We conclude with an explicit calculation of the values represented by $x^2 - dy^2$ for $x = A_k + cA_{k+1}$ and $y = B_k + cB_{k+1}$ where $1 \leqslant c \leqslant a_{k+2} - 1$. Since $x^2 - dy^2 = (y\sqrt{d} - x)(y\sqrt{d} - x)^*$ where $(z)^*$ denotes the conjugate of z, we have that for $x = A_k$ and $y = B_k$, $x^2 - dy^2 = D_k(D_k)^* = (-1)^{k+1} Q_{k+1}$ (see Perron [2]). Thus for $x = A_k + cA_{k+1}$ and $y = B_k + cB_{k+1}$,

$$\begin{aligned} x^2 - dy^2 &= D_k(D_k)^*(1 - c/\zeta_{k+2})(1 - c/(\zeta_{k+2})^*) \\ &= (-1)^{k+1}(Q_{k+1} + cQ_{k+2}(2P_{k+2}/Q_{k+2} - c)) \\ &= (-1)^{k+1} V_{k+1}(c). \end{aligned}$$

Since $2P_{k+2}/Q_{k+2} = \zeta_{k+2} + (\zeta_{k+2})^*$ we have $\zeta_{k+2} > 2P_{k+2}/Q_{k+2} > \zeta_{k+2} - 1 > c$ and so $V_{k+1}(c) > Q_{k+1}$ for $1 \leqslant c \leqslant a_{k+2} - 1$. The maximum of V_{k+1} occurs when $c = \|P_{k+2}/Q_{k+2}\|$ and is approximately $Q_{k+1} + (P_{k+2})^2/Q_{k+2} = d/Q_{k+2}$. Since $Q_{k+2} = 1$ at the end of each period of $\sqrt{d}$, V_{k+1} can take values as large as d.

For an integer of the form (2) with $m > 2$, we see from the recursion formula $B_{k+1} = a_{k+1} B_k + B_{k-1}$ that it can be rewritten as $sB_k + tB_{k+1}$ where $s > 1$ and $t > 0$ are integers. Then the previous calculations may be repeated to find the value of $x^2 - dy^2$ in terms of Q_{k+1}, Q_{k+2} and P_{k+2}.

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A generalization of Atkinson's formula to L-functions

by

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1. Introduction. Let

$$I(q, T) = \sum_{\chi \bmod q} \int_0^T |L(\tfrac{1}{2} + it, \chi)|^2 \, dt \tag{1.1}$$

and define $E(q, T)$ via the identity

$$I(q, T) = \frac{\varphi^2(q)}{q} T \left(\log \frac{qT}{2\pi} + \sum_{p|q} \frac{\log p}{p-1} + 2\gamma - 1 \right) + E(q, T), \tag{1.2}$$

where φ is Euler's function and γ is his constant.

Consider first the case $q = 1$. Atkinson [1] has established for $E(1, T)$ a very precise explicit expression in terms of two sums involving the divisor function $d(n)$. Recently Jutila [7] found a new interesting application of this formula by showing that it yields in a simple manner Balasubramanian's [2] estimate $E(1, T) \ll T^{1/3+\varepsilon}$, valid for any positive ε.

The more general function $E(q, T)$ has been studied by Rane [9] (in fact, he considers $E(q, T) - E(q, 1)$) who proved

$$E(q, T) \ll qT^{1/2} \log T. \tag{1.3}$$

A simpler proof of this is due to Balasubramanian and Ramachandra [3].

Our object is to generalize Atkinson's formula to $E(q, T)$ (Theorem 1). Then we deduce by Jutila's method a new inequality for $E(q, T)$ (Corollary 1). In turn, this implies immediately new mean value estimates for L-functions (Corollary 2), which can be applied to estimate the density of the zeros in small rectangles (Corollary 3).

We proceed to state the main results. Let

$$e(T, u) = \left(1 + \frac{\pi u}{2T}\right)^{-1/4} \left(\left(\frac{2T}{\pi u}\right)^{1/2} \operatorname{arsinh}\left(\left(\frac{\pi u}{2T}\right)^{1/2}\right) \right)^{-1}, \tag{1.4}$$

$$f(T, u) = 2T \operatorname{arsinh}\left(\left(\frac{\pi u}{2T}\right)^{1/2}\right) + (\pi^2 u^2 + 2\pi u T)^{1/2} - \frac{\pi}{4}, \tag{1.5}$$

$$g(T, u) = T \log \frac{T}{2\pi u} - T + 2\pi u + \frac{\pi}{4}. \tag{1.6}$$

In terms of these functions define

$$\Sigma_1(k, T) = \left(\frac{2T}{\pi}\right)^{1/4} \sum_{n \le N} (-1)^{kn} d(n)(kn)^{-3/4} e(T, kn) \cos f(T, kn), \tag{1.7}$$

$$\Sigma_2(k, T) = -2 \sum_{n \le N'} d(n)(kn)^{-1/2} \left(\log \frac{kT}{2\pi n}\right)^{-1} \cos g\left(T, \frac{n}{k}\right), \tag{1.8}$$

where

$$N' = N'(k, T, N) = k\left(\frac{T}{2\pi} + \frac{kN}{2} - \left(\left(\frac{kN}{2}\right)^2 + \frac{kNT}{2\pi}\right)^{1/2}\right). \tag{1.9}$$

Let $\mu(n)$ be the Möbius function.

THEOREM 1. *If q is a positive integer, $T > 3$ and $T \ll N \ll T$, then*

$$E(q, T) = \frac{\varphi(q)}{q} \sum_{k | q} k\mu(q/k)(\Sigma_1(k, T) + \Sigma_2(k, T))$$
$$+ O\left(d(q)\varphi(q) q^{-1/2} \log^2 T\right) + O(q/T).$$

Our first corollary improves on (1.3) for all values of q and T.

COROLLARY 1. *For any $\varepsilon > 0$,*

$$E(q, T) \ll \begin{cases} (qT)^{1/3+\varepsilon} + q^{1+\varepsilon} & \text{if } q \ll T, \\ (qT)^{1/2+\varepsilon} + qT^{-1} & \text{if } T \ll q. \end{cases}$$

COROLLARY 2. *For $H \gg 1$ and any $\varepsilon > 0$,*

$$\sum_{\chi \bmod q} \int_T^{T+H} |L(\tfrac{1}{2}+it, \chi)|^2 \, dt \ll \left(qH + (qT)^{1/3}\right)\left(q(T+H)\right)^{\varepsilon}. \tag{1.10}$$

For $t > 1$ and any $\varepsilon > 0$,

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2}+it, \chi)|^2 \ll \left(q + (qt)^{1/3}\right)(qt)^{\varepsilon}. \tag{1.11}$$

In (1.10) qH dominates if $T \ll q^2 H^3$. In (1.11) q dominates if $t \ll q^2$. Note that (1.11) trivially contains the inequality $L(\frac{1}{2}+it, \chi) \ll q^{a+\varepsilon} + (qt)^{1/6+\varepsilon}$ with $a = \frac{1}{2}$. However, Heath-Brown [5] has proved a similar bound with $a = \frac{1}{4}$, which is stronger when $q \gg t^{1/2}$.

