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A localization theorem in the theory of diophantine approximation and an application to Pell's equation

by

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Introduction. Some thirty years ago the second of us showed (Szüsz [3]) that for any irrational number α the inequality

$$\|\alpha x\| < x^{\delta-1} \qquad (0 < \delta < 1)$$

has a solution in any interval $(n, n^{1/\delta})$ where $\|z\|$ denotes the distance from z to the nearest integer. Since, as is well known, $\|\alpha x\| < x^{-1}$ is solvable with natural numbers x, it would be interesting to give a "localization" for the numbers x satisfying $\|\alpha x\| < x^{-1}$ but it is easy to see that one can give a counterexample to any such statement. Further, one could ask for a characterization of the natural numbers x for which

$$\|\alpha x\| < K/x \tag{1}$$

holds where K is a positive constant.

Let the regular continued fraction representation of α be $[a_0; a_1, a_2, \ldots]$ and let the denominators of the convergents be $B_0 = 1, B_1, B_2, \ldots$ In Section 1 we prove

THEOREM 1. *Let $K \geqslant 1/\sqrt{5}$ and let α be an irrational number. Then all positive integers x satisfying* (1) *have the form*

$$c_{n+1} B_n + c_{n+2} B_{n+1} + \ldots + c_{n+m} B_{n+m-1} \tag{2}$$

with $m < C\log(2K+1)+3$, where C is an absolute constant and the coefficients c_{k+1} satisfy $0 \leqslant c_i < a_i$, $0 \leqslant c_{k+1} \leqslant a_{k+1}$ for $k > 0$ and if $c_{k+1} = a_{k+1}$ then $c_k = 0$.

This result extends the classical result of Legendre that $m = 1$ for $K \leqslant 1/2$ and that $m \leqslant 2$ for $K = 1$ (see Perron [2], Sections 13 and 16). In Section 2 we apply our result to Pell's equation $x^2 - dy^2 = N$. While this equation has been treated for $|N| < \sqrt{d}$ (see Perron [2] for references), we can drop this restriction and thus generalize the classical results.

1. Proof of the main theorem. We use the notations of Perron [2] for regular continued fractions:

$$\alpha = [a_0; a_1, a_2, \dots],$$
$$[a_0; a_1, \dots, a_k] = A_k/B_k \quad \text{where} \quad (A_k, B_k) = 1,$$
$$\zeta_k = [a_k; a_{k+1}, \dots].$$

We set

$$D_k = B_k\alpha - A_k = (-1)^k/(B_k\zeta_{k+1} + B_{k-1}).$$

Since $A_{k+1} = a_{k+1}A_k + A_{k-1}$ and $B_{k+1} = a_{k+1}B_k + B_{k-1}$, it follows that $D_{k+1} = a_{k+1}D_k + D_{k-1}$. Since $D_{k+1} = -D_k/\zeta_{k+2}$, the D_k's alternate in sign and their absolute values decrease monotonically to zero.

LEMMA 1.1 (Ostrowski [1]). *Every positive integer x has a unique representation as*

$$x = \sum_{k=0}^{N} c_{k+1}B_k \tag{3}$$

where $0 \leqslant c_1 < a_1$, $0 \leqslant c_{k+1} \leqslant a_{k+1}$ for $k > 0$ and if $c_{k+1} = a_{k+1}$ then $c_k = 0$.

The proof can be done by induction on x.

Let c_{n+1} be the first nonzero coefficient in the representation (3) of x so that $c_{n+1} > 0$ and $c_{k+1} = 0$ for $0 \leqslant k < n \leqslant N$.

LEMMA 1.2. *We have*

$$|(c_{n+1}-1)D_n - D_{n+1}| < \Big|\sum_{k=0}^{N} c_{k+1}D_k\Big| < |c_{n+1}D_n - D_{n+1}|.$$

Proof. Since the D_k's alternate in sign,

$$\Big|\sum_{k=0}^{N} c_{k+1}D_k\Big| > |c_{n+1}D_n + (a_{n+2}-1)D_{n+1} + a_{n+4}D_{n+3} + \dots|$$

and the lower estimate follows since $a_{k+1}D_k = D_{k+1} - D_{k-1}$. The upper estimate is obtained similarly by considering $|c_{n+1}D_n + a_{n+3}D_{n+2} + a_{n+5}D_{n+4} + \dots|$.

