H) \ll \sum_{\substack{l_1|\lambda_1 \\ l_3|\lambda_3}} \sum_{\substack{\tilde{m}_1, \tilde{m}_3 \\ 0.5\tilde{M}_i \leqslant \tilde{m}_i \leqslant 2\tilde{M}_i \\ (\tilde{m}_1, \tilde{m}_3)=1}} \frac{1}{1 + \tilde{H}\left\|\frac{\tilde{m}_3}{\tilde{m}_1} + \frac{x}{\tilde{m}_1 \tilde{m}_3}\right\|} + \frac{M_1 M_3}{H}.$$

Now we are in the situation of [1], hence

$$R^{\infty}_{(j)}(\lambda_1, \lambda_3, x, H) \ll \left(1 + \lambda_1 \lambda_3 \frac{M_1 M_3}{H}\right) x^{\varepsilon}. \tag{4.2}$$

This allows us to assume henceforth

$$M_1 M_3 \geqslant x^{2/3}, \tag{4.3}$$

because in the opposite case, Theorem I is a direct consequence of Lemma 3, (3.1) and (4.2) with $H = 1/2$.

5. Transformation of $R_{(j)}(\lambda_1, \lambda_3, x, H)$. First we treat the summation over m_3, denoted by $\sum$, in order to apply Poisson's summation formula.

$$\sum := \sum_{\substack{m_3 \equiv 2(4) \\ ([m_1,\lambda_1],[m_3,\lambda_3])=1}} b_3(m_3) b_4\left(\frac{x}{m_3}\right)$$
$$\times e\left(h\bar{4}\frac{\overline{[m_3, \lambda_3]}}{[m_1, \lambda_1]}\right) e\left(\frac{hx}{4[m_1, \lambda_1][m_3, \lambda_3]}\right)$$
$$= \sideset{}{^*}\sum_{d \bmod [m_1,\lambda_1]} e\left(h\bar{4}\frac{\bar{d}}{[m_1, \lambda_1]}\right)$$
$$\times \sum_{\substack{m_3 \equiv 2(4) \\ [m_3,\lambda_3] \equiv d([m_1,\lambda_1])}} b_3(m_3) b_4\left(\frac{x}{m_3}\right) e\left(\frac{hx}{4[m_1, \lambda_1][m_3, \lambda_3]}\right), \tag{5.1}$$

where $d\bar{d} \equiv 1([m_1, \lambda_1])$.

The inner sum may be written as

$$\sum_{k_3 l_3 | \lambda_3} \mu(k_3) \sum_{\substack{m_3 \equiv 2(4) \\ m_3 k_3 \lambda_3 \equiv d([m_1,\lambda_1])}} b_3(m_3 k_3 l_3) b_4\left(\frac{x}{m_3 k_3 l_3}\right) e\left(\frac{hx}{4m_3 k_3 \lambda_3 [m_1, \lambda_1]}\right).$$

Since the congruence conditions in the last sum are equivalent to

$$m_3 \equiv 2[m_1, \lambda_1]\overline{[m_1, \lambda_1]} + d\overline{k_3 \lambda_3}\, 4\bar{4}\ (4[m_1, \lambda_1]),$$

where

$$k_3 \lambda_3 \overline{k_3 \lambda_3} \equiv 1\ ([m_1, \lambda_1]),$$

(cf. (2.8)), the application of Poisson's summation formula to the sum over m_3 finally gives

$$\sum = \frac{1}{4[m_1, \lambda_1]} \sum_{k_3 l_3 | \lambda_3} \frac{\mu(k_3)}{k_3 l_3} \sum_{K=-\infty}^{\infty} e\left(-\frac{K}{2}\right)$$
$$\times \sideset{}{^*}\sum_{d \bmod [m_1,\lambda_1]} e\left(h\bar{4}\frac{\bar{d}}{[m_1, \lambda_1]} - K\bar{4}\overline{k_3 \lambda_3}\frac{d}{[m_1, \lambda_1]}\right) I(\tilde{h}, \tilde{K}, [m_1, \lambda_1]) \tag{5.2}$$

where

$$I(\tilde{h}, \tilde{K}, u) = \int_0^{M_3} b_3(t) b_4\left(\frac{x}{t}\right) e\left(\frac{\tilde{h}x}{u}\frac{1}{t} + \frac{\tilde{K}}{u} t\right) dt \tag{5.3}$$

with

$$\tilde{h} = \frac{hl_3}{4\lambda_3} \quad \text{and} \quad \tilde{K} = \frac{K}{4k_3 l_3}. \tag{5.4}$$