Gallagher [4] has proved an inequality of the sort (1.11) with t in place of $(qt)^{1/3}$ (and $\log qt$ in place of $(qt)^{\varepsilon}$). Hence (1.11) is an improvement if $t \gg q$. Following Gallagher, we may apply Corollary 2 to give a local density estimate. Let $N_\chi(\alpha, T)$ be the number of zeros of $L(s, \chi)$ in the rectangle $\alpha < \sigma < 1$, $|t| < T$.

COROLLARY 3. *For $H \gg 1$ and any $\varepsilon > 0$,*

$$\sum_{\chi \bmod q} \left(N_\chi(\alpha, T+H) - N_\chi(\alpha, T)\right) \ll (qH)^{1-\alpha}\left(qH + (qT)^{1/3}\right)^{2(1-\alpha)+\varepsilon}.$$

In particular, for $T \ll q^2$, we have

$$\sum_{\chi \bmod q} \left(N_\chi(\alpha, T+1) - N_\chi(\alpha, T)\right) \ll q^{3(1-\alpha)+\varepsilon}.$$

The second estimate was previously known for $T \ll q$ (Gallagher [4]). We omit the proof of Corollary 3, since it can easily be proved as indicated by Gallagher.

Theorem 1 may be regarded as an analogue of the sum formula for a certain modified divisor function, in the same way as Atkinson's formula is an analogue of Voronoi's summation formula for $\sum_{n<x} \frac{1}{2}(-1)^n d(n)$. (See Jutila [7], where the arithmetic function is given in a different form.) We deal with this question in the last section.

In the case $q = 1$ the topics of this paper are extensively discussed by Ivić [6]. In particular, Atkinson's formula is proved there.

All the constants implied by "O", "$\ll$" etc. will depend at most on ε. We always assume $T > 3$.

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2. Proof of Theorem 1. Let u and v be complex variables. For $\mathrm{Re}(u) > 1$, $\mathrm{Re}(v) > 1$, we have

$$\sum_{\chi \bmod q} L(u, \chi) L(v, \bar{\chi}) = \sum_{\chi \bmod q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi(m)\bar{\chi}(n) m^{-u} n^{-v}.$$

The terms of the double series are classified according to whether $m = n$, $m > n$ or $m < n$. Also the orthogonality of the characters is used. Hence, for $\mathrm{Re}(u) > 1$, $\mathrm{Re}(v) > 1$,

$$\sum_{\chi \bmod q} L(u, \chi) L(v, \bar{\chi}) = \varphi(q)\left(L(u+v, \chi_0) + f_q(u, v) + f_q(v, u)\right), \tag{2.1}$$

where χ_0 is the principal character modulo q and

$$f_q(u, v) = \sum_{\substack{r=1 \\ (r,q)=1}}^{\infty} \sum_{s=1}^{\infty} r^{-u}(r+qs)^{-v}.$$

This double series is convergent for $\mathrm{Re}(u+v) > 2$, $\mathrm{Re}(v) > 1$. We have

$$f_q(u, v) = \sum_{k|q} \mu(k) \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} (kr)^{-u}(kr+qs)^{-v}.$$

Now let $\mathrm{Re}(u) < -1$ and $\mathrm{Re}(u+v) > 2$. Then, by Poisson's summation formula,

$$\sum_{r=1}^{\infty}(kr)^{-u}(kr+qs)^{-v}$$

$$= \int_0^{\infty}(kx)^{-u}(kx+qs)^{-v}dx+2\sum_{m=1}^{\infty}\int_0^{\infty}(kx)^{-u}(kx+qs)^{-v}\cos(2\pi mx)\,dx$$

$$= k^{-1}(qs)^{1-u-v}\Big(\int_0^{\infty}y^{-u}(1+y)^{-v}dy+2\sum_{m=1}^{\infty}\int_0^{\infty}y^{-u}(1+y)^{-v}\cos(2\pi m(q/k)sy)\,dy\Big).$$

Summing with respect to s and k and using the identities

$$\int_0^{\infty}y^{-u}(1+y)^{-v}dy = \Gamma(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}, \qquad \sum_{k|q}\mu(k)k^{-1} = \varphi(q)q^{-1},$$

this gives

$$f_q(u,v) = \varphi(q)q^{-u-v}\zeta(u+v-1)\Gamma(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}+g_q(u,v), \tag{2.2}$$

where

$$g_q(u,v) \tag{2.3}$$

$$= 2q^{1-u-v}\sum_{k|q}\mu(k)k^{-1}\sum_{s=1}^{\infty}s^{1-u-v}\sum_{m=1}^{\infty}\int_0^{\infty}y^{-u}(1+y)^{-v}\cos(2\pi m(q/k)sy)\,dy.$$

Now (2.2) gives an analytic continuation of $f_q(u,v)-g_q(u,v)$ to a meromorphic function of u and v. Hence,

$$f_q(u,v)-g_q(u,v)+f_q(v,u)-g_q(v,u)$$

is meromorphic and is given by (2.2). This gives an expression for $f_q(u,v)+f_q(v,u)$ if this sum is analytically continuable. But, by (2.1), it is. The expression so obtained is substituted in (2.1) to give

$$\sum_{\chi \bmod q}L(u,\chi)L(v,\bar{\chi})$$

$$= \varphi(q)\bigg(L(u+v,\chi_0)+\varphi(q)q^{-u-v}\zeta(u+v-1)\Gamma(u+v-1)\Big(\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma(1-v)}{\Gamma(u)}\Big)$$
$$+g_q(u,v)+g_q(v,u)\bigg).$$

Write $u+v = 1+\delta$, $|\delta| < 1/2$. The first two terms on the right-hand side give us

$$\varphi(q)L(1+\delta,\chi_0)+\varphi^2(q)q^{-1-\delta}\zeta(\delta)\Gamma(\delta)\Big(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)}+\frac{\Gamma(u-\delta)}{\Gamma(u)}\Big).$$

We use the functional equation for $\zeta(s)$ and write

$$\gamma_q = \gamma+\sum_{p|q}\frac{\log p}{p-1}. \tag{2.4}$$

Then we obtain

$$\varphi(q)L(1+\delta,\chi_0)+\frac{\varphi^2(q)}{q}\zeta(1-\delta)\Big(\frac{q}{2\pi}\Big)^{-\delta}\frac{1}{2}\sec\Big(\frac{\pi\delta}{2}\Big)\Big(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)}+\frac{\Gamma(u-\delta)}{\Gamma(u)}\Big)$$

$$= \frac{\varphi^2(q)}{q}\bigg(\Big(\frac{1}{\delta}+\gamma_q\Big)-\Big(\frac{1}{\delta}-\gamma\Big)\Big(1-\delta\log\frac{q}{2\pi}\Big)\frac{1}{2}\Big(1-\delta\frac{\Gamma'}{\Gamma}(1-u)+1-\delta\frac{\Gamma'}{\Gamma}(u)\Big)\bigg)$$
$$+O(|\delta|)$$

$$= \frac{\varphi^2(q)}{q}\Big(\frac{1}{2}\Big(\frac{\Gamma'}{\Gamma}(1-u)+\frac{\Gamma'}{\Gamma}(u)\Big)+\gamma+\gamma_q+\log\frac{q}{2\pi}\Big)+O(|\delta|).$$