LEMMA 1.3. *For any integer $x > 1$ either*

$$\|\alpha x\| = \Big|\sum_{k=0}^{N} c_{k+1}D_k\Big| \tag{4}$$

or $\|\alpha x\| > D_2$.

Proof. A simple calculation shows that if $c_1 = c_2 = 0$ then the right-hand side of (4) is $< 1/2$. The exceptional cases occur when $a_1 = 1$ and $c_2 > 0$ and when $c_1 > 0$.

We now prove Theorem 1. From Lemmas 1.2 and 1.3 we see that $\|\alpha x\|$ is minimized for integers of the form

$$x = B_n + (a_{n+2}-1)B_{n+1} + a_{n+4}B_{n+3} + \dots + a_{n+2m}B_{n+2m-1} = B_{n+2m} - B_{n+1}$$

and for such numbers we have that

$$\|\alpha x\|\, x > |-D_{n+1}|\, x > (B_{n+2m} - B_{n+1})/(B_{n+2} + B_{n+1}) > (G^{2m-3}-1)/2$$

where $G = (1+\sqrt{5})/2$ since $B_{k+2m}/B_{k+2} > G^{2m-3}$ for any α. Thus if $\|\alpha x\|\, x < K$ we must have $(G^{2m-3}-1)/2 < K$ and our result follows.

We note that an upper estimate for $\|\alpha x\|\, x$ would require an upper bound for the ratios B_{k+1}/B_k and these ratios are unbounded for almost all α.

2. An application to Pell's equation. We now consider the positive integer solutions of $x^2 - dy^2 = N$ where the integer $d > 1$ is not a perfect square. For $N > 0$, we have that $\|y\sqrt{d}\|^2 + 2\sqrt{d}\|y\sqrt{d}\|\, y = N$ and so $\|y\sqrt{d}\|\, y < N/2\sqrt{d}$. If $0 < N < \sqrt{d}$ then $\|y\sqrt{d}\|\, y < 1/2$ and we obtain the classical result that the only solutions are those given by the convergents of $\sqrt{d}$ (the case $N < 0$ follows in a similar manner since the convergents of $\sqrt{d}$ and $1/\sqrt{d}$ coincide with one trivial exception). For $|N| > \sqrt{d}$, it follows that the solutions will be as described by Theorem 1 (together with the corresponding sums of A_k's).

With the notations of Perron [2] for the regular continued fraction of $\sqrt{d}$, we have

$$\zeta_0 = (\sqrt{d}+0)/1, \quad \dots, \quad \zeta_k = (\sqrt{d}+P_k)/Q_k, \quad \dots$$

where $a_kQ_k = P_k + P_{k+1}$ and $d - (P_{k+1})^2 = Q_kQ_{k+1}$. Thus the partial quotients satisfy $a_k < 2\sqrt{d}$ and we have the estimate $B_{k+1}/B_k < 1 + 2\sqrt{d}$; such an estimation does not hold in general but it does hold for quadratic surds.

From the upper estimate in Lemma 1.2 we see that $\|y\sqrt{d}\|$ is maximized for integers of the form

$$y = a_{n+1}B_n + a_{n+3}B_{n+2} + \dots + a_{n+2m+1}B_{n+2m} = B_{n+2m+1} - B_{n-1}.$$

For such numbers

$$\|y\sqrt{d}\|\, y < |a_{n+1}D_n - D_{n+1}|\, y = |-D_{n-1}|\, y < B_{n+2m+1}/B_n < (1+2\sqrt{d})^{2m+1}$$

and we have shown

THEOREM 2. *The positive integer solutions of $x^2 - dy^2 = N$ where $2K_1\sqrt{d} < |N| < 2K_2\sqrt{d}$ are given by (2) and the corresponding sums of A_k's with $C_1 \log K_1 < m < C_2 \log(2K_2+1) + 3$ where C_1 depends only on d and C_2 is an absolute constant.*