Thus, by the definition of $\sum$ in (5.1) and by (5.2) with the usual notation for Kloosterman sums, we have

$$|R_{(j)}(\lambda_1, \lambda_3, x, H)| \tag{5.5}$$
$$\leqslant \sum_{k_3 l_3 | \lambda_3} \frac{1}{k_3 l_3} \left| \sum_{1 \leqslant h \leqslant H} \frac{\chi(-h)}{2\pi i h} \sum_{K=-\infty}^{\infty} e\left(-\frac{K}{2}\right) \sum_{(m_1, 2\lambda_3)=1} \frac{b_1(m_1) b_2(x/m_1)}{[m_1, \lambda_1]}\right.$$
$$\left. \times S(h\bar{4}\bar{4}\overline{k_3 \lambda_3}, -K, [m_1, \lambda_1]) \cdot I(\tilde{h}, \tilde{K}, [m_1, \lambda_1]) \right|$$
$$\leqslant \sum_{k_1 l_1 | \lambda_1} \frac{l_1}{\lambda_1} \sum_{k_3 l_3 | \lambda_3} \frac{1}{k_3 l_3} \left| \sum_{1 \leqslant h \leqslant H} \frac{\chi(-h)}{2\pi i h} \sum_{K=-\infty}^{\infty} e\left(-\frac{K}{2}\right)\right.$$
$$\left. \times \sum_{(m_1, 2\lambda_3)=1} \frac{b_1(m_1 k_1 l_1) b_2(x/m_1 k_1 l_1)}{m_1 k_1 l_1} S(h\overline{16 k_3 \lambda_3}, -K, \tilde{m}_1) \cdot I(\tilde{h}, \tilde{K}, \tilde{m}_1) \right|,$$

where

$$(5.6)\qquad 16k_3\lambda_3\,\overline{16k_3\lambda_3} \equiv 1\ (\tilde m_1) \quad \text{with} \quad \tilde m_1 = m_1 k_1 \lambda_1 .$$

Now we are in a position to proceed as in [1].

For the integral we obtain by partial integration q times, $q \geq 0$

$$(5.7)\qquad I(\tilde h, \tilde K, \tilde m_1) \ll M_3^{1-q}\left(\frac{\tilde m_1}{|\tilde K| + \tilde h x/M_3^2}\right)^q, \quad \text{if} \quad \frac{\tilde K M_3^2}{x\tilde h} \notin \left[\frac{1}{4}, 4\right].$$

The terms with $K = 0$ and $|K| > x^{1+\varepsilon}\frac{h}{M_3^2}\frac{\lambda_1}{l_1}k_3 l_3$ contribute therefore to $R_{(j)}(\lambda_1, \lambda_3, x, H)$ at most $O(x^{1/2+\varepsilon})$, since $M_i \ll \sqrt{x}$, $i = 1, 3$.