Hence, making $\delta\to 0$, we have

$$\sum_{\chi \bmod q}L(u,\chi)L(1-u,\bar{\chi}) = \frac{\varphi^2(q)}{q}\Big(\frac{1}{2}\Big(\frac{\Gamma'}{\Gamma}(1-u)+\frac{\Gamma'}{\Gamma}(u)\Big)+\gamma+\gamma_q+\log\frac{q}{2\pi}\Big)$$
$$+\varphi(q)\big(g_q(u,1-u)+g_q(1-u,u)\big).$$

Hence (see (1.1))

$$2iI(q,T) = \sum_{\chi \bmod q}\int_{1/2-iT}^{1/2+iT}L(u,\chi)L(1-u,\bar{\chi})\,du$$

$$= \frac{\varphi^2(q)}{q}\Big(\log\frac{\Gamma(\frac{1}{2}+iT)}{\Gamma(\frac{1}{2}-iT)}+2iT\Big(\gamma+\gamma_q+\log\frac{q}{2\pi}\Big)\Big)$$

$$+2\varphi(q)\int_{1/2-iT}^{1/2+iT}g_q(u,1-u)\,du.$$

By Stirling's formula this gives (1.2) with

$$E(q,T) = \frac{1}{i}\varphi(q)\int_{1/2-iT}^{1/2+iT}g_q(u,1-u)\,du+O(q/T). \tag{2.5}$$

We return to (2.3) and investigate the convergence of its right-hand side. We have, for $\mathrm{Re}(u) < 1$, $\mathrm{Re}(u+v) > 0$ and $n \geqslant 1$,

$$2\int_0^{\infty}y^{-u}(1+y)^{-v}\cos(2\pi ny)\,dy$$

$$= n^{u-1}\int_0^{\infty}y^{-u}\Big(1+\frac{y}{n}\Big)^{-v}(e^{2\pi iy}+e^{-2\pi iy})\,dy$$

$$= n^{u-1}\int_0^{i\infty} y^{-u}\left(1+\frac{y}{n}\right)^{-v} e^{2\pi i y}\,dy + n^{u-1}\int_0^{-i\infty} y^{-u}\left(1+\frac{y}{n}\right)^{-v} e^{-2\pi i y}\,dy$$

$$\ll \left|\frac{n^{u-1}}{u-1}\right|$$

uniformly for bounded u and v. It follows that the double series (2.3) is absolutely convergent for $\mathrm{Re}(u) < 0$, $\mathrm{Re}(v) > 1$, $\mathrm{Re}(u+v) > 0$, by comparison with

$$\sum_{s=1}^{\infty} |s^{-v}| \sum_{m=1}^{\infty} |m^{u-1}|.$$

In particular, (2.3) holds when $u+v = 1$ and $\mathrm{Re}(u) < 0$, and then (replace k by q/k)

$$g_q(u,\, 1-u) = 2\sum_{k|q}\frac{k}{q}\mu\left(\frac{q}{k}\right)\sum_{n=1}^{\infty} d(n)\int_0^{\infty} y^{-u}(1+y)^{u-1}\cos(2\pi nky)\,dy.$$

In view of (2.5) we need an analytic continuation of $g_q(u, 1-u)$ valid when $\mathrm{Re}(u) = 1/2$. This is obtained by using the formula

$$\sum_{n\leqslant x} d(n) = x(\log x + 2\gamma - 1) + 1/4 + \Delta(x)$$

with the classical estimate

$$\Delta(x) \ll x^{1/3}\log x. \tag{2.6}$$

Following Atkinson with obvious modifications (see [1], Sec. 4, pp. 357–361) we find

$$E(q,\, T) = \varphi(q)\sum_{k|q}\frac{k}{q}\mu\left(\frac{q}{k}\right)(I_{k,1} - I_{k,2} + I_{k,3} - I_{k,4}) + O\left(\frac{q}{T}\right), \tag{2.7}$$

where

$$I_{k,1} = 4\sum_{n\leqslant N} d(n)\int_0^{\infty}\frac{\sin\left(T\log\frac{1+y}{y}\right)\cos(2\pi kny)}{y^{1/2}(1+y)^{1/2}\log\frac{1+y}{y}}\,dy,$$

$$I_{k,2} = 4\Delta(N)\int_0^{\infty}\frac{\sin\left(T\log\frac{1+y}{y}\right)\cos(2\pi kNy)}{y^{1/2}(1+y)^{1/2}\log\frac{1+y}{y}}\,dy,$$

$$I_{k,3} = -2\frac{\log N + 2\gamma}{\pi k} I_{k,3,1} + \frac{1}{\pi i k} I_{k,3,2},$$

$$I_{k,3,1} = \int_0^{\infty}\frac{\sin\left(T\log\frac{1+y}{y}\right)\sin(2\pi kNy)}{y^{3/2}(1+y)^{1/2}\log\frac{1+y}{y}}\,dy,$$

$$I_{k,3,2} = \int_0^{\infty}\frac{\sin(2\pi kNy)}{y}\,dy\int_{1/2-iT}^{1/2+iT}\left(\frac{1+y}{y}\right)^{u}\frac{du}{u},$$

$$I_{k,4} = 4\int_N^{\infty}\frac{\Delta(x)}{x}\,dx\int_0^{\infty}\frac{\cos(2\pi kxy)}{y^{1/2}(1+y)^{3/2}\log\frac{1+y}{y}}$$

$$\times\left(T\cos\left(T\log\frac{1+y}{y}\right) - \sin\left(T\log\frac{1+y}{y}\right)\left(\frac{1}{2} + \left(\log\frac{1+y}{y}\right)^{-1}\right)\right)dy,$$

where $N > 1$. From now on assume $T \ll N \ll T$. To evaluate these integrals we need the following lemma (see [1], p. 365). Note that our U is twice that of Atkinson.

LEMMA 1. *Let $\alpha, \beta, \gamma, a, b, m, T$ be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < 1/2$, $a < T/(8\pi m)$, $b \geqslant T$, $m \geqslant 1$, $T \geqslant 1$. Then*

$$\int_a^b y^{-\alpha}(1+y)^{-\beta}\left(\log\frac{1+y}{y}\right)^{-\gamma}\exp\left\{i\left(T\log\frac{1+y}{y} + 2\pi my\right)\right\}dy$$

$$= 2^{\alpha+\beta-1}\left(\frac{\pi m}{2T}\right)^{\alpha-1/2}(mU)^{-1/2}(U+1)^{\alpha-\beta}V^{-\gamma}\exp\{i(TV + \pi m(U-1) + \pi/4)\}$$

$$+ O(a^{1-\alpha}T^{-1}) + O(b^{\gamma-\alpha-\beta}m^{-1}) + R(T, m),$$

uniformly for $|\alpha - 1| > \varepsilon > 0$, where

$$U = \sqrt{\frac{2T}{\pi m} + 1}, \qquad V = 2\,\mathrm{arsinh}\sqrt{\frac{\pi m}{2T}}$$

and

$$R(T, m) \ll \begin{cases} T^{(\gamma-\alpha-\beta)/2-1/4}\, m^{-(\gamma-\alpha-\beta)/2-5/4} & (1 \leqslant m \leqslant T), \\ T^{-1/2-\alpha}\, m^{\alpha-1} & (m \geqslant T). \end{cases}$$

A similar result holds for the corresponding integral with $-m$ in place of m, except that here the explicit term on the right-hand side is to be omitted.