We conclude with an explicit calculation of the values represented by x^2-dy^2 for $x = A_k+cA_{k+1}$ and $y = B_k+cB_{k+1}$ where $1 \leqslant c \leqslant a_{k+2}-1$. Since $x^2-dy^2 = (y\sqrt{d}-x)(y\sqrt{d}-x)^*$ where $(z)^*$ denotes the conjugate of z, we have that for $x = A_k$ and $y = B_k$, $x^2-dy^2 = D_k(D_k)^* = (-1)^{k+1}Q_{k+1}$ (see Perron [2]). Thus for $x = A_k+cA_{k+1}$ and $y = B_k+cB_{k+1}$,

$$\begin{aligned} x^2-dy^2 &= D_k(D_k)^*(1-c/\zeta_{k+2})(1-c/(\zeta_{k+2})^*) \\ &= (-1)^{k+1}(Q_{k+1}+cQ_{k+2}(2P_{k+2}/Q_{k+2}-c)) \\ &= (-1)^{k+1}V_{k+1}(c). \end{aligned}$$

Since $2P_{k+2}/Q_{k+2} = \zeta_{k+2}+(\zeta_{k+2})^*$ we have $\zeta_{k+2} > 2P_{k+2}/Q_{k+2} > \zeta_{k+2}-1 > c$ and so $V_{k+1}(c) > Q_{k+1}$ for $1 \leqslant c \leqslant a_{k+2}-1$. The maximum of V_{k+1} occurs when $c = \|P_{k+2}/Q_{k+2}\|$ and is approximately $Q_{k+1}+(P_{k+2})^2/Q_{k+2} = d/Q_{k+2}$. Since $Q_{k+2} = 1$ at the end of each period of $\sqrt{d}$, V_{k+1} can take values as large as d.

For an integer of the form (2) with $m > 2$, we see from the recursion formula $B_{k+1} = a_{k+1}B_k+B_{k-1}$ that it can be rewritten as sB_k+tB_{k+1} where $s > 1$ and $t > 0$ are integers. Then the previous calculations may be repeated to find the value of x^2-dy^2 in terms of Q_{k+1}, Q_{k+2} and P_{k+2}.

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A generalization of Atkinson's formula to *L*-functions

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1. Introduction. Let

$$I(q, T) = \sum_{\chi \bmod q} \int_0^T |L(\tfrac{1}{2}+it, \chi)|^2\, dt \tag{1.1}$$

and define $E(q, T)$ via the identity

$$I(q, T) = \frac{\varphi^2(q)}{q} T\left(\log\frac{qT}{2\pi}+\sum_{p\mid q}\frac{\log p}{p-1}+2\gamma-1\right)+E(q, T), \tag{1.2}$$

where φ is Euler's function and γ is his constant.

Consider first the case $q = 1$. Atkinson [1] has established for $E(1, T)$ a very precise explicit expression in terms of two sums involving the divisor function $d(n)$. Recently Jutila [7] found a new interesting application of this formula by showing that it yields in a simple manner Balasubramanian's [2] estimate $E(1, T) \ll T^{1/3+\varepsilon}$, valid for any positive ε.

The more general function $E(q, T)$ has been studied by Rane [9] (in fact, he considers $E(q, T)-E(q, 1)$) who proved

$$E(q, T) \ll qT^{1/2}\log T. \tag{1.3}$$

A simpler proof of this is due to Balasubramanian and Ramachandra [3].

Our object is to generalize Atkinson's formula to $E(q, T)$ (Theorem 1). Then we deduce by Jutila's method a new inequality for $E(q, T)$ (Corollary 1). In turn, this implies immediately new mean value estimates for L-functions (Corollary 2), which can be applied to estimate the density of the zeros in small rectangles (Corollary 3).

We proceed to state the main results. Let

$$e(T, u) = \left(1+\frac{\pi u}{2T}\right)^{-1/4}\left(\left(\frac{2T}{\pi u}\right)^{1/2}\operatorname{arsinh}\left(\left(\frac{\pi u}{2T}\right)^{1/2}\right)\right)^{-1}, \tag{1.4}$$

$$f(T, u) = 2T\operatorname{arsinh}\left(\left(\frac{\pi u}{2T}\right)^{1/2}\right)+(\pi^2u^2+2\pi uT)^{1/2}-\frac{\pi}{4}, \tag{1.5}$$

$$g(T, u) = T\log\frac{T}{2\pi u}-T+2\pi u+\frac{\pi}{4}. \tag{1.6}$$