In the remaining part, we split up the range of the variables h and K into $\ll \log^2 x$ subintervals of the type $\frac{5}{4}H_1 \leq h < \frac{7}{4}H_1$, $\frac{5}{4}K_1 \leq \pm K < \frac{7}{4}K_1$ with

$$(5.8)\qquad 1/2 \leq H_1 \leq H, \qquad 1/2 \leq K_1 \leq 2x^{1+\varepsilon}\frac{\lambda_1}{l_1}k_3 l_3 M_3^{-2} H_1,$$

obtaining

$$(5.9)\qquad R_{(j)}(\lambda_1, \lambda_3, x, H) \ll \max_{+,-}\left(\sum_{H_1, K_1} R^{\pm}_{(j)}(\lambda_1, \lambda_3, x, H_1, K_1)\right) + x^{1/2+\varepsilon},$$

where for H_1, K_1 satisfying (5.8)

$$(5.10)\qquad R^{\pm}_{(j)}(\lambda_1, \lambda_3, x, H_1, K_1)$$

$$= \frac{1}{H_1}\sum_{\substack{k_1 l_1 | \lambda_1 \\ k_3 l_3 | \lambda_3}} \frac{l_1}{\lambda_1}\frac{1}{k_3 l_3}\sum_h a_h \sum_K b_K \sum_{(m_1, 2\lambda_3) = 1} \frac{b_1(m_1 k_1 l_1)\, b_2\left(\frac{x}{m_1 k_1 l_1}\right)}{m_1 k_1 l_1}$$

$$\times S(h\overline{16k_3\lambda_3}, \mp K, \tilde m_1)\cdot I(\tilde h, \pm\tilde K, \tilde m_1),$$

with

$$(5.11)\qquad a_h = \frac{\chi(-h)}{2\pi i h} H_1\, \chi_{[(5/4)H_1, (7/4)H_1]} \quad \text{and} \quad b_K = e(-K/2)\,\chi_{[(5/4)K_1, (7/4)K_1]}$$

(cf. (5.5), $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$).

The proof of Theorem I now reduces to showing that

$$(5.12)\qquad R^{\pm}_{(j)}(\lambda_1, \lambda_3, x, H_1, K_1) \ll (\lambda_1\lambda_3)^{5/4} x^{2/3+\varepsilon}.$$

If $H_1 \geq x^{\varepsilon-1} M_1 M_3 \frac{\lambda_1}{l_1}\frac{\lambda_3}{l_3}$ and $\frac{K_1 M_3^2}{x H_1}\frac{\lambda_3}{k_3 l_3^2} \notin \left[\frac{1}{8}, 8\right]$, (5.12) follows directly from (5.7).

In the next two sections we prove (5.12) in the remaining cases, i.e.,

$$(\xi_1)\qquad H_1 < x^{\varepsilon-1} M_1 M_3 \frac{\lambda_1}{l_1}\frac{\lambda_3}{l_3}$$

and

$$(\xi_2)\qquad H_1 \geq x^{\varepsilon-1} M_1 M_3 \frac{\lambda_1}{l_1}\frac{\lambda_3}{l_3}, \qquad \frac{K_1 M_3^2}{x H_1}\frac{\lambda_3}{k_3 l_3^2} \in \left[\frac{1}{8}, 8\right].$$

6. The case (ξ_1). In this case, we have to replace Lemma 3 of [1] by Theorem 9 of [2]. For the application of Theorem 9 we choose

$$g(h, K, m_1)$$

$$= \frac{1}{M_3 x^{6\varepsilon}}\frac{M_1}{m_1 k_1 l_1} b_1(m_1 k_1 l_1)\, b_2\left(\frac{x}{m_1 k_1 l_1}\right) f_1(h)\cdot f_2(K)\cdot I(\tilde h, \tilde K, \tilde m_1),$$

where $f_1(h)$ is a function of C^∞-class with

$$\operatorname{supp} f_1(h) = [H_1, 2H_1],$$
$$f_1(h) = 1 \quad \text{for} \quad h \in [\tfrac{5}{4}H_1, \tfrac{7}{4}H_1],$$
$$f_1^{(q)}(h) \ll H_1^{-q}, \quad q = 0, 1, 2, \ldots$$

and $f_2(K)$ is of the same type as $f_1(h)$.