To evaluate $I_{k,1}$ we apply Lemma 1 with $\alpha=\beta=1/2$, $\gamma=1$, $m=kn$ and make $a\to 0$, $b\to\infty$. Then the integral in $I_{k,1}$ equals

$$\tfrac{1}{2}(knU)^{-1/2}V^{-1}\sin(TV+\pi kn(U-1)+\pi/4)+O((kn)^{-1})+O(T^{-1}(kn)^{-1/2}).$$

Hence (recall (1.7)),

$$I_{k,1}=\Sigma_1(k,T)+O(k^{-1}\log^2 T)+O((kT)^{-1/2}\log T). \tag{2.8}$$

Like above, we get

$$I_{k,2}\ll|\Delta(N)|(kT)^{-1/2}\ll k^{-1/2}T^{-1/6}\log T, \tag{2.9}$$

where we have used (2.6).

Next consider $I_{k,3,1}$. Divide the range of integration at $Y=(2kN)^{-1}$. The integral over $[0,Y]$ gives, as in [1],

$$\ll(kN)^{1/2}T^{-1}.$$

The integral over $[Y,\infty)$ is estimated by Lemma 1 with $\alpha=3/2$, $\beta=1/2$, $\gamma=1$, $m=kN$, $a=Y$, and making $b\to\infty$. This gives

$$\ll k^{1/2}N^{-1/2}+(kN)^{1/2}T^{-1}$$

so that

$$I_{k,3,1}\ll k^{1/2}T^{-1/2}.$$

Consider $I_{k,3,2}$. We divide the range of integration with respect to y at $y=1$, and proceed like in [1]. The integral over $[0,1]$ is

$$\pi^2 i+O((kN)^{1/2}T^{-1})$$

and $[1,\infty)$ contributes

$$\ll(kN)^{-1}\log T.$$

Hence,

$$I_{k,3,2}=\pi^2 i+O(k^{1/2}T^{-1/2})$$

and so, altogether,

$$I_{k,3}=\pi/k+O((kT)^{-1/2}\log T). \tag{2.10}$$

It remains to evaluate $I_{k,4}$. To estimate the inner integrals we use Lemma 1 with $\alpha=1/2$, $\beta=3/2$, $\gamma=1,2$, $m=kx$, making $a\to 0$, $b\to\infty$. We have then, for $kx\gg T$,

$$\int_0^\infty\frac{\cos\left(T\log\frac{1+y}{y}\right)\cos(2\pi kxy)}{y^{1/2}(1+y)^{3/2}\log\frac{1+y}{y}}\,dy$$

$$=(kxU)^{-1/2}(U+1)^{-1}V^{-1}\cos(TV+\pi kx(U-1)+\pi/4)+O(T^{-1}(kx)^{-1/2}),$$

and similarly, for $\gamma=1,2$,

$$\int_0^\infty\frac{\sin\left(T\log\frac{1+y}{y}\right)\cos(2\pi kxy)}{y^{1/2}(1+y)^{3/2}\left(\log\frac{1+y}{y}\right)^\gamma}\,dy\ll(kx)^{-1/2}.$$

Hence, writing

$$f(x)=\frac{x^2}{2}-x\sqrt{\frac{T}{2\pi}+\frac{x^2}{4}}-\frac{T}{\pi}\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right),$$

$$g_\alpha(x)=\left(x^\alpha\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)\left(\sqrt{\frac{T}{2\pi x^2}+\frac14}+\frac12\right)\left(\frac{T}{2\pi x^2}+\frac14\right)^{1/4}\right)^{-1},$$

we have

$$I_{k,4}=\frac{1}{\sqrt2}\int_N^\infty\frac{\Delta(x)}{x}\left(Tg_1(\sqrt{kx})\cos(-2\pi f(\sqrt{kx})+\pi/4)+O((kx)^{-1/2})\right)dx.$$

The O-term here gives, by (2.6),

$$\ll k^{-1/2}\int_N^\infty|\Delta(x)|x^{-3/2}\,dx\ll k^{-1/2}T^{-1/6}\log T.$$

Next we shall use Voronoi's formula for $\Delta(x)$. We have (see e.g. [6], eq. (15.24))

$$\Delta(x)=\frac{x^{1/4}}{\pi\sqrt2}\sum_{n=1}^\infty d(n)n^{-3/4}\left(\cos(4\pi\sqrt{nx}-\pi/4)-\frac{3}{32\pi\sqrt{nx}}\sin(4\pi\sqrt{nx}-\pi/4)\right)+O(x^{-3/4})$$

except when x is an integer, the series being boundedly convergent in any finite x-interval. The error term $O(x^{-3/4})$ here contributes

$$\ll k^{-1/2}T\int_N^\infty x^{-9/4}\,dx\ll k^{-1/2}T^{-1/4}.$$

Hence, changing the variable from x to $\sqrt{kx}$,

$$I_{k,4} = \frac{T}{\pi} k^{-1/4} \sum_{n=1}^{\infty} d(n) n^{-3/4} J_{n,k} + O(k^{-1/2} T^{-1/6} \log T),$$

where

$$J_{n,k} = \int_{\sqrt{kN}}^{\infty} g_{3/2}(x) \cos(-2\pi f(x) + \pi/4)$$
$$\times \left(\cos\left(4\pi x \sqrt{\frac{n}{k}} - \frac{\pi}{4}\right) - \frac{3}{32\pi x}\sqrt{\frac{k}{n}} \sin\left(4\pi x \sqrt{\frac{n}{k}} - \frac{\pi}{4}\right)\right) dx.$$

This will be evaluated by the next modification of Atkinson's Lemma 3 ([1], p. 372).

LEMMA 2. *Let*

$$Z = \frac{T}{2\pi} + \frac{a^2}{2} - a\sqrt{\frac{T}{2\pi} + \frac{a^2}{4}}$$

and recall the definition (1.6). *For* $T \gg 1$, $a \gg \sqrt{T}$, $v > 0$ *and* $\alpha > 1$, *we have*

$$\int_{a}^{\infty} g_\alpha(x) e^{2\pi i (f(x) + 2x\sqrt{v})} dx$$
$$= \frac{4\pi}{T} v^{(\alpha-1)/2} \left(\log \frac{T}{2\pi v}\right)^{-1} \left(\frac{T}{2\pi} - v\right)^{3/2-\alpha} e^{-ig(T,v) + i\pi/2}$$
$$+ O\left(a^{-\alpha} \min\left(\frac{a}{\sqrt{T}}, |\sqrt{v} - \sqrt{Z}|^{-1}\right)\right) + O\left(v^{(\alpha-1)/2}\left(\frac{T}{2\pi} - v\right)^{1-\alpha} T^{-3/2}\right),$$

provided that $v < Z$. *If* $v \geq Z$ *or if* $\sqrt{v}$ *is replaced by* $-\sqrt{v}$, *then the main term and the last error term on the right-hand side are to be omitted.*

Remark. In the corresponding lemma of Atkinson the lower limit of integration is assumed to satisfy $\sqrt{T} \ll a \ll \sqrt{T}$. It is easy to see that weakening this restriction to $a \gg \sqrt{T}$ results in that Atkinson's $O(T^{-\alpha/2} \min(1, |\sqrt{v} - \sqrt{Z}|^{-1}))$ is to be replaced by the O-term given above, which is in fact better, since we have assumed $\alpha > 1$. This last assumption is made in order to be able to weaken the condition $v \geq 1$ to $v > 0$. Apart from these modifications, Lemma 2 is the same as the original one.

Now Lemma 2 is applied with $a = \sqrt{kN}$, $v = n/k$ and $\alpha = 3/2$ or $\alpha = 5/2$. Then

$$kZ = N',$$

where N' is defined by (1.9). The main term comes from the integral with $\alpha = 3/2$. Note also that

$$N' < A\frac{kT}{2\pi}, \tag{2.11}$$

for some $A < 1$. Hence, for $n < N'$,

$$J_{n,k} = \frac{2\pi}{T}\left(\frac{n}{k}\right)^{1/4}\left(\log\frac{kT}{2\pi n}\right)^{-1} \cos g\left(T, \frac{n}{k}\right)$$
$$+ O(k^{-1/4} T^{-3/4} \min(1, |\sqrt{n} - \sqrt{N'}|^{-1}))$$
$$+ O\left(\left(\frac{n}{k}\right)^{1/4} T^{-3/2}\left(\frac{T}{2\pi} - \frac{n}{k}\right)^{-1/2}\right)$$
$$+ O\left(\left(\frac{n}{k}\right)^{1/4} T^{-1}\left(\frac{T}{2\pi} - \frac{n}{k}\right)^{-1}\left(\log\frac{kT}{2\pi n}\right)^{-1}\right),$$

but if $n \geq N'$, the main term and the last two error terms are to be omitted. Hence,

$$I_{k,4} = \sum_{j=1}^{4} I_{k,4,j} + O(k^{-1/2} T^{-1/6} \log T),$$

where $I_{k,4,1} = -\Sigma_2(k, T)$ (see (1.8)),

$$I_{k,4,2} \ll k^{1/2} \sum_{n<N'} d(n) n^{-1/2} (kT - 2\pi n)^{-1},$$
$$I_{k,4,3} \ll k^{-1/2} T^{1/4} \sum_{n=1}^{\infty} d(n) n^{-3/4} \min(1, |\sqrt{n} - \sqrt{N'}|^{-1}),$$
$$I_{k,4,4} \ll T^{-1/2} \sum_{n<N'} d(n) n^{-1/2} (kT - 2\pi n)^{-1/2}.$$

Since $T \ll N \ll T$ we have $N' \ll T$. Using also (2.11) it is easily seen that

$$I_{k,4,j} \ll (kT)^{-1/2} \log T$$

for $j = 2, 4$. In $I_{k,4,3}$ the sum is split up at $\frac{1}{2}N'$, $N' - \sqrt{N'}$, $N' + \sqrt{N'}$ and $2N'$. Then easily

$$I_{k,4,3} \ll k^{-1/2} \log^2 T,$$

giving the critical error term. Hence,

$$I_{k,4} = -\Sigma_2(k, T) + O(k^{-1/2} \log^2 T).$$

Combining this with (2.8)–(2.10) yields

$$I_{k,1} - I_{k,2} + I_{k,3} - I_{k,4} = \Sigma_1(k, T) + \Sigma_2(k, T) + O(k^{-1/2} \log^2 T).$$

Substituting this in (2.7) completes the proof of Theorem 1.

3. A smoothed form of $E(q, T)$. For the proof of Corollary 1 we need the lemma of this section, whose proof is based on the method of [7]. Let $L = \log T$, $G > 0$. Consider the function

$$E_1(q, x) = \frac{1}{\sqrt{\pi} G} \int_{-GL}^{GL} E(q, x+u) e^{-(u/G)^2} du$$

defined for $x > GL$.

LEMMA 3. *Let* $T \ll x \ll T$ *and*

$$L^2 \ll G \ll T^{1/2} L^{-2}. \tag{3.1}$$

Let $M_k = k^{-1} T G^{-2} L^2$ *and*

$$r(x, m) = \exp\left(-G^2 \operatorname{arsinh}^2\left(\left(\frac{\pi m}{2x}\right)^{1/2}\right)\right).$$

Then, for any $\varepsilon > 0$,

$$E_1(q, x) = \left(\frac{2x}{\pi}\right)^{1/4} \frac{\varphi(q)}{q} \sum_{k|q} k\mu\left(\frac{q}{k}\right) \times \sum_{n \leqslant M_k} (-1)^{kn} d(n) (kn)^{-3/4} e(x, kn) r(x, kn) \cos f(x, kn) + O(q^{1/2} T^{\varepsilon}) + O(q/T) + O(G^{3/2} T^{-1/2+\varepsilon}).$$

Proof. We use Theorem 1 with $N = T$ and encounter the expressions

$$S_j = \int_{-GL}^{GL} \Sigma_j(k, x+u) e^{-(u/G)^2} du \quad (j = 1, 2). \tag{3.2}$$

Consider first S_1. By (1.7),

$$S_1 = \left(\frac{2}{\pi}\right)^{1/4} \int_{-GL}^{GL} (x+u)^{1/4} \sum_{n \leqslant T} a_{k,n} e(x+u, kn) \cos f(x+u, kn) e^{-(u/G)^2} du,$$

where we have written $a_{k,n} = (-1)^{kn} d(n) (kn)^{-3/4}$ for brevity.

Since

$$(x+u)^{1/4} - x^{1/4} \ll |u| x^{-3/4},$$

$$e(x+u, kn) - e(x, kn) \ll |u| x^{-1} (1 + knx^{-1})^{1/4}$$

(see (1.4)), we have

$$S_1 = \left(\frac{2x}{\pi}\right)^{1/4} \sum_{n \leqslant T} a_{k,n} e(x, kn) \int_{-GL}^{GL} \cos f(x+u, kn) e^{-(u/G)^2} du + O(G^2 (kT)^{-1/2} L). \tag{3.3}$$

Write, for brevity, $w(x, m) = \operatorname{arsinh}\left(\left(\frac{\pi m}{2x}\right)^{1/2}\right)$. By (1.5),

$$f'(t, m) = 2w(t, m),$$

$$f''(t, m) = -t^{-1}\left(1 + \frac{2t}{\pi m}\right)^{-1/2},$$

$$f'''(t, m) = t^{-1}\left(1 + \frac{2t}{\pi m}\right)^{-1/2} (t^{-1} + (\pi m + 2t)^{-1}).$$