Since in the case (ξ_1) $I(\tilde h, \tilde K, \tilde m_1)$ is of C^∞-class with derivatives

$$\frac{\partial^{q_1+q_2+q_3}}{\partial h^{q_1}\partial K^{q_2}\partial m_1^{q_3}} I(\tilde h, \tilde K, \tilde m_1) \ll M_3 H_1^{-q_1} K_1^{-q_2}\left(\frac{M_1}{k_1 l_1}\right)^{-q_3} x^{\varepsilon(q_1+q_2+q_3)},$$

the assumptions of [2], Theorem 9, are fulfilled and if we replace there $\theta_{r,s}$ by 1/2, we deduce easily from (5.10) and (5.11)

$$R^{\pm}_{(j)}(\lambda_1, \lambda_3, x, H_1, K_1) \ll (\lambda_1\lambda_3)^{5/4} x^{5/8+8\varepsilon}$$

and therefore (5.12).

7. The case (ξ_2). If (ξ_2) holds, then

$$I(\tilde h, \tilde K, \tilde m_1) = e\left(\frac{2\sqrt{\tilde h\tilde K x}}{\tilde m_1}\right)\left(\frac{M_1 M_3^3}{x H_1}\right)^{1/2}\left(\frac{\lambda_1}{l_1}\frac{\lambda_3}{l_3}\right)^{1/2} g(\tilde h, \tilde K, \tilde m_1),$$

where g is a function of C^∞-class with derivatives

$$(7.1)\qquad \frac{\partial^{q_1+q_2+q_3}}{\partial h^{q_1}\partial K^{q_2}\partial m_1^{q_3}} g(\tilde h, \tilde K, \tilde m_1) \ll H_1^{-q_1} K_1^{-q_2}\left(\frac{M_1}{k_1 l_1}\right)^{-q_3}.$$

This is a direct consequence of [1], Lemma 4.

Hence by (5.10)

$$R^{\pm}_{(j)}(\lambda_1, \lambda_3, x, H_1, K_1) = \sum_{\substack{k_1 l_1 | \lambda_1 \\ k_3 l_3 | \lambda_3}} \left(\frac{\lambda_1}{l_1}\right)^{1/2} \frac{\lambda_3}{l_3} \frac{1}{\sqrt{k_3 l_3}} \left(\frac{M_1 M_3^3}{x H_1^3}\right)^{1/2} \sum_h a_h \sum_K b_K \tag{7.2}$$

$$\times \sum_{\substack{m_1 \\ (m_1, 16 k_3 \lambda_3) = 1}} \Phi(\tilde{h}, \tilde{K}, \tilde{m}_1)\, e\left(\frac{2\sqrt{\tilde{h}\tilde{K}x}}{\tilde{m}_1}\right) \frac{1}{4 m_1 k_1 \lambda_1 \sqrt{k_3 \lambda_3}} S(h\overline{16 k_3 \lambda_3}, -K, \tilde{m}_1)$$

with some $\Phi(\tilde{h}, \tilde{K}, \tilde{m}_1)$ satisfying (7.1).

Like Iwaniec and Deshouillers, we now represent $\Phi(\tilde{h}, \tilde{K}, \tilde{m}_1)$ as the Fourier integral

$$\Phi(\tilde{h}, \tilde{K}, \tilde{m}_1) = \iint \hat{\Phi}\left(\delta_1, \delta_2, \frac{4\pi\sqrt{\tilde{h}\tilde{K}}}{\tilde{m}_1}\right) e(\delta_1 \tilde{h} + \delta_2 \tilde{K})\, d\delta_1\, d\delta_2$$

with

$$\frac{\partial^q}{\partial t^q} \hat{\Phi}(\delta_1, \delta_2, t) \ll \frac{\tilde{H}_1}{\tilde{H}_1^2 \delta_1^2 + 1} \frac{\tilde{K}_1}{\tilde{K}_1^2 \delta_2^2 + 1} t^{-q},$$

$$q = 0, 1, \ldots, \qquad \tilde{H}_1 = H_1 \frac{l_3}{\lambda_3}, \qquad \tilde{K}_1 = \frac{K_1}{k_3 l_3}. \tag{7.3}$$