Hence,

$$f(x+u, kn) = f(x, kn) + 2w(x, kn) u + A(x, kn) u^2 + O(|u|^3 x^{-2}),$$

where $A(x, kn) \ll x^{-1}$. We substitute this in (3.3), omitting the error term. This effects an error

$$\ll k^{-1/2} T^{-3/2} G^4 L.$$

Hence, the integral in (3.3) may be replaced by

$$\operatorname{Re}\left(e^{if(x,kn)} \int_{-GL}^{GL} e^{i2w(x,kn)u - B(x,kn)u^2} du\right),$$

where

$$B(x, kn) = G^{-2} - iA(x, kn) = G^{-2} + O(x^{-1}).$$

The integral is extended over the real line with a negligible error. Then it is evaluated by the formula

$$\int_{-\infty}^{\infty} e^{au - bu^2} du = \left(\frac{\pi}{b}\right)^{1/2} \exp\left(\frac{a^2}{4b}\right) \quad (\operatorname{Re}(b) > 0).$$

This gives

$$\left(\frac{\pi}{B(x, kn)}\right)^{1/2} \exp\left(-\frac{w^2(x, kn)}{B(x, kn)}\right) \ll G \exp(-AG^2 \min(kn/x, 1)) \tag{3.4}$$

for some positive A. Hence the terms with $n > M_k$ may be omitted with a negligible error. For the remaining values of n the left-hand side of (3.4) is, by (3.1),

$$\sqrt{\pi} G r(x, kn) + O(G^3 x^{-1} L^2).$$

Thus,

$$S_1 = \sqrt{\pi} G \left(\frac{2x}{\pi}\right)^{1/4} \sum_{n \leqslant M_k} a_{k,n} e(x, kn) r(x, kn) \cos f(x, kn) + O(G^2 (kT)^{-1/2} L) + O(k^{-1} T^{-1/2} G^{5/2} L^{7/2}). \tag{3.5}$$

Consider now S_2 as given by (3.2). First we want to replace $N' = N'(k, x+u, T)$ by $N'' = N'(k, x, T)$. By (1.9), we have $T \ll N' \ll T$ and $N' - N'' \ll |u|$. By (2.11) and (3.1), we have $N', N'' \leqslant Akx/2\pi$ for some $A < 1$. Moreover,

$$\left(\log\frac{k(x+u)}{2\pi n}\right)^{-1} - \left(\log\frac{kx}{2\pi n}\right)^{-1} \ll |u|\, x^{-1}.$$

By these remarks, we have for any $\varepsilon > 0$,

$$S_2 \ll \sum_{n<N''} d(n)(kn)^{-1/2} \Big| \int_{-GL}^{GL} e^{ig(x+u,n/k)-(u/G)^2}\, du \Big| + G^2 k^{-1/2} T^{-1/2+\varepsilon}. \tag{3.6}$$

We substitute

$$g(x+u, n/k) = g(x, n/k) + \left(\log\frac{kx}{2\pi n}\right)u + \frac{1}{2x}u^2 + O(|u|^3 x^{-2}).$$

The O-term contributes

$$\ll k^{-1/2} T^{-3/2} G^4 L.$$

The integral may now be extended over the real line with a negligible error and the new integral is evaluated as above, giving, for some positive A,

$$\ll G e^{-AG^2 \log^2 (kx/2\pi n)},$$

which is also negligible. Hence the second term on the right of (3.6) dominates. Using this and (3.5) and the definitions of $E(q, T)$ and $E_1(q, T)$ we get Lemma 3.

4. Proof of Corollary 1. We have trivially

$$I(q, t_1) \leqslant I(q, T) \leqslant I(q, t_2)$$

for $1 \leqslant t_1 \leqslant T \leqslant t_2 \leqslant 2T$. Therefore, by (1.2),

$$E(q, t_1) + O\big(q(T-t_1)\log qT\big) \leqslant E(q, T) \leqslant E(q, t_2) + O\big(q(t_2 - T)\log qT\big).$$

(Note that $q^{-1}\varphi(q)\sum_{p|q}(\log p)/(p-1) \ll \log q$.) This yields

$$E_1(q, T-GL) + O(qGL\log qT) \leqslant E(q, T)\frac{1}{\sqrt{\pi}G}\int_{-GL}^{GL} e^{-(u/G)^2}\, du$$

$$\leqslant E_1(q, T+GL) + O(qGL\log qT),$$

where G, L and $E_1(q, x)$ are as in Section 3. Hence,

$$E(q, T) \ll |E_1(q, T-GL)| + |E_1(q, T+GL)| + qGL\log qT.$$

Now Lemma 3 is applied to the right-hand side. Estimating the main term trivially we arrive at

$$E(q, T) \ll (q^{1/2} + qT^{-1} + G^{3/2}T^{-1/2} + G^{-1/2}T^{1/2} + qG)(qT)^\varepsilon$$

for any positive ε. The optimal choice of G is

$$G = q^{-2/3}T^{1/3} + L^2,$$

which satisfies (3.1). This gives

$$E(q, T) \ll (qT)^{1/3+\varepsilon} + q^{1+\varepsilon}.$$

On the other hand, it follows trivially from Theorem 1 that

$$E(q, T) \ll (qT)^{1/2+\varepsilon} + qT^{-1}.$$

This completes the proof.

5. Proof of Corollary 2. By (1.1), the left-hand side of (1.10) is $I(q, T+H) - I(q, T)$. By (1.2), this is

$$\ll qH\log q(T+H) + |E(q, T+H)| + |E(q, T)|.$$

Thus (1.10) follows from Corollary 1.

We now prove (1.11). For $q \gg t$ this is already known by [4]. Let $q \ll t$. Generalizing a lemma of Heath-Brown we have

$$L^2(\tfrac{1}{2}+it, \chi) \ll \log^2 t\Big(1 + \int_{-\log^2 t}^{\log^2 t} |L(\tfrac{1}{2}+i(t+v), \chi)|^2 e^{-|v|}\, dv\Big),$$

when χ is primitive. For a proof we refer to [8], Lemma 6. The primitivity is not an essential restriction. Thus the result follows from (1.10).

6. An analogue of Theorem 1. Let

$$d(n, q) = \sum_{\substack{k|n \\ (k,q)=1}} 1, \qquad c(n, q) = \sum_{k|n,q} k\mu\left(\frac{q}{k}\right) d\left(\frac{n}{k}\right),$$

$$\Omega_2(n) = \max\{a \mid 2^a \text{ divides } n\},$$

$$d^*(n, q) = \begin{cases} \tfrac{1}{2}(-1)^n d(n, q) & \text{if } 2 \nmid q, \\ \tfrac{1}{2}(\Omega_2(n) - (-1)^n) d(n, q) & \text{if } 2 \| q, \\ d(n, q) & \text{if } 4 | q, \end{cases}$$

$$D^*(x, q) = \sum_{n \leqslant x} d^*(n, q).$$

Here $2^a \| q$ means that $2^a | q$ but $2^{a+1} \nmid q$.

THEOREM 2. *For $q \ll x$, $1 \leqslant N \ll qx$ and any $\varepsilon > 0$, we have*

$$D^*(x, q) = \frac{\varphi(q)}{q}\frac{x}{b_q}\left(\log\frac{x}{b_q}+\gamma_q+\gamma-1\right)+\Delta^*\left(\frac{x}{b_q}, q\right),$$

where

$$\Delta^*(x, q) = \frac{1}{\pi\sqrt{2}}\left(\frac{x}{q}\right)^{1/4}\sum_{n\leqslant N}(-1)^n c(n, q)\, n^{-3/4}\cos\left(4\pi\sqrt{\frac{nx}{q}}-\frac{\pi}{4}\right)$$
$$+O\left(\left(\frac{qx}{N}\right)^{1/2}x^{\varepsilon}\right),$$

γ_q is defined by (2.4) and

$$b_q = \begin{cases} 4 & \text{if} \quad 2\nmid q, \\ 2 & \text{if} \quad 2\| q, \\ 1 & \text{if} \quad 4| q. \end{cases}$$

Remark. A comparison of Theorem 2 and (1.2) shows that the analogue (in the sense of Jutila [7]) of $I(q, T)$ is

$$2\pi\frac{\varphi(q)}{q}D^*\left(b_q\frac{qT}{2\pi}, q\right).$$

Indeed, the main term of $2\pi\frac{\varphi(q)}{q}\Delta^*\left(\frac{qT}{2\pi}, q\right)$ is

$$\frac{\varphi(q)}{q}\left(\frac{2T}{\pi}\right)^{1/4}\sum_{n\leqslant N}(-1)^n c(n, q)\, n^{-3/4}\cos(2\sqrt{2\pi nT}-\pi/4).$$

On the other hand, we get from Lemma 3, interchanging summations, that the main term of $E_1(q, T)$ is

$$\frac{\varphi(q)}{q}\left(\frac{2T}{\pi}\right)^{1/4}\sum_{n\leqslant M_1}(-1)^n c(n, q)\, n^{-3/4}e(T, n)\, r(T, n)\cos f(T, n).$$

In the last two expressions the terms with $n = o(T^{1/3})$ are asymptotically equal, as $T\to\infty$.

Remark. We have

$$c(n, q) = \sum_{m|n} c_q(m),$$

where $c_q(m)$ is Ramanujan's sum. Moreover, $q\prod_{p|q}p^{-1}$ divides n if $c(n, q)\neq 0$.

Before proving Theorem 2 we give two lemmas. Let $G(s, q)$ and $F(s, q)$ be the generating Dirichlet series of $d(n, q)$ and $c(n, q)$, respectively, and let $\chi(s)$ be as in the functional equation

(6.1) $$\zeta(s) = \chi(s)\zeta(1-s).$$

LEMMA 4. *The functions $G(s, q)$ and $F(s, q)$ are analytically continuable to the whole complex plane. They are related by*

$$G(s, q) = q^{-s}\chi^2(s)F(1-s, q).$$

Proof. Obviously, we have $G(s, q) = \zeta(s)L(s, \chi_0)$. This gives the analytic continuation of $G(s, q)$. Now it suffices to prove the identity for $\sigma < 0$. Then, by (6.1),

$$\zeta(s)L(s, \chi_0) = \chi^2(s)\zeta^2(1-s)\prod_{p|q}(1-p^{-s}).$$

Here

$$\begin{aligned}\zeta^2(1-s)\prod_{p|q}(1-p^{-s}) &= \sum_{n=1}^{\infty}d(n)n^{s-1}\sum_{k|q}\mu(k)k^{1-s}k^{-1}\\ &= \sum_{m=1}^{\infty}\left(\frac{m}{q}\right)^{s-1}\sum_{\substack{k|q\\ q|mk}}d\left(\frac{mk}{q}\right)\mu(k)k^{-1}\\ &= \sum_{m=1}^{\infty}\left(\frac{m}{q}\right)^{s-1}\sum_{n|q,m}d\left(\frac{mn}{(q, m)}\right)\mu\left(\frac{qn}{(q, m)}\right)\frac{(q, m)}{qn}\\ &= q^{-s}F(1-s, q),\end{aligned}$$

which completes the proof.

Let $G^*(s, q)$ and $F^*(s, q)$ be the generating functions of $d^*(n, q)$ and $(-1)^n c(n, q)$, respectively.

LEMMA 5. *The functions $G^*(s, q)$ and $F^*(s, q)$ are analytically continuable to the whole complex plane. Moreover*

(6.2) $$G^*(s, q) = (b_q q)^{-s}\chi^2(s)F^*(1-s, q),$$

(6.3) $$G^*(s, q) = H(s)G(s, q),$$

where

$$H(s) = \begin{cases} -\frac{1}{2}+2^{1-s}-2^{-2s} & \text{if} \quad 2\nmid q, \\ (\frac{1}{2}-2^{-s}+2^{-2s})(1-2^{-s})^{-1} & \text{if} \quad 2\| q, \\ 1 & \text{if} \quad 4| q. \end{cases}$$

Proof. In what follows we assume $\sigma > 1$. The final results are obtained by analytic continuation.

If $4|q$ and $2\nmid n$, then $c(n, q) = 0$, so that $c(n, q) = (-1)^n c(n, q)$. Therefore Lemma 5 is trivial if $4|q$.

Assume now $2\nmid q$. Using the identities

(6.4) $$\begin{aligned} d(2n, q) &= 2d(n, q) \quad \text{if} \quad 2\nmid n,\\ d(4n, q)-2d(2n, q)+d(n, q) &= 0,\end{aligned}$$

we get

$$G^*(s, q) = \sum_{\substack{n=1\\ 2|n}}^{\infty} d(n, q)\, n^{-s} - \tfrac{1}{2} G(s, q)$$

$$= 2^{-s} \sum_{n=1}^{\infty} \big(d(2n, q) - 2d(n, q)\big) n^{-s} + 2^{1-s} \sum_{n=1}^{\infty} d(n, q)\, n^{-s} - \tfrac{1}{2} G(s, q)$$

$$= 2^{-2s} \sum_{n=1}^{\infty} \big(d(4n, q) - 2d(2n, q)\big) n^{-s} + (2^{1-s} - \tfrac{1}{2})\, G(s, q)$$

$$= H(s)\, G(s, q),$$

which proves (6.3) for odd values of q. Since $c(n, q)$ also satisfies (6.4), we get similarly

$$F^*(s, q) = (-1 + 2^{2-s} - 2^{1-2s})\, F(s, q),$$

which in combination with Lemma 4 and (6.3) proves (6.2) for odd values of q.