The Kloosterman sum in (7.2) may be represented as (cf. [2], (1.6)):

$$S(h\bar{r}, -K, m_1 s) = e\left(K \frac{\bar{s}}{r}\right) S_{\infty, 1/s}(h, -K, \gamma),$$

where

$$s = k_1 \lambda_1, \quad r = 16 k_3 \lambda_3, \quad \gamma = s\sqrt{r}\, m_1 = 4 k_1 \lambda_1 \sqrt{k_3 \lambda_3}\, m_1 \quad \text{and} \quad s\bar{s} \equiv 1\,(r).$$

Thus by (7.2) (for the notation of the innermost sum cf. [2], Th. I)

$$R^{\pm}_{(j)}(\lambda_1, \lambda_3, x, H_1, K_1) \tag{7.4}$$

$$\ll \sum_{\substack{k_1 l_1 | \lambda_1 \\ k_3 l_3 | \lambda_3}} \left(\frac{\lambda_1}{l_1}\right)^{1/2} \frac{\lambda_3}{l_3} \frac{1}{\sqrt{k_3 l_3}} \left(\frac{M_1 M_3^3}{x H_1^3}\right)^{1/2}$$

$$\times \left| \iint \sum_h a_h e\left(\delta_1 \frac{h l_3}{4\lambda_3}\right) \sum_K b_K e\left(\frac{\delta_2 K}{4 k_3 l_3} + K \frac{\overline{k_1 \lambda_1}}{16 k_3 \lambda_3}\right) \right.$$

$$\left. \times \sum^{r} \frac{1}{\gamma} S_{\infty, 1/(k_1 \lambda_1)}(h, -K, \gamma)\, f\left(\frac{4\pi\sqrt{hK}}{\gamma}\right) d\delta_1\, d\delta_2 \right|.$$

where

$$f\left(\frac{4\pi\sqrt{hK}}{\gamma}\right) = e\left(\frac{2\sqrt{hKx}}{\gamma}\right) \hat{\Phi}\left(\delta_1, \delta_2, \frac{4\pi\sqrt{hK}}{\gamma}\right), \tag{7.5}$$

with

$$\operatorname{supp} f(t) = [\tau, 8\tau], \qquad \tau = \frac{\pi}{2} \frac{\sqrt{H_1 K_1}}{M_1} \frac{l_1}{\lambda_1 \sqrt{k_3 \lambda_3}}. \tag{7.6}$$

We are now in a position to apply Kuznietsov's formula (cf. [2], (1.20)). The double integral in (7.4) may therefore be estimated by

$$\iint \left\{ \sum_{j=1}^{\infty} \left| \frac{\check{f}(\varkappa_j)}{\cosh \pi \varkappa_j} \right| \left| \sum_h a_h e\left(\delta_1 \frac{h l_3}{4\lambda_3}\right) \varrho_{j\infty}(h) \right| \right. \tag{7.7}$$

$$\times \left| \sum_K b_K e\left(\frac{\delta_2 K}{4 k_3 l_3} + K \frac{\overline{\lambda_1 k_1}}{16 k_3 \lambda_3}\right) \varrho_{j\frac{1}{k_1 \lambda_1}}(K) \right|$$

$$+ \sum_c \int_{-\infty}^{\infty} |\check{f}(r)| \left| \sum_h a_h e\left(\delta_1 \frac{h l_3}{4\lambda_3}\right) h^{ir} \varphi_{c\infty h}\left(\frac{1}{2} + ir\right) \right|$$

$$\left. \times \left| \sum_K b_K e\left(\frac{\delta_2 K}{4 k_3 l_3} + K \frac{\overline{\lambda_1 k_1}}{16 k_3 l_3}\right) K^{ir} \varphi_{c\frac{1}{k_1 \lambda_1} K}\left(\frac{1}{2} + ir\right) \right| dr \right\} d\delta_1\, d\delta_2$$

where $\varkappa_j$, $\varphi_{abn}(s)$ and $\check{f}(r)$ are defined in [2], (1.16) and (1.23) respectively.

Next we have to estimate $\check{f}(r)$ and $\check{f}(ir)$ for real r. Since $\varkappa_j^2 = \lambda_j - \frac{1}{4}$, where λ_j is an eigenvalue of the Laplacian, $\varkappa_j$ is real or purely imaginary, i.e., $\varkappa_j = ir$ with $0 < r < \frac{1}{2}$.

LEMMA 4. *Let τ be defined as in (7.6), $Q = \tau\sqrt{x}/2$ and let r be real. Then, for $f(t)$ given by (7.5), we have*

(i) $$\check{f}(ir) \ll \frac{\tilde{H}_1 \tilde{K}_1}{(\tilde{H}_1^2 \delta_1^2 + 1)(\tilde{K}_1^2 \delta_2^2 + 1)} \frac{1}{r^2 + Q^2} \quad \text{if} \quad 0 < r < 1/2,$$

(ii) $$\check{f}(r) \ll \frac{\tilde{H}_1 \tilde{K}_1}{(\tilde{H}_1^2 \delta_1^2 + 1)(\tilde{K}_1^2 \delta_2^2 + 1)} \begin{cases} \dfrac{1}{r^2 + 1} & \text{if} \quad Q \leqslant |r| \leqslant 8Q, \\[2ex] \dfrac{1}{Q^2 + r^2} & \text{otherwise}, \end{cases}$$

(iii) $$\check{f}(r) \ll \frac{\tilde{H}_1 \tilde{K}_1}{(\tilde{H}_1^2 \delta_1^2 + 1)(\tilde{K}_1^2 \delta_2^2 + 1)} \frac{Q\sqrt{x}}{|r|^{5/2}} \quad \text{if} \quad |r| \geqslant 1.$$

Proof. (ii) is a direct consequence of [1], Lemma 6, if we modify the definition of $\check{f}(r)$ in a suitable manner (cf. [1], (22) and [2], (1.23)).

(iii) is a direct consequence of [2], (7.4).

For the proof of (i) we recall the definition of $\check{f}(ir)$:

$$\check{f}(ir) = \frac{4}{\pi}\cos \pi r \int_{\tau}^{8\tau} K_{-2r}(t) f(t)\frac{dt}{t}$$
$$= \frac{4}{\pi}\frac{\tilde{H}_1 \tilde{K}_1}{(\tilde{H}_1^2 \delta_1^2+1)(\tilde{K}_1^2 \delta_2^2+1)}\cos \pi r \int_{\tau}^{8\tau} K_{-2r}(t)\, e\left(\frac{\sqrt{xt}}{2\pi}\right) c(t)\frac{dt}{t},$$

where

$$c(t) = \left(\frac{\tilde{H}_1 \tilde{K}_1}{(\tilde{H}_1^2 \delta_1^2+1)(\tilde{K}_1^2 \delta_2^2+1)}\right)^{-1} e\left(-\frac{\sqrt{xt}}{2\pi}\right) f(t)$$

with

$$\operatorname{supp} c(t) = [\tau, 8\tau], \tag{7.8}$$
$$\frac{\partial^q}{\partial t^q} c(t) \ll t^{-q}, \quad q \geqslant 0$$

(cf. (7.3), (7.5) and (7.6)).

We now appeal to the formula (cf. [3], p. 82):

$$K_{-2r}(t) = \int_{0}^{\infty} e^{-t\cosh\xi}\cosh(-2r\xi)\, d\xi.$$

Since by partial integration q-times (cf. (7.8))

$$\left|\int_{\tau}^{8\tau} e^{-t(\cosh\xi - i\sqrt{x})}\, c(t)\frac{dt}{t}\right| \ll \frac{1}{(\tau\sqrt{x})}q\, e^{-\tau\cosh\xi},$$

(i) follows, by choosing q large enough.

If we replace Lemma 7 and Lemma 8 of [1] by [2], Theorem 2, we deduce from (7.4), (7.7) and Lemma 4 that

$$R_{(j)}^{\pm}(\lambda_1, \lambda_3, x, H_1, K_1) \ll \sum_{\substack{k_1 l_1 | \lambda_1 \\ k_3 l_3 | \lambda_3}} \left(\frac{\lambda_1}{l_1}\right)^{1/2} \frac{\lambda_3}{l_3}\frac{1}{\sqrt{k_3 l_3}} x^{\varepsilon}\left(\frac{M_1 M_3^3}{x H_1^3}\right)^{1/2}$$
$$\times\left(\frac{(Q^2+H_1)(Q^2+K_1)}{Q^2} H_1 K_1\right)^{1/2}$$
$$\ll x^{\varepsilon}\left(\lambda_1 \lambda_3 M_1 + x\left(\frac{H}{M_1 M_3}\right)^{1/2}\right.$$
$$\left. +(\lambda_1 \lambda_3)^{1/2}(M_1 M_3)^{1/2} + (\lambda_1 \lambda_3)^{1/2}\left(\frac{x M_1}{M_3}\right)^{1/2}\right).$$

This implies (5.12) by choosing

$$H = M_1 M_3 / x^{2/3}$$

(cf. (4.3)). Theorem I now follows from (2.1), (2.2), Lemma 3, (3.1), (4.1), (4.2), (5.9) and (5.12).

Proof of Theorem II. For this proof we need as a direct consequence of Theorem I the following

LEMMA 5. *Let λ be as in Theorem* I *and squarefree and let $\sum'$ indicate a summation over values of n for which $n(n+1)$ is not divisible by the squares of primes that are congruent to* 1 mod 4. *Then for $\lambda \leqslant x^{\varepsilon}$ and $y \geqslant x^{2/3+11\varepsilon}$*

$$\sum_{\substack{x-y<n\leqslant x \\ n(n+1)\equiv 0(\lambda)}}' r(n) r(n+1) = \frac{8B_2\, 2^{\omega(\lambda)}}{\lambda Y(\lambda)} y + O(y x^{-16/\varepsilon 15}),$$

where

$$Y(\lambda) = 2^{-\omega(\lambda)}\prod_{p|\lambda}\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{2}{p}-\frac{4}{p^2}\right)$$

and

$$B_2 = \prod_{p\equiv 1(4)}\left(1-\frac{2}{p^2}\left(3-\frac{2}{p}\right)\left(1+\frac{1}{p}\right)^{-1}\right).$$

The proof is similar to that of Lemma 7 in [5].

Replacing in [5], (16) the exponent 1/16 by α, gives

$$\sum\nolimits_1 = \frac{8B_2\, 2^{\omega(\lambda)}}{\lambda Y(\lambda)} y + O(y x^{-\alpha+\varepsilon/1000}) + O(x^{2/3+9\varepsilon/4+7\alpha/2})$$

and

$$\sum\nolimits_2 = O(y x^{-\alpha/2+\varepsilon/1000}) + O(x^{1/2+\varepsilon}).$$

Lemma 5 now follows with the choice $\alpha = \frac{15}{7}\varepsilon$.

The next lemma is a summary of results of Hooley (cf. [5], [7]).

LEMMA 6. *Let $\omega_v'(n)$ be the number of distinct prime factors p of n such that $p \equiv 1(4)$ and $p \leqslant v$. The parameter v will be chosen later. Further, let δ always indicate a squarefree integer (possibly* 1) *composed entirely of prime factors $p \equiv 1(4)$, $p \leqslant v$. Then*

(i) $r_1(n) \geqslant (1/4)\cdot 2^{-\log n/\log v} r(n)\, 2^{-\omega_v'(n)}$, *if n be not divisible by the square of a prime $p \equiv 1(4)$.*

(ii) *there is a constant $a_2 < 1$, such that if $v = \xi^{a_2}$, then there exists a function $\varrho(\delta) = \varrho_{v,\xi}(\delta)$ with the properties:*

$$\varrho(\delta) = 0 \quad \textit{for } \delta > \xi \quad \textit{and} \quad \varrho(\delta) \ll 1,$$

$$\sum_{\delta}\frac{\varrho(\delta)\, 2^{\omega(\delta)}}{\delta} \gg \frac{1}{\log v},$$

$$2^{-\omega_v'(n)} \geqslant \sum_{\delta \mid n} \varrho(\delta)\, Y(\delta).$$

Theorem II is now a direct consequence of Lemma 5 and Lemma 6, if we choose herein $\xi = x^{\varepsilon}$.

References

[1] J.-M. Deshouillers and H. Iwaniec, *An additive divisor problem*, J. London Math. Soc. (2) 26 (1982), pp. 1–14.

[2] — — *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. 70 (1982), pp. 219–288.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions II*, Mc Graw Hill, New York–Toronto–London 1953.

[4] T. Estermann, *An asymptotic formula in the theory of numbers*, Proc. London Math. Soc. (2) 34 (1932), pp. 280–292.

[5] C. Hooley, *On the intervals between numbers that are sums of two squares III*, J. Reine Angew. Math. 267 (1974), pp. 207–218.

[6] K. H. Indlekofer, *Scharfe untere Abschätzung für die Anzahlfunktion der B-Zwillinge*, Acta Arith. 26 (1974), pp. 207–212.

[7] P. J. Kelly, *The number of B-twins in an interval*, Dissertation, Nottingham 1978.

[8] G. J. Rieger, *Aufeinanderfolgende Zahlen als Summe von 2 Quadraten*, Indag. Math. 27 (1965), pp. 208–220.

UNIVERSITÄT ULM
MATH. III
OBERER ESELSBERG
D-7900 ULM

Received on 3.8.1984
and in revised form on 25.6.1985 (1443)

ACTA ARITHMETICA
XLVII (1986)

Geometrische Reihen in algebraischen Zahlkörpern

von

ULRICH RAUSCH (Marburg)

Einleitung. Unter geometrischen Reihen in einem algebraischen Zahlkörper K verstehen wir Reihen der Form

$$G(z) = \sum_{\mu} \exp\{-\sum_{p=1}^{r+1} |\mu^{(p)}|\, z_p\},$$

wo μ die total positiven ganzen Zahlen von K oder, allgemeiner, die total positiven Zahlen einer gewissen Restklasse durchläuft. Die z_p sind komplexe Variable mit positivem Realteil. (Zu den Bezeichnungen vergleiche man die Zusammenstellung am Schluß dieser Einleitung.)

Diese Reihen wurden zuerst von Hecke [5] für reell-quadratische Zahlkörper eingeführt. Rademacher [9] verallgemeinerte Heckes Ergebnisse weitgehend bei seiner Übertragung des Goldbach-Problems auf beliebige algebraische Zahlkörper. Ausgehend von Rademachers Arbeit, verwandte Friedrich [2] dann geometrische Reihen zur asymptotischen Auswertung gewisser Partitionenfunktionen in Zahlkörpern. Weitere Anwendungen geometrischer Reihen auf die additive Theorie reell-quadratischer Zahlkörper findet man bei Schaal [13], [14].

Immer spielt das Verhalten der geometrischen Reihen in der Umgebung des Nullpunktes eine zentrale Rolle, und es ist das Ziel der vorliegenden Arbeit(1), die hierfür bekannten asymptotischen Entwicklungen zu verschärfen. Ein wenig erweitert wird die Problemstellung noch durch die Einführung von Koeffizienten in der Form verallgemeinerter Größencharaktere.

Außerdem untersuche ich unter demselben Gesichtspunkt die „Einheitenreihen" $E(z; U)$, die sich als natürliche Vorstufe zur geometrischen Reihe präsentieren und wie diese definiert sind mit dem Unterschied, daß der Summationsbereich für μ eine torsionsfreie Untergruppe U der

(1) Diese Arbeit ist eine gekürzte Fassung meiner Dissertation. Herrn Prof. Dr. W. Schaal, der das Thema anregte, bin ich für seine vielfältige Unterstützung aufrichtig dankbar.