Finally, consider the remaining case $2\|q$. We have

$$H(s)\, G(s, q) = \Big(\tfrac{1}{2} \sum_{n=1}^{\infty} a(n)\, n^{-s}\Big) G(s, q),$$

where $a(n)$ is a multiplicative function defined by $a(1) = 1$, $a(2) = -1$, $a(2^m) = 1$ if $m \geqslant 2$ and $a(n) = 0$ otherwise. The right-hand side is

$$\sum_{n=1}^{\infty} \tfrac{1}{2} \Big(\sum_{k|n} a(k)\, d(n/k, q)\Big) n^{-s},$$

which equals $G^*(s, q)$, since $d(2^{-m} n, q) = d(n, q)$ for $2^m | n$. This proves (6.3) in the case $2\|q$. Since

$$c(2^a m, q) = (1-a)\, c(m, q) \quad \text{if} \quad 2 \nmid m,\ a \geqslant 0,$$

we have

$$F^*(s, q) = \sum_{a=0}^{\infty} \sum_{\substack{n=1\\ 2^a \| n}}^{\infty} (-1)^n c(n, q)\, n^{-s}$$

$$= \sum_{a=0}^{\infty} (-1)^{2^a} 2^{-as} \sum_{\substack{m=1\\ 2 \nmid m}}^{\infty} c(2^a m, q)\, m^{-s}$$

$$= \sum_{a=0}^{\infty} (-1)^{2^a} (1-a)\, 2^{-as} \sum_{\substack{m=1\\ 2 \nmid m}}^{\infty} c(m, q)\, m^{-s}$$

$$= -\big(1 + (2^s - 1)^{-2}\big) \tfrac{1}{2} \big(F(s, q) - F^*(s, q)\big).$$

Hence,

$$F^*(s, q) = (2^{1-2s} - 2^{1-s} + 1)(2^{1-s} - 1)^{-1} F(s, q),$$

which in combination with Lemma 4 and (6.3) proves (6.2) for $2\|q$.

We proceed to prove Theorem 2. Using (6.2) the method in Titchmarsh [10], Ch. 12, yields

$$\text{(6.5)} \qquad D^*(x, q) = \operatorname*{Res}_{s=1}\big(G^*(s, q)\, x^s s^{-1}\big) + \frac{1}{2\pi i} \sum_{n=1}^{\infty} (-1)^n c(n, q) \int_{-\varepsilon-iT}^{-\varepsilon+iT} \chi^2(s)\, n^{s-1} \Big(\frac{x}{b_q q}\Big)^s \frac{ds}{s} + O\Big(\frac{q^\varepsilon x^{1+\varepsilon}}{T}\Big) + O\Big(\Big(\frac{qT^2}{x}\Big)^\varepsilon\Big) + O(x^\varepsilon).$$

The residue may be evaluated using (6.3) and the identity $G(s, q) = \zeta(s)\, L(s, \chi_0)$. This gives the main term of Theorem 2. Let $N_0 = N + m$, where m (and N) is an integer, $1 \leqslant m \leqslant q$, and choose T to satisfy

$$\frac{b_q q T^2}{4\pi^2 x} = N_0 + \frac{1}{2}.$$

Then the O-terms above give

$$\ll \Big(\frac{qx}{N_0}\Big)^{1/2} x^\varepsilon \ll \Big(\frac{qx}{N}\Big)^{1/2} x^\varepsilon.$$

Following Titchmarsh, we see that the series on the right of (6.5) is (note that $q \ll x$)

$$\frac{1}{\pi\sqrt{2}} \Big(\frac{x}{b_q q}\Big)^{1/4} \sum_{n \leqslant N_0} (-1)^n c(n, q)\, n^{-3/4} \cos\Big(4\pi \sqrt{\frac{nx}{b_q q}} - \frac{\pi}{4}\Big) + O\Big(\Big(\frac{x}{q}\Big)^{-1/4}\Big) + O\Big(N_0^\varepsilon \sum_{n=1}^{\infty} |c(n, q)|\, n^{-1-\varepsilon}\Big) + O\Big(N_0^\varepsilon \sum_{n \leqslant 2N_0} |c(n, q)|\, n^{-1} \Big|\log \frac{n}{N_0 + \frac{1}{2}}\Big|^{-1}\Big).$$

In the first term N_0 may be replaced by N, since

$$\Big(\frac{x}{q}\Big)^{1/4} \sum_{n=N+1}^{N+q} n^{-3/4} |c(n, q)| \ll \Big(\frac{x}{q}\Big)^{1/4} N^{-3/4+\varepsilon} \sum_{k|q} k \cdot \frac{q}{k} \ll \Big(\frac{qx}{N}\Big)^{1/2} x^\varepsilon.$$

The first O-term is $\ll 1$. The second O-term is

$$\ll N_0^\varepsilon \sum_{k|q} k \sum_{n=1}^{\infty} d(n)(kn)^{-1-\varepsilon} \ll (qN_0)^\varepsilon \ll x^\varepsilon.$$

The third O-term is averaged with respect to m. We have

$$q^{-1}\sum_{m=1}^{q}\sum_{n\leqslant 2N+2m}|c(n,q)|\,n^{-1}\left|\log\frac{n}{N+m+\frac{1}{2}}\right|^{-1}$$

$$\ll q^{-1}\sum_{k|q}k\sum_{m=1}^{q}\sum_{n\leqslant(2/k)(N+m)}d(n)(kn)^{-1}\frac{N+m}{|kn-N-m-\frac{1}{2}|}\ll x^{\varepsilon}.$$

The proof is complete.

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G-fonctions et théorème d'irreductibilite de Hilbert

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Jusqu'à l'article de Bombieri sur les G-fonctions [1], on ne disposait pas de résultats généraux sur la nature arithmétique des valeurs en des points algébriques de G-fonctions satisfaisant des équations différentielles linéaires, de tels résultats ayant pourtant été énoncés dans le célèbre article de C. L. Siegel de 1929 [20]. La méthode de Siegel, fructueuse dans le cadre des E-fonctions, présente en effet de nombreuses difficultés quand on cherche à l'appliquer aux G-fonctions. Bombieri est parvenu à y faire face, au moyen d'arguments sophistiqués comme le théorème de Dwork–Robba. Nous proposons ici une approche différente du problème, basée sur la méthode de Gel'fond, qui évite les complications de la méthode de Siegel, et conduit à un nouvel énoncé sur l'irrationalité et l'indépendance linéaire des valeurs de G-fonctions (théorème principal).

Les fonctions algébriques constituent un exemple typique de G-fonctions vérifiant des équations différentielles linéaires; on obtient dans ce cas particulier un énoncé (théorème 2) qui généralise simultanément des résultats de P. Bundschuh [3], T. Schneider [17], [18] et de V. G. Sprindžuk [21]–[24] sur le théorème d'irréductibilité de Hilbert.

Présenté ici comme le fruit d'une méthode analytique, le théorème 2 possède en fait une origine purement algébrique: le paragraphe 2.4, qui s'appuie sur un article de Bombieri sur le théorème de décomposition de Weil ([2], voir aussi [7]), explique qu'il provient essentiellement de la quadraticité de la hauteur sur les variétés abéliennes. Le théorème 3, version géométrique du théorème 2, permet d'autre part de donner corps au lien mis en évidence par M. Fried [12] entre les travaux de V. G. Sprindžuk et ceux de R. Weissauer.

La dernière partie est consacrée au théorème d'irréductibilité de Hilbert; le théorème 2 conduit à une nouvelle version qui montre en gros que toute partie hilbertienne d'un corps de nombres contient "beaucoup" de progressions géométriques.

Notations.

Valeurs absolues. Nous adopterons les normalisations suivantes des valeurs absolues $|\ |_v$ associées aux places v d'un corps de nombres F: