STUDIA MATHEMATICA, T. LXXXIII. (1986)

Generalized convolutions IV

by

K. URBANIK (Wrocław)

Abstract. The paper is a continuation of the author's earlier work [9]. It is a study of ∘-stable probability measures and some Banach algebras associated with generalized convolutions. The main result of the paper is the existence of weak characteristic functions for every generalized convolution.

1. Notation and preliminaries. Generalized convolutions were introduced in [8]. Let us recall some definitions. We denote by P the set of all probability measures defined on Borel subsets of the positive half-line $\boldsymbol{R}_+$. The set P is endowed with the topology of weak convergence. For $\mu \in P$ and $a > 0$ we define the map T_a by setting $(T_a \mu)(E) = \mu(a^{-1} E)$ for all Borel subsets E of $\boldsymbol{R}_+$. By δ_c we denote the probability measure concentrated at the point c. Further we put $T_0 \mu = \delta_0$ for all $\mu \in P$.

A commutative and associative P-valued binary operation $\circ$ on P, continuous in each variable separately, is called a *generalized convolution* if it is distributive with respect to convex combinations and maps T_a $(a > 0)$ with δ_0 as the unit element. Moreover, the key axiom postulates the existence of norming constants c_n and a measure $\gamma \in P$ other than δ_0 such that

$$T_{c_n} \delta_1^{\circ n} \to \gamma \tag{1.1}$$

where $\delta_1^{\circ n}$ is the nth power of δ_1 under $\circ$. For basic properties of generalized convolutions we refer to [8] and [11]. In particular, every generalized convolution is continuous in both variables ([11], Theorem 2.1). Moreover, generalized convolution algebras admitting a nonconstant continuous homomorphism into the algebra of real numbers with the operations of multiplication and convex combinations admit characteristic functions ([8], Theorem 6). The characteristic function plays the same fundamental role for a generalized convolution as the Laplace transform for the ordinary one. The aim of this paper is to prove that every generalized convolution algebra admits an analogue of the characteristic function which will be called a weak characteristic function. The main results of this paper are based on two techniques: one uses ∘-stable measures, the other uses Banach algebra arguments. The idea of generating some Banach algebras by generalized convolutions is due to V. E. Vol'kovich in [12] and [13].

In the sequel we shall use the following notation. $\bar{R}_+$ will denote the compactified half-line $[0, \infty]$. $\bar{P}$ will denote the set of all Borel probability measures on $\bar{R}_+$ and $P_\infty = \bar{P} \setminus P$.

We begin with the following simple lemma.

LEMMA 1.1. *Suppose that $\mu_n, \mu \in P$, $\mu \neq \delta_0$, $T_{a_n}\mu_n \to \mu$ and the sequence $T_{b_n}\mu_n$ is conditionally compact in P. Then the sequence b_n/a_n is bounded and the set of limit points of $T_{b_n}\mu_n$ coincides with the set of measures $T_c\mu$ where c is any limit point of the sequence b_n/a_n.*

Proof. Suppose that $d_k = b_{n_k}/a_{n_k} \to \infty$ for a subsequence $n_1 < n_2 < \ldots$ Then, by Proposition 2.2 in [11], all limit points of the sequence $T_{d_k} T_{a_{n_k}} \mu_{n_k} = T_{b_{n_k}} \mu_{n_k}$ in $\bar{P}$ belong to P_∞, which contradicts the assumption. Thus the sequence b_n/a_n is bounded. Let ν be a limit point of the sequence $T_{b_n}\mu_n$ and $T_{b_{m_k}}\mu_{m_k} \to \nu$. Without loss of generality we may assume that the sequence $c_k = b_{m_k}/a_{m_k}$ is convergent, say to a limit c. Then the formula

$$T_{b_{m_k}}\mu_{m_k} = T_{c_k}(T_{a_{m_k}}\mu_{m_k}) \quad (k = 1, 2, \ldots)$$

yields $\nu = T_c\mu$. Conversely, if c is a limit point of b_n/a_n and $c_k = b_{m_k}/a_{m_k} \to c$, then $T_{b_{m_k}}\mu_{m_k} \to T_c\mu$, which completes the proof.

LEMMA 1.2. *Let $\mu, \nu \in P$. Then for any bounded Borel function f on R_+ the function $(u, v) \to \int_0^\infty f(x)(\delta_u \circ \delta_v)(dx)$ is Borel on $R_+ \times R_+$ and the formula*

$$\int_0^\infty f(x)(\mu \circ \nu)(dx) = \int_0^\infty \int_0^\infty \int_0^\infty f(x)(\delta_u \circ \delta_v)(dx)\,\mu(du)\,\nu(dv)$$

holds.

Proof. Let F be the set of all bounded Borel functions f for which the above assertion is true. By formula (2.13) in [11] the set F contains all bounded continuous functions on R_+. Moreover, the set F is closed under bounded convergence, which shows that it contains all bounded Borel functions on R_+. The lemma is thus proved.

For any pair μ, ν from P by $\mu\nu$ we shall denote the probability distribution of the product XY of two independent random variables with probability distributions μ and ν respectively. The operation $\mu\nu$ is a commutative semigroup operation with the following properties:

$$(T_a\mu)\nu = T_a(\mu\nu) \quad (a \in R_+), \tag{1.2}$$

$$T_a\mu = \delta_a\mu \quad (a \in R_+), \tag{1.3}$$

$$(c\mu + (1-c)\nu)\lambda = c(\mu\lambda) + (1-c)(\nu\lambda) \quad (0 \leqslant c \leqslant 1). \tag{1.4}$$

Moreover, we have the following lemmas.

LEMMA 1.3. *Let $\mu, \nu \in P$. Then for any bounded Borel function f on R_+ the formula*

$$\int_0^\infty f(x)(\mu\nu)(dx) = \int_0^\infty \int_0^\infty f(xy)\,\mu(dx)\,\nu(dy)$$

holds.

LEMMA 1.4. *If μ is absolutely continuous with respect to the Lebesgue measure on R_+ and $\nu(\{0\}) = 0$, then $\mu\nu$ is also absolutely continuous with respect to the Lebesgue measure on R_+. Conversely, if the Lebesgue measure on R_+ is absolutely continuous with respect to μ and $\nu \neq \delta_0$, then the Lebesgue measure on R_+ is also absolutely continuous with respect to $\mu\nu$.*

Let m denote the Lebesgue measure on R_+. Put $m_0 = \delta_0 + m$. By P_0 we shall denote the subset of P consisting of all measures absolutely continuous with respect to m_0. Further by Q we shall denote the subset of P_0 consisting of all measures equivalent to the Lebesgue measure on R_+. It is evident that both sets P_0 and Q are invariant under all transformations T_a $(a > 0)$. As a direct consequence of Lemma 1.4 we have the following statement.

PROPOSITION 1.1. *If $\mu \in P_0$ and $\nu \in P$, then $\mu\nu \in P_0$.*

By $L_\infty(m_0)$ we shall denote the space of all complex-valued Borel functions on R_+ with the finite norm $\|f\|_\infty = \operatorname{vrai\,sup}_{m_0}\{|f(x)|: x \in R_+\}$. Of course, we identify two functions equal m_0-almost everywhere. We observe that the equality $\|g\|_\infty = 0$ yields $g(0) = 0$ and $\int_0^\infty |g(xy)|\,dx = 0$ for all $y \in R_+$. Then $\int_0^\infty \int_0^\infty |g(xy)|\,dx\,\mu(dy) = 0$ for any $\mu \in P$, and consequently $\int_0^\infty g(xy)\,\mu(dy) = 0$ m_0-almost everywhere. Thus for every $\mu \in P$ the formula

$$(U_\mu f)(x) = \int_0^\infty f(xy)\,\mu(dy) \quad (x \in R_+) \tag{1.5}$$

defines the operator U_μ from $L_\infty(m_0)$ into itself. Of course

$$(U_\mu f)(0) = f(0), \tag{1.6}$$

$$(U_{T_a\mu} f)(x) = (U_\mu f)(ax) \quad m_0\text{-almost everywhere}, \tag{1.7}$$

$$U_{c\mu + (1-c)\nu} = cU_\mu + (1-c)U_\nu \quad (0 \leqslant c \leqslant 1) \tag{1.8}$$

for all $\mu \in P$ and $f \in L_\infty(m_0)$. Further, by Lemma 1.3 we have the formula

$$U_{\mu\nu} = U_\mu U_\nu \quad (\mu, \nu \in P). \tag{1.9}$$

Substituting $x = e^u$ $(-\infty \leqslant u < \infty)$ into (1.5) we get, by Theorem 3.6.4 in [4], the following statement.

PROPOSITION 1.2. *For every $\mu \in P_0$ and $f \in L_\infty(m_0)$ the function $(U_\mu f)(x)$ is continuous for $x > 0$.*

As a direct consequence of the above proposition and (1.9) we obtain the following assertion.

COROLLARY 1.1. *If $\mu_n, \mu \in P$, $\nu \in P_0$, $\mu_n \to \mu$ and $\mu(\{0\}) = 0$, then, for every $f \in L_\infty(m_0)$, $U_{\nu\mu_n} f \to U_{\nu\mu} f$ pointwise. If in addition $U_\nu f$ is continuous at the origin, then the above relation holds without any restriction on $\mu(\{0\})$.*

In what follows ω_h $(h > 0)$ will denote the uniform distribution on the interval $[1, 1+h]$. Of course $\omega_h \in P_0$.

PROPOSITION 1.3. *For any $\mu \in P$ and $f \in L_\infty(m_0)$*

$$U_{\omega_h \mu} f \to U_\mu f$$

m_0-almost everywhere as $h \to 0$.

Proof. We note, by (1.6), that $(U_{\omega_h\mu} f)(0) = f(0) = (U_\mu f)(0)$. Further, by (1.9),

$$(U_{\omega_h\mu} f)(x) = \frac{1}{hx} \int_x^{x(1+h)} (U_\mu f)(u)\,du \quad \text{for } x > 0,$$

which yields our assertion.

LEMMA 1.5. *Suppose that $\mu, \nu \in P$, $\nu(\{0\}) = 0$, $f, g \in L_\infty(m_0)$, and*

$$\int_0^\infty \frac{|(U_\mu g)(x)|}{x}\,dx < \infty, \tag{1.10}$$

$$\int_0^\infty \frac{|(U_\nu g)(x)|}{x}\,dx < \infty. \tag{1.11}$$

Then

(1.12)

$$\int_0^\infty x^{-1} (U_\mu g)(x)(U_\nu f)(x^{-1})\,dx = \int_0^\infty \int_0^\infty x^{-1} f(x^{-1})(U_\nu g)(xy)\,dx\,\mu(dy).$$

Proof. Denote by I the left-hand side of formula (1.12). Then, by (1.5),

$$I = \int_0^\infty \int_0^\infty z^{-1} (U_\mu g)(z) f(z^{-1} y)\,\nu(dy)\,dz.$$

By assumption (1.10) we may change the order of integration. Setting $z = xy$

and taking into account that ν has no atom at the origin we get, by virtue of (1.9), the formula

$$I = \int_0^\infty x^{-1} f(x^{-1})(U_{\mu\nu} g)(x)\,dx = \int_0^\infty x^{-1} f(x^{-1}) \int_0^\infty (U_\nu g)(xy)\,\mu(dy)\,dx.$$

Changing, by (1.11), the order of integration we get the assertion of the lemma.

LEMMA 1.6. *Suppose that $\mu_n, \nu \in P$, $\nu(\{0\}) = 0$, $f, g \in L_\infty(m_0)$, $\mu_n \to \delta_a$ $(a > 0)$,*

$$\int_0^\infty \frac{|(U_{\mu_n} g)(x)|}{x}\,dx < \infty \quad (n = 1, 2, \ldots),$$

$$\int_0^\infty \frac{|(U_\nu g)(x)|}{x}\,dx < \infty \tag{1.13}$$

and $U_\nu(f) = 0$ m_0-almost everywhere. Then

$$\int_0^\infty x^{-1} f(x^{-1})(U_\nu g)(ax)\,dx = 0.$$

Proof. By Lemma 1.5 we have the equality

$$\int_0^\infty w(y)\,\mu_n(dy) = 0 \quad (n = 1, 2, \ldots) \tag{1.14}$$

where

$$w(y) = \int_0^\infty x^{-1} f(x^{-1})(U_\nu g)(xy)\,dx.$$

By (1.13) the function w is bounded on $\boldsymbol{R}_+$. Moreover, by Theorem 3.6.4 in [4] it is continuous on $(0, \infty)$. Consequently, (1.14) yields the equation $w(a) = 0$, which completes the proof.

2. ○-stable measures. A measure λ from P is said to be ○-*stable* if $\lambda \neq \delta_0$ and

$$T_{a_n} \mu^{\circ n} \to \lambda \tag{2.1}$$

for a measure $\mu \in P$ and a norming sequence a_n of positive numbers tending to 0; the measure μ which can arise belongs to the *domain of attraction* of λ. By S we shall denote the set of all ○-stable measures from P. The set S is nonvoid because the measure γ defined by (1.1) belongs to S. Moreover,

$$T_a S = S \quad (a > 0). \tag{2.2}$$

Given p $(0 < p \leqslant \infty)$, by S_p we shall denote the set of all measures λ from P other than δ_0 satisfying for all $a, b \in R_+$ the equality

$$T_a \lambda \circ T_b \lambda = T_{g_p(a,b)} \lambda \tag{2.3}$$

where

$$g_p(a, b) = (a^p + b^p)^{1/p} \quad (0 < p < \infty)$$

and $g_\infty(a, b) = \max(a, b)$. It is clear that the sets S_p $(0 < p \leqslant \infty)$ are disjont and

$$T_a S_p = S_p \quad (a > 0). \tag{2.4}$$

Moreover, setting $c_n = n^{-1/p}$ $(0 < p < \infty)$ we have the formula $T_{c_n} \lambda^{\circ n} = \lambda$ for $\lambda \in S_p$. This yields the inclusion

$$S_p \subset S \quad (0 < p < \infty). \tag{2.5}$$

For any pair $\mu, \nu \in P$ and $0 < p \leqslant \infty$ by $\mu *_p \nu$ we shall denote the probability distribution of $g_p(X, Y)$ where the random variables X and Y are independent and have the probability distributions μ and ν respectively. It is clear that the operations $*_p$ are generalized convolutions and

$$\delta_a *_p \delta_b = \delta_{g_p(a,b)} \quad (a, b \in R_+). \tag{2.6}$$

Moreover, for any complex number z with $\mathrm{Re}\, z < 0$, $0 < p < \infty$ and $\mu, \nu \in P$,

$$\int_0^\infty \exp(zx^p)(\mu *_p \nu)(dx) = \int_0^\infty \exp(zx^p)\,\mu(dx) \int_0^\infty \exp(zx^p)\,\nu(dx). \tag{2.7}$$

PROPOSITION 2.1. *Let $\lambda \in S_p$ $(0 < p < \infty)$. Then for any pair $\mu, \nu \in P$ the formula*

$$\lambda\mu \circ \lambda\nu = \lambda(\mu *_p \nu)$$

is true.

Proof. By (1.3) and (2.6) for any $u, v \in R_+$ we have the equalities

$$\lambda\delta_u \circ \lambda\delta_v = T_u \lambda \circ T_v \lambda = T_{g_p(u,v)} \lambda = \delta_{g_p(u,v)} \lambda = \lambda(\delta_u *_p \delta_v).$$

Applying Lemmas 1.2 and 1.3 for any pair $\mu, \nu \in P$ we get the formula

$$\lambda\mu \circ \lambda\nu = \int_0^\infty \int_0^\infty (\lambda\delta_u \circ \lambda\delta_v)\,\mu(du)\,\nu(dv) = \int_0^\infty \int_0^\infty \lambda(\delta_u *_p \delta_v)\,\mu(du)\,\nu(dv)$$
$$= \lambda(\mu *_p \nu),$$

which completes the proof.

LEMMA 2.1. *$S_\infty \neq \emptyset$ if and only if $\circ = *_\infty$. Moreover, $S_\infty = \{\delta_a \colon a > 0\} \subset S$.*

Proof. Since all measures from S_∞ are idempotents under the operation $\circ$, the sufficiency of our condition is a direct consequence of Theorems 4.1 and 4.2 in [11]. The necessity follows from the formula $S_\infty = \{\delta_a \colon a > 0\}$ for the operation $*_\infty$. To prove the inclusion $S_\infty \subset S$ it suffices, by (2.4), to show that $\delta_1 \in S$ for the operation $*_\infty$. Setting $a_1 = 1$, $a_n = (\log n)^{-1}$ $(n \geqslant 2)$, $\mu(E) = \int_{E \cap [1,\infty)} e^{1-u}\,du$, we get, by a simple calculation, $T_{a_n} \mu^{*_\infty n} \to \delta_1$, which completes the proof.

The relationship between $\circ$-stable measures and the family S_p $(0 < p \leqslant \infty)$ is given by the following statement.

PROPOSITION 2.2. *For any generalized convolution the equality*

$$S = \bigcup_p S_p$$

is true.

Proof. The inclusion $S \supset \bigcup_p S_p$ follows from (2.5) and Lemma 2.1. To prove the converse inclusion let us suppose that $\lambda \in S$ and (2.1) holds for a measure $\mu \in P$ and a norming sequence a_n tending to 0. Then $T_{a_n} \mu \to \delta_0$ and consequently $T_{a_n} \mu^{\circ(n+1)} \to \lambda$, which, by Lemma 1.1, yields $\lim_{n\to\infty} (a_{n+1}/a_n) = 1$. Since $a_n \to 0$, the above relation implies for any pair x, y of positive numbers the existence of subsequences a_{n_k} and a_{m_k} satisfying the condition

$$\lim_{k\to\infty} \frac{a_{n_k}}{a_{m_k}} = \frac{y}{x}.$$

Put $b_k = x a_{n_k}/a_{m_k}$, $d_k = x a_{n_k}/a_{n_k+m_k}$. Then

$$T_{d_k} T_{a_{n_k+m_k}} \mu^{\circ(n_k+m_k)} = T_x(T_{a_{n_k}} \mu^{\circ n_k}) \circ T_{b_k}(T_{a_{m_k}} \mu^{\circ m_k}).$$

The right-hand side of the above equality tends to $T_x \lambda \circ T_y \lambda$ as $k \to \infty$. Consequently, by Lemma 1.1, the limit $d = \lim_{k\to\infty} d_k$ exists and the left-hand side of the equality in question converges to $T_d \lambda$ as $k \to \infty$. We define the function g by setting $g(x, y) = d$ if $x, y > 0$ and $g(0, x) = g(x, 0) = x$. This function fulfils the equality

$$T_x \lambda \circ T_y \lambda = T_{g(x,y)} \lambda \quad (x, y \in R_+). \tag{2.8}$$

By the continuity of the operation $\circ$ in both variables and Lemma 1.1 we infer that the function g is continuous on $R_+ \times R_+$. Moreover, it fulfils the conditions

$$g(x, y) = g(y, x), \tag{2.9}$$

(2.10) $$g(g(x, y), z) = g(x, g(y, z)),$$

(2.11) $$g(zx, zy) = zg(x, y)$$

for all $x, y, z \in \boldsymbol{R}_+$.

Now we shall prove the inequality

(2.12) $$g(x, y) \geqslant \max(x, y) \qquad (x, y \in \boldsymbol{R}_+).$$

Suppose the contrary for a pair $a \geqslant b$, i.e. $g(a, b) < a$. Since $g(0, x) = x$, we have $b > 0$. Then, by Lemma 2.3 in [11], $g(a, b) > 0$, and consequently, by (2.11), without loss of generality we may assume that $g(a, b) = 1$. In this case we have

(2.13) $$a > 1$$

and $\lambda = T_a \lambda \circ T_b \lambda$. Setting $c = b/a$ we get the equality $\lambda = T_a(\lambda \circ T_c \lambda)$, which, by induction, yields the formula

(2.14) $$\lambda = T_{a^n}(\lambda \circ \nu_n) \qquad (n = 1, 2, \ldots)$$

where $\nu_1 = T_c \lambda$ and $\nu_{n+1} = T_{a^{-1}} \nu_n \circ T_c \lambda$.

Let μ be a limit point in $\bar{P}$ of the sequence $\lambda \circ \nu_n$ and $\lambda \circ \nu_{n_k} \to \mu$ in $\bar{P}$. By Corollary 2.4 in [11], $\mu \neq \delta_0$ because $\lambda \neq \delta_0$. Further, by (2.13) and Proposition 2.2 in [11], all limit points of the right-hand side of (2.14) belong to P_∞, which yields a contradiction. The inequality (2.12) is thus proved.

Now we shall prove that for all $x \in \boldsymbol{R}_+$

(2.15) $$g(x, y_1) \geqslant g(x, y_2) \quad \text{whenever } y_1 > y_2.$$

Suppose that $y_1 > y_2$. Then, by (2.12), $g(y_1, y_2) \geqslant y_1$. Since $g(0, y_2) = y_2$, we conclude, by the continuity of g, that there exists a number y_0 lying between 0 and y_1 and satisfying the equality $g(y_0, y_2) = y_1$. Taking into account (2.9), (2.10) and (2.12) we have the inequality

$$g(x, y_1) = g(x, g(y_0, y_2)) = g(g(x, y_2), y_0) \geqslant g(x, y_2)$$

for all $x \in \boldsymbol{R}_+$, which completes the proof of (2.15).

F. Bohnenblust proved in [2] (pp. 630–632) that any continuous function g satisfying equalities (2.9), (2.10), (2.11), inequality (2.15) and the boundary condition $g(0, x) = x$ is of the form $g = g_p$ $(0 < p \leqslant \infty)$. This, by (2.8), shows that $\lambda \in \bigcup_p S_p$, which completes the proof.

LEMMA 2.2. *No $\circ$-stable measure has an atom at the origin.*

Proof. Suppose that $\lambda \in S$. Then, by Proposition 2.2, $\lambda \in S_p$ for some p $(0 < p \leqslant \infty)$. For $p = \infty$ our assertion follows immediately from Lemma

2.1. Consider the case $p < \infty$. Then the measure λ can be written in the form $\lambda = c\delta_0 + (1-c)\lambda'$, where $\lambda' \in P$, $\lambda'(\{0\}) = 0$ and

(2.16) $$0 \leqslant c < 1.$$

Consequently,

$$\lambda \circ T_n \lambda = T_{b_n} \lambda = c\delta_0 + (1-c) T_{b_n} \lambda'$$

where $b_n = (1+n^p)^{1/p}$. This yields, by Proposition 2.1 in [11],

(2.17) $$\lambda \circ T_n \lambda \to c\delta_0 + (1-c)\delta_\infty \quad \text{in } \bar{P}.$$

On the other hand

$$\lambda \circ T_n \lambda = c^2 \delta_0 + c(1-c) T_n \lambda' + c(1-c)\lambda' + (1-c)^2 \lambda' \circ T_n \lambda'.$$

By Propositions 2.1 and 2.4 in [11], $T_n \lambda' \to \delta_\infty$ and $\lambda' \circ T_n \lambda' \to \delta_\infty$ in $\bar{P}$. Thus

$$\lambda \circ T_n \lambda \to c^2 \delta_0 + c(1-c)\lambda' + (1-c)\delta_\infty \quad \text{in } \bar{P}.$$

Comparing the above relation with (2.17) we infer that $c^2 = c$, which, by (2.16), yields $c = 0$. This shows that λ has no atom at the origin.

LEMMA 2.3. *Suppose that $\lambda_n \in S_{p_n}$, $p_n \to p > 0$ and $\lambda_n \to \lambda \neq \delta_0$ in $\bar{P}$. Then either $\lambda \in S_p$ or $\lambda(\{\infty\}) > \frac{1}{2}$.*

Proof. First suppose that $\lambda \in P$. Then using equality (2.3) for λ_n and the continuity of $\circ$ in both variables we get the corresponding equality (2.3) for λ. This shows that $\lambda \in S_p$. Suppose now that $\lambda \in P_\infty$, i.e.

$$\lambda = c\delta_\infty + (1-c)\lambda'$$

where $\lambda' \in P$ and

(2.18) $$0 < c \leqslant 1.$$

Then the measures λ_n have a representation $\lambda_n = c_n \lambda_n'' + (1-c_n)\lambda_n'$ where $c_n \to c$, $\lambda_n'' \to \delta_\infty$ and $\lambda_n' \to \lambda'$ in $\bar{P}$. Consequently,

(2.19) $$\lambda_n \circ \lambda_n = T_{2^{1/p_n}} \lambda_n \to c\delta_\infty + (1-c) T_{2^{1/p}} \lambda' \quad \text{in } \bar{P}.$$

On the other hand

(2.20) $$\lambda_n \circ \lambda_n = c_n^2 \lambda_n'' \circ \lambda_n'' + 2c_n(1-c_n)\lambda_n' \circ \lambda_n'' + (1-c_n)^2 \lambda_n' \circ \lambda_n'.$$

By Propositions 2.4 and 2.5 in [11], $\lambda_n' \circ \lambda_n'' \to \delta_\infty$, $\lambda_n' \circ \lambda_n' \to \lambda' \circ \lambda'$ and each limit point of $\lambda_n'' \circ \lambda_n''$ belongs to P_∞. Since, by (2.19), the sequence $\lambda_n \circ \lambda_n$ is convergent in $\bar{P}$ and inequality (2.18) holds, we infer by (2.20) that the sequence $\lambda_n'' \circ \lambda_n''$ is convergent to a measure ν belonging to P_∞. Thus, by (2.20),

$$\lambda_n \circ \lambda_n \to c^2 \nu + 2c(1-c)\delta_\infty + (1-c)^2 \lambda' \circ \lambda'.$$

Comparing the above relation with (2.19) we get the equality for the mass of the limit measure at ∞:

$$c = c^2 \nu(\{\infty\}) + 2c(1-c).$$

Since $\nu(\{\infty\}) > 0$, we have, by (2.18), $c > 2c(1-c)$, which yields $c > \frac{1}{2}$. Thus $\lambda(\{\infty\}) > \frac{1}{2}$, which completes the proof.

LEMMA 2.4. *If* $p_n \to p > 0$ *and* $S_{p_n} \neq \emptyset$ $(n = 1, 2, \ldots)$, *then* $S_p \neq \emptyset$.

Proof. Let $m(\mu)$ denote any median of μ. Suppose that $\lambda_n \in S_{p_n}$. Then, by Lemma 2.2 and Proposition 2.2, $m(\lambda_n) > 0$. Since $m(T_a \mu) = am(\mu)$, we may assume by (2.4) without loss of generality that $m(\lambda_n) = 1$. Passing to a subsequence if necessary we may also assume that the sequence λ_n is convergent in $\bar{P}$, say to λ. Of course, $m(\lambda) = 1$, which shows that $\lambda \neq \delta_\infty$ and $\lambda(\{\infty\}) \leqslant 1 - \lambda([0, 1]) \leqslant \frac{1}{2}$. Applying Lemma 2.3 we infer that $\lambda \in S_p$, which completes the proof.

In what follows ϱ_s $(0 < s < 1)$ will denote the measure from P with the Laplace transform $\exp(-z^s)$ $(z \in \boldsymbol{R}_+)$. In other words, ϱ_s is the stable measure with exponent s in the ordinary convolution algebra on $\boldsymbol{R}_+$. It is clear that

$$\varrho_s \to \delta_1 \quad \text{as } s \to 1. \tag{2.21}$$

Moreover, it is well known that

$$\varrho_s(E) = \int_E r_s(x^{-s}) \frac{dx}{x}$$

where r_s is an entire function ([5], Theorem 2.3.1). Hence it follows that $\varrho_s \in Q$. Let X_s be a random variable with the probability distribution ϱ_s. For any p $(0 < p < \infty)$ and q $(0 < q < p)$ by $\pi_{p,q}$ we shall denote the probability distribution of the random variable $X_{q/p}^{1/p}$. Of course $\pi_{p,q} \in Q$ and, by formula (2.7),

$$T_a \pi_{p,q} *_p T_b \pi_{p,q} = T_{g_q(a,b)} \pi_{p,q} \tag{2.22}$$

for all $a, b \in \boldsymbol{R}_+$.

LEMMA 2.5. *Let* $\lambda \in S_p$ $(0 < p < \infty)$. *Then for any* q $(0 < q < p)$

$$\lambda \pi_{p,q} \in S_q \cap Q, \quad \lambda \pi_{p,q} \to \lambda \quad \text{as } q \to p.$$

Proof. By (2.21) we have $\lambda \pi_{p,q} \to \lambda$ as $q \to p$. Put $\nu = \lambda \pi_{p,q}$. By Lemma 2.2, λ has no atom at the origin. Applying Lemma 1.4 we infer that $\nu \in Q$. Further, by Proposition 2.1 and formula (2.22) we get the equality

$$T_a \nu \circ T_b \nu = T_{g_q(a,b)} \nu$$

for all $a, b \in \boldsymbol{R}_+$. Thus $\nu \in S_q$, which completes the proof.

Remark 2.1. If $S_\infty \neq \emptyset$, then for any q $(0 < q < \infty)$ the Weibull–

Gnedenko measures

$$\sigma_q(E) = q \int_E u^{-q-1} \exp(-u^{-q})\, du$$

belong to $S_q \cap Q$.

In fact, by Lemma 2, $\circ = *_\infty$ and our assertion can be verified by a simple calculation.

We know that $S \neq \emptyset$. By Proposition 2.2, $S_p \neq \emptyset$ for some p $(0 < p \leqslant \infty)$. Now as a direct consequence of Lemmas 2.4, 2.5 and Remark 2.1 we get the following statement.

PROPOSITION 2.3. *For every generalized convolution there exists an index* $\varkappa$ $(0 < \varkappa \leqslant \infty)$ *such that* $S_p = \emptyset$ *for* $p > \varkappa$ *and* $S_p \neq \emptyset$ *for* $0 < p \leqslant \varkappa$. *Moreover,* $S_p \cap Q \neq \emptyset$ *for* $0 < p < \varkappa$.

The index $\varkappa$ will be called the *characteristic exponent* of the generalized convolution $\circ$ and denoted by $\varkappa(\circ)$.

We now proceed to the study of norming sequences.

LEMMA 2.6. *For any measure* μ *belonging to the domain of attraction of a measure* λ *from* S_p $(0 < p < \infty)$ *there exists a monotone nonincreasing norming sequence* b_n *such that* $T_{b_n} \mu^{\circ n} \to \lambda$.

Proof. Let μ belong to the domain of attraction of $\lambda \in S_p$ $(0 < p < \infty)$. Taking an arbitrary norming sequence a_n fulfilling (2.1) we put $b_n = \min\{a_j\colon j = 1, 2, \ldots, n\}$. Of course, $b_n \to 0$, $b_n \geqslant b_{n+1}$,

$$b_n \leqslant a_n \quad (n = 1, 2, \ldots), \tag{2.23}$$

$b_n = a_{j_n}$ for some j_n $(1 \leqslant j_n \leqslant n)$ and

$$T_{b_n} \mu^{\circ n} = T_{a_{j_n}} \mu^{\circ j_n} \circ T_{a_{j_n}} \mu^{\circ (n-j_n)} \quad (n = 1, 2, \ldots). \tag{2.24}$$

Here we use the notation $\mu^{\circ 0} = \delta_0$. By (2.23) the sequence b_n/a_n is conditionally compact, which yields the conditional compactness of $T_{b_n} \mu^{\circ n}$. Applying Lemma 1.1 we conclude that all its limit points are of the form $T_c \lambda$ for some numbers c from the unit interval $[0, 1]$. By (2.24) and Corollary 2.3 in [11] the sequence $T_{a_{j_n}} \mu^{\circ (n-j_n)}$ is also conditionally compact and, by Lemma 1.1, all its limit points are of the form $T_d \lambda$ for some nonnegative numbers d. Moreover, for any such number c there exists a number d such that, by (2.24),

$$T_c \lambda = \lambda \circ T_d \lambda = T_{(1+d^p)^{1/p}} \lambda.$$

Thus $c = (1+d^p)^{1/p}$, which shows that $c = 1$. In other words, $T_{b_n} \mu^{\circ n} \to \lambda$, which completes the proof.

Let s be a real number. A sequence a_n of positive numbers is said to be *regularly varying of index* s ([1], p. 94) if

$$\lim_{n\to\infty} (a_{[xn]}/a_n) = x^s \tag{2.25}$$

for every $x > 0$. The square brackets here denote the integral part.

LEMMA 2.7. *Let* $\lambda \in S_p$ $(0 < p < \infty)$. *Then each monotone nonincreasing norming sequence corresponding to any measure belonging to the domain of attraction of* λ *is regularly varying with index* $-1/p$.

Proof. It is evident that for monotone nonincreasing sequences a_n it suffices to prove (2.25) for positive rational numbers x only. Suppose that $a_n \geqslant a_{n+1}$, $a_n \to 0$ and (2.1) holds for a measure μ.

First we shall prove (2.25) with $s = -1/p$ for positive integers x. By (2.1) we have

$$T_{a_n}\mu^{\circ kn} = (T_{a_n}\mu^{\circ n})^{\circ k} \to \lambda^{\circ k} = T_{k^{1/p}}\lambda \quad (k = 1, 2, \ldots),$$

which yields

$$T_{k^{-1/p}a_n}\mu^{\circ kn} \to \lambda \quad (k = 1, 2, \ldots).$$

This, by Lemma 1.1, implies

$$\lim_{n\to\infty} (k^{-1/p} a_n/a_{kn}) = 1 \quad (k = 1, 2, \ldots).$$

Hence it follows that (2.25) is true for positive integers.

Let r be a positive integer. Put $q_n = n - r[n/r]$. Then $0 \leqslant q_n < r$ and consequently $T_{a_n}\mu^{\circ q_n} \to \delta_0$, which, by (2.1), yields

$$T_{a_n}\mu^{\circ r[n/r]} \to \lambda. \tag{2.26}$$

On the other hand

$$T_{a_{[n/r]}}\mu^{\circ r[n/r]} \to \lambda^{\circ r} = T_{r^{1/p}}\lambda,$$

which yields the relation

$$T_{r^{-1/p}a_{[n/r]}}\mu^{\circ r[n/r]} \to \lambda.$$

Comparing this with (2.26) we have, by Lemma 1.1,

$$\lim_{n\to\infty} \frac{r^{-1/p} a_{[n/r]}}{a_n} = 1 \quad (r = 1, 2, \ldots).$$

This gives (2.25) for $x = 1/r$ $(r = 1, 2, \ldots)$.

Now let x be a positive rational number k/r. Then

$$\frac{a_{[kn/r]}}{a_n} = \frac{a_{[kn/r]}}{a_{kn}} \cdot \frac{a_{kn}}{a_n} \to \left(\frac{k}{r}\right)^{-1/p},$$

which proves (2.25) for positive rational numbers. The lemma is thus proved.

From Lemma 2.7 and the corollary to Theorem 3 in [1] we get the following statement.

COROLLARY 2.1. *Let* $\lambda \in S_p$ $(0 < p < \infty)$. *Then each monotone nonincreasing norming sequence* a_n *corresponding to any measure belonging to the domain of attraction of* λ *has the properties*:

$$n^s a_n \to \infty \quad \text{for } s > 1/p,$$
$$n^s a_n \to 0 \quad \text{for } s < 1/p.$$

COROLLARY 2.2. *If* μ *belongs to the domain of attraction of a measure from* S_p $(0 < p < \infty)$, *then* $T_{n^{-s}}\mu^{\circ n} \to \delta_0$ *for all* $s > 1/p$.

Proof. By Lemma 2.6 we can take a monotone nonincreasing norming sequence a_n corresponding to the measure μ. By Corollary 2.1 we have $n^s a_n \to \infty$ for all $s > 1/p$. Now our assertion is a direct consequence of the formula

$$T_{n^{-s}}\mu^{\circ n} = T_{n^{-s}a_n^{-1}}(T_{a_n}\mu^{\circ n}).$$

PROPOSITION 2.4. *For any measure* μ *belonging to the domain of attraction of a measure from* S_p $(0 < p < \infty)$ *the moments* $\int_0^\infty x^q \mu(dx)$ *are finite for all* q *satisfying the inequality* $0 < q < p$.

Proof. Given a positive number $q < p$, we take a number s satisfying the inequality

$$1/p < s < 1/q. \tag{2.27}$$

Then, by Corollary 2.2, $T_{n^{-s}}\mu^{\circ n} \to \delta_0$, which, by Lemma 4.2 in [11], yields $\mu^n([0, n^s]) \to 1$. From the above relation we get easily

$$n\mu([n^s, \infty)) \to 0. \tag{2.28}$$

Put $r = 1/s$ and $n_k = [k^r]$ $(k = 1, 2, \ldots)$. Then, by (2.27),

$$q < r \tag{2.29}$$

and

$$n_k^s \leqslant k \quad (k = 1, 2, \ldots). \tag{2.30}$$

Moreover, by (2.28),

$$n_k \mu([n_k^s, \infty)) \to 0. \tag{2.31}$$

Since, by (2.30), $\mu([n_k^s, \infty)) \geqslant \mu([k, \infty))$ and $n_k \geqslant k^r - 1$ $(k = 1, 2, \ldots)$, the relation (2.31) yields $k^r \mu([k, \infty)) \to 0$ as $k \to \infty$. Hence and from (2.29) one can easily get the finiteness of the moment $\int_0^\infty x^q \mu(dx)$, which completes the proof.

3. Banach algebras associated with generalized convolutions. Let V be the set of all complex-valued bounded countably additive measures on R_+. As the norm $\|\alpha\|$ in V we take the total variation of α. By Lemma 1.2 for any Borel subset E of R_+ the function $\delta_u \circ \delta_v(E)$ is Borel measurable on $R_+ \times R_+$. We extend the generalized convolution $\circ$ from P onto V by setting

$$(\alpha \circ \beta)(E) = \int_0^\infty \int_0^\infty \delta_u \circ \delta_v(E)\, \alpha(du)\, \beta(dv)$$

for any pair $\alpha, \beta \in V$. It is clear that $\alpha \circ \beta \in V$ and $\|\alpha \circ \beta\| \leqslant \|\alpha\|\, \|\beta\|$.

Let V_0 denote the subset of V consisting of all set functions absolutely continuous with respect to the measure m_0. It is easy to check that V_0 is a subspace of V invariant under all transformations T_a $(a > 0)$. Moreover, T_a $(a > 0)$ are linear isometries on V_0 and for any $\alpha \in V_0$ the mapping $(0, \infty) \ni a \to T_a \alpha$ is continuous. Further, the Banach space V_0 is isomorphic to $L_1(m_0)$, and consequently each continuous linear functional l on V_0 is of the form

$$l(\alpha) = \int_0^\infty f(x)\, \alpha(dx) \quad (\alpha \in V_0) \tag{3.1}$$

where $f \in L_\infty(m_0)$.

LEMMA 3.1. *The space V_0 is closed under the generalized convolution $\circ$.*

Proof. By Proposition 2.3 there exists a measure $\sigma \in S_p \cap Q$ for some index p $(0 < p < \infty)$. From Lemma 1.2 we get the equality

$$(T_{2^{1/p}}\sigma)(E) = (\sigma \circ \sigma)(E) = \int_0^\infty \int_0^\infty (\delta_u \circ \delta_v)(E)\, \sigma(du)\, \sigma(dv) \tag{3.2}$$

for all Borel subsets E of R_+. Suppose that $m_0(E) = 0$. Since $T_{2^{1/p}}\sigma \in Q$, we have $(T_{2^{1/p}})(E) = 0$, which, by (3.2), implies $(\delta_u \circ \delta_v)(E) = 0$ for $\sigma \times \sigma$-almost all $(u, v) \in R_+ \times R_+$. Hence we conclude by Lemma 1.2 that

$$(\alpha' \circ \beta')(E) = \int_0^\infty \int_0^\infty (\delta_u \circ \delta_v)(E)\, \alpha'(du)\, \beta'(dv) = 0 \tag{3.3}$$

for any pair α', β' of set functions absolutely continuous with respect to the Lebesgue measure on R_+. Each pair $\alpha, \beta \in V_0$ has a representation $\alpha = a\delta_0 + (1-a)\alpha'$, $\beta = b\delta_0 + (1-b)\beta'$ where $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$ and the measures α', β' are absolutely continuous with respect to the Lebesgue measure on R_+. Then

$$(\alpha \circ \beta)(E) = ab\delta_0(E) + (1-a)b\alpha'(E) + a(1-b)\beta'(E) + (1-a)(1-b)(\alpha' \circ \beta')(E),$$

which, by (3.3), yields $(\alpha \circ \beta)(E) = 0$. In other words, the set function $\alpha \circ \beta$ is absolutely continuous with respect to m_0, which completes the proof.

COROLLARY 3.1. *V_0 is the Banach algebra with the unit δ_0 under the operation $\circ$.*

In what follows by H we shall denote the set of all continuous homomorphisms from the Banach algebra V_0 onto the field of complex numbers. For any $h \in H$ we have the formula $h(\alpha \circ \beta) = h(\alpha) h(\beta)$ $(\alpha, \beta \in V_0)$ and $h(\delta_0) = 1$. Since h is a linear functional we have also the representation

$$h(\alpha) = \int_0^\infty k(x)\, \alpha(dx) \quad (\alpha \in V_0) \tag{3.4}$$

where $k \in L_\infty(m_0)$. The kernel k has the following properties:

$$k(0) = 1, \tag{3.5}$$

$$\|k\|_\infty = 1. \tag{3.6}$$

Moreover, the map (3.4) from H into $L_\infty(m_0)$ is one-to-one.

In the sequel H^s will denote the subset of H consisting of symmetric homomorphisms, i.e. of homomorphisms h fulfilling the condition $h(\bar{\alpha}) = \overline{h(\alpha)}$ where the bar denotes the complex conjugate. It is clear that $h \in H^s$ if and only if the corresponding kernel k is real-valued. We always have two trivial symmetric homomorphisms h_0 and h_∞ defined by the formulas

$$h_0(\alpha) = \alpha(R_+), \quad h_\infty(\alpha) = \alpha(\{0\}) \quad (\alpha \in V_0).$$

We shall also use the notation $H^s_+ = H^s \setminus \{h_0, h_\infty\}$. Further, for any $\lambda \in S \cap P_0$ we put

$$H_\lambda = \{h\colon h \in H,\ h(T_2 \lambda - \lambda) \neq 0\}.$$

We note that, by Proposition 2.3, $S \cap P_0 \neq \emptyset$. Moreover, $h_0, h_\infty \notin H_\lambda$.

LEMMA 3.2. *For every $\lambda \in S \cap P_0$ the set H_λ is nonvoid.*

Proof. By Lemma 2.1 and Proposition 2.1 we may assume that $\lambda \in S_p \cap P_0$ for some p $(0 < p < \infty)$. Further, by Proposition 2.1, we have the formula

$$(T_2 \lambda - \lambda)^{\circ n} = \lambda(\delta_2 - \delta_1)^{*p^n},$$

which, by Lemma 1.3 and formula (2.7), yields

(3.7)
$$\int_0^\infty \exp(-x^p)(T_2\lambda - \lambda)^{\circ 2n}(dx) = \int_0^\infty \left(\exp(-2^p x^p) - \exp(-x^p)\right)^{2n} \lambda(dx)$$
$$\geqslant \int_0^\infty \exp(-2nx^p)\left(1 - \exp[(1-2^p)x^p]\right)^{2n} \lambda(dx)$$
$$\geqslant \lambda([a, b]) \exp(-2nb^p)\left(1 - \exp[(1-2^p)a^p]\right)^{2n}$$

for any pair of positive numbers $a < b$. By Lemma 2.2, $\lambda([a, b]) > 0$ for suitably chosen a and b. Then the right-hand side of (3.7) is greater than c^{2n} for a positive constant c. On the other hand the left-hand side of (3.7) is less than $\|(T_2\lambda-\lambda)^{\circ 2n}\|$. Thus

$$\|(T_2\lambda-\lambda)^{\circ 2n}\|^{1/2n} \geqslant c \quad (n=1, 2, \ldots),$$

and consequently the spectral norm of $T_2\lambda-\lambda$ is positive. This shows that $T_2\lambda-\lambda$ does not belong to the radical of V_0, which yields the assertion of the lemma.

LEMMA 3.3. *Let $\sigma\in S_p\cap Q$ $(0<p<\infty)$. Then for every $h\in H\setminus\{h_0\}$ we have either $h(T_t\sigma)=0$ for all $t>0$ or $h(T_t\sigma)=\exp(ct^p)$ $(t\geqslant 0)$ for a complex constant c with* $\operatorname{Re}c<0$.

Proof. Put $f(x)=h(T_{x^{1/p}}\sigma)$ $(x>0)$. The continuity of the mapping $(0,\infty)\ni t\to T_t\sigma$ yields the continuity of the function f on $(0,\infty)$. By (2.3) the function f fulfils the equation

$$f(x)f(y)=f(x+y) \quad (x, y>0).$$

It is well known ([4], VIII.8.1) that all solutions of the above equation not identically equal to 0 are of the form $f(x)=e^{cx}$ where c is a complex constant. Thus either $h(T_t\sigma)=0$ for all $t>0$ or $h(T_t\sigma)=\exp(ct^p)$ for a complex constant c. By (3.5), we have $\operatorname{Re}c\leqslant 0$. It remains to prove that $\operatorname{Re}c<0$. Suppose the contrary, i.e. $\operatorname{Re}c=0$. Then taking the kernel k corresponding to h, we have, by (3.4),

$$\int_0^\infty k(tx)\sigma(dx)=\exp(ibt^p) \quad (t\geqslant 0)$$

for a real constant b. Setting $t_0=1$ if $b=0$ and $t_0=(2\pi/|b|)^{1/p}$ otherwise we have $\int_0^\infty k(t_0x)\sigma(dx)=1$, which, by (3.6), yields $\operatorname{Re}k(t_0x)=1$ and consequently $k(t_0x)=1$ for σ-almost all x. Since $\sigma\in Q$, we conclude by (3.5) that $k(x)=1$ m_0-almost everywhere. This yields, by (3.4), $h(\alpha)=\alpha(\boldsymbol{R}_+)=h_0(\alpha)$ for all $\alpha\in V_0$, which contradicts the assumption. The lemma is thus proved.

We note that for $h\in H_\sigma$, $h(T_2\sigma)\neq h(\sigma)$ and consequently the function $h(T_t\sigma)$ does not vanish identically on $(0,\infty)$. As a direct consequence of Lemma 3.3 we get the following corollary.

COROLLARY 3.2. *Let $\sigma\in S_p\cap Q$ $(0<p<\infty)$. Then for every $h\in H_\sigma$, $h(T_t\sigma)=\exp(ct^p)$ $(t\geqslant 0)$ where c is a complex constant with* $\operatorname{Re}c<0$.

LEMMA 3.4. *Let $\sigma\in S_p\cap Q$ $(0<p<\infty)$. Then $\lim_{t\to 0+}h(T_t\alpha)=\alpha(\boldsymbol{R}_+)$ for all $h\in H_\sigma$ and $\alpha\in V_0$.*

Proof. By Corollary 3.2, $h(T_t\sigma)=\exp(ct^p)$ $(t\geqslant 0)$ for a complex constant c with $\operatorname{Re}c<0$. Let k be the kernel corresponding by (3.4) to h. For positive numbers ε and t we put

$$A(\varepsilon, t)=\{x:\ 1-\operatorname{Re}k(tx)\geqslant\varepsilon\}.$$

Then, by (3.4) and (3.6),

$$1-\operatorname{Re}\exp(ct^p)=\int_0^\infty(1-\operatorname{Re}k(tx))\sigma(dx)\geqslant\varepsilon\sigma(A(\varepsilon, t)),$$

which yields

$$\lim_{t\to 0+}\sigma(A(\varepsilon, t))=0 \quad (\varepsilon>0).$$

By (3.5), $0\notin A(\varepsilon, t)$. Thus

$$\lim_{t\to 0+}(\delta_0+\sigma)(A(\varepsilon, t))=0,$$

which implies

$$\lim_{t\to 0+}\mu(A(\varepsilon, t))=0$$

for all $\mu\in P_0$. The above relation, formula (3.6) and the inequality

$$1-\operatorname{Re}h(T_t\mu)=\int_0^\infty(1-\operatorname{Re}k(tx))\mu(dx)\leqslant 2\mu(A(\varepsilon, t))+\varepsilon$$

give the relation

$$\lim_{t\to 0+}h(T_t\mu)=1$$

for all $\mu\in P_0$, which by the Jordan decomposition of set functions yields the assertion of the lemma.

Our next aim is to establish some basic properties of kernels corresponding by (3.4) to homomorphisms from H_σ. In what follows K_σ will denote the set of all such kernels. As a direct consequence of Proposition 1.2, Lemma 3.4 and formula (3.5) we get the following statement.

LEMMA 3.5. *Let $\sigma\in S\cap Q$. Then for every $k\in K_\sigma$ the function $U_\mu k$ is continuous on $\boldsymbol{R}_+$ for every $\mu\in P_0$.*

LEMMA 3.6. *Let $\sigma\in S\cap Q$. Then for all $k\in K_\sigma$ and $\mu, \nu\in P$ the equality*

$$U_{\mu\circ\nu}k=(U_\mu k)(U_\nu k)$$

holds m_0-almost everywhere.

Proof. We note that for $\lambda\in P_0$, $(U_\lambda k)(t)=h(T_t\lambda)$ $(t\geqslant 0)$, which yields, by Lemma 3.5, the continuity of $U_\lambda k$ on $\boldsymbol{R}_+$. Given $\mu, \nu\in P$, we have, by Proposition 1.1, $\omega_h\mu$, $\omega_h\nu\in P_0$ where ω_h is the uniform distribution on the interval $[1, 1+h]$. Consequently,

$$U_{\omega_h\mu\circ\omega_h\nu}k = (U_{\omega_h\mu}k)(U_{\omega_h\nu}k). \tag{3.8}$$

Further, by (1.9),

$$(U_{\omega_u(\omega_h\mu\circ\omega_h\nu)}k)(x) = u^{-1}\int_1^{1+u}(U_{\omega_h\mu\circ\omega_h\nu}k)(xy)\,dy,$$

which, by (3.8), implies the equality

$$(U_{\omega_u(\omega_h\mu\circ\omega_h\nu)}k)(x) = u^{-1}\int_1^{1+u}(U_{\omega_h\mu}k)(xy)(U_{\omega_h\nu}k)(xy)\,dy.$$

Applying Proposition 1.3 to the right-hand side of the above equality as $h \to 0$ we obtain the relation

$$(U_{\omega_u(\omega_h\mu\circ\omega_h\nu)}k)(x) \to u^{-1}\int_1^{1+u}(U_\mu k)(xy)(U_\nu k)(xy)\,dy \tag{3.9}$$

as $h \to 0$. On the other hand, by the continuity of $U_{\omega_u}k$ on $\boldsymbol{R}_+$ and Corollary 1.1,

$$U_{\omega_u(\omega_h\mu\circ\omega_h\nu)}k \to U_{\omega_u(\mu\circ\nu)}k$$

as $h \to 0$. Comparing the above relation with (3.9) and applying formula (1.9) we get the equality

$$u^{-1}\int_1^{1+u}(U_{\mu\circ\nu}k)(xy)\,dy = u^{-1}\int_1^{1+u}(U_\mu k)(xy)(U_\nu k)(xy)\,dy,$$

which yields the assertion of the lemma as $u \to 0$.

LEMMA 3.7. *Let $\sigma \in S_p \cap Q$ $(0 < p < \infty)$ and $k \in \boldsymbol{K}_\sigma$. Then*

$$\int_0^1 \frac{1-\operatorname{Re}k(x)}{x^r}\,dx < \infty \quad \textit{for all } r < 1+p.$$

Proof. Put

$$I = \int_0^\infty\int_0^1 \frac{1-\operatorname{Re}k(xy)}{x^r}\,dx\,\sigma(dy).$$

Since the integrand is nonnegative, we can change the order of integration. Then, by Corollary 3.2, we obtain

$$I = \int_0^1 \frac{1-\operatorname{Re}\exp(cx^p)}{x^r}\,dx.$$

The above integral is finite for $r < 1+p$. Thus the integral

$$\int_0^1 \frac{1-\operatorname{Re}k(xy)}{x^r}\,dx$$

is finite for σ-almost all y, which yields the assertion of the lemma.

The measure γ defined by condition (1.1) is $\circ$-stable. By Proposition 2.2 there exists an index p_0 $(0 < p_0 \leqslant \infty)$ such that $\gamma \in S_{p_0}$.

LEMMA 3.8. *Let $\sigma \in S_p \cap Q$ $(0 < p < \infty)$ and $k \in K_\sigma$. The integral*

$$\int_0^1 \frac{(\operatorname{Im}k(x))^2}{x^r}\,dx$$

is finite whenever $1 < r < \min(1+p, 1+p_0)$.

Proof. Put $\varrho = \delta_1 \circ \delta_1$. By Lemma 3.6, $U_\varrho k = k^2$ m_0-almost everywhere. Thus, taking into account (3.6), we have the inequality

$$(\operatorname{Im}k)^2 = (\operatorname{Re}k)^2 - \operatorname{Re}U_\varrho k \leqslant 1 - \operatorname{Re}U_\varrho k \tag{3.10}$$

m_0-almost everywhere. From Lemma 3.7 we get the inequality

$$\int_0^1 \frac{1-\operatorname{Re}k(xy)}{x^r}\,dx \leqslant by^{r-1} \qquad (y \in \boldsymbol{R}_+)$$

for $r < 1+p$ where b is a positive constant. Consequently,

$$\int_0^1 \frac{1-\operatorname{Re}(U_\varrho k)(xy)}{x^r}\,dx = \int_0^\infty\int_0^1 \frac{1-\operatorname{Re}k(xy)}{x^r}\,dx\,\varrho(dy) \leqslant b\int_0^\infty y^{r-1}\,\varrho(dy). \tag{3.11}$$

If $p_0 = \infty$, then, by Lemma 2.1, $\circ = *_\infty$ and consequently $\varrho = \delta_1$. In this case the right-hand side of (3.11) is obviously finite. Consider the case $p_0 < \infty$. Since the measure ϱ belongs to the domain of attraction of the measure γ, we infer, by virtue of Proposition 2.4, that the right-hand side of (3.11) is finite whenever $1 < r < 1+p_0$. Now for r fulfilling the condition $1 < r < \min(1+p, 1+p_0)$ our assertion is an immediate consequence of inequality (3.10). The lemma is thus proved.

LEMMA 3.9. *Let $\sigma \in S_p \cap Q$ $(0 < p < \infty)$. Then for every $k \in K_\sigma$ the integral*

$$\int_0^1 \frac{|1-k(x)|}{x}\,dx$$

is finite.

Proof. Since, by Lemma 3.7,

$$\int_0^1 \frac{1-\operatorname{Re} k(x)}{x}\,dx$$

is finite, it suffices to prove the inequality

$$\int_0^1 \frac{|\operatorname{Im} k(x)|}{x}\,dx < \infty. \tag{3.12}$$

Let r be a real number satisfying the inequality $1 < r < \min(1+p,\, 1+p_0)$. Then, by the Schwarz inequality,

$$\left(\int_0^1 \frac{|\operatorname{Im} k(x)|}{x}\,dx\right)^2 \leqslant \int_0^1 \frac{dx}{x^{2-r}} \int_0^1 \frac{(\operatorname{Im} k(x))^2}{x^r}\,dx.$$

Taking into account the inequality $2-r < 1$ and Lemma 3.8, we conclude that both integrals on the right-hand side of the above inequality are finite. This yields (3.12), which completes the proof.

We can now formulate a result which plays a crucial role in our considerations.

PROPOSITION 3.1. *Let $\sigma \in S_p \cap Q$ $(0 < p < \infty)$. If $f \in L_\infty(m_0)$ and $U_\sigma f = 0$ m_0-almost everywhere, then $f = 0$ m_0-almost everywhere.*

Proof. By (1.6) we have $f(0) = 0$. Consequently, to prove our assertion it suffices to show that $f = 0$ almost everywhere with respect to the Lebesgue measure on $\boldsymbol{R}_+$.

We define an auxiliary function F analytic in the half-plane $\operatorname{Re} z < 0$ by setting

$$F(z) = \int_0^\infty f(x^{-1})\,x^{p-1}\exp(zx^p)\,dx.$$

By Lemma 3.2 the set H_σ is nonvoid. Taking a kernel k from K_σ and an arbitrary positive number b we put $g(x) = k(x) - k\big((1+b)^{1/p}x\big)$ $(x \in \boldsymbol{R}_+)$. Of course, $g \in L_\infty(m_0)$. Further, for any positive number a we put $\mu_n = T_{1/n}\sigma \circ \delta_a$ $(n = 1, 2, \ldots)$. Finally, setting $\nu = \sigma$ we shall show that the conditions of Lemma 1.6 are fulfilled. The conditions $\sigma(\{0\}) = 0$ and $\mu_n \to \delta_a$ are obvious. By Corollary 3.2

$$(U_\sigma g)(x) = \exp(cx^p) - \exp[c(1+b)x^p] \tag{3.13}$$

for a complex constant c with $\operatorname{Re} c < 0$, which yields the inequality

$$\int_0^\infty \frac{(U_\sigma g)(x)}{x}\,dx < \infty.$$

By Lemma 3.6 we have the formula

$$(U_{\mu_n} g)(x) = \exp(cn^{-p}x^p)\,k(ax) - \exp[c(1+b)n^{-p}x^p]\,k\big(a(1+b)^{1/p}x\big),$$

which yields the inequality

$$\begin{aligned}|(U_{\mu_n} g)(x)| \leqslant{}& |1-k(ax)|\exp(cn^{-p}x^p)\\ &+\big|1-k\big(a(1+b)^{1/p}x\big)\big|\exp[c(1+b)n^{-p}x^p]\\ &+|1-\exp(cbn^{-p}x^p)|\exp(cn^{-p}x^p).\end{aligned}$$

Applying Lemma 3.9 we infer that the integral

$$\int_0^\infty \frac{|(U_{\mu_n} g)(x)|}{x}\,dx$$

is finite. Thus we have proved that all conditions of Lemma 1.6 are fulfilled. Now, by virtue of this lemma, we get the equality

$$\int_0^\infty x^{-1} f(x^{-1})(U_\sigma g)(ax)\,dx = 0$$

for all $a > 0$. By (3.13) the above equality can be written in the form

$$\int_0^\infty x^{-1} f(x^{-1})\exp(ca^p x^p)[1-\exp(cba^p x^p)]\,dx = 0$$

for all $a, b > 0$. Dividing the left-hand side of this equality by b, changing the order of integration and passing to the limit as $b \to 0$, which is of course justified, we get the equality $F(ca^p) = 0$ for all $a > 0$. This shows that the function F vanishes in the half-plane $\operatorname{Re} z < 0$. Now our assertion is an immediate consequence of the Uniqueness Theorem for the Laplace transform.

LEMMA 3.10. *Let $\sigma \in S_p \cap Q$ $(0 < p < \infty)$. Then $H_\sigma = H \setminus \{h_0, h_\infty\}$.*

Proof. Suppose that $h \neq h_0$ and $h \notin H_\sigma$. Then, by Lemma 3.3, $h(T_t\sigma) = 0$ for $t > 0$. Let k be the kernel corresponding to h. Then $(U_\sigma k)(t) = 0$ for $t > 0$ and, by (3.5), $(U_\sigma k)(0) = 1$. The kernel k_∞ corresponding to h_∞ is simply the indicator of the one-point set $\{0\}$ and $(U_\sigma k_\infty)(t) = 0$ for $t > 0$, $(U_\sigma k_\infty)(0) = 1$. Thus $U_\sigma k = U_\sigma k_\infty$, which, by Proposition 3.1, yields $k = k_\infty$ m_0-almost everywhere or, equivalently, $h = h_\infty$. The lemma is thus proved.

From Lemma 3.10 we get immediately the following corollary.

COROLLARY 3.3. *For any* $\sigma \in S_p \cap Q$ $(0 < p < \infty)$ *the equality* $H^s_+ = H^s \cap H_\sigma$ *is true.*

An extension of the operation $(\mu, \nu) \to \mu\nu$ $(\mu, \nu \in P)$ to the space V can be defined by the formula

$$\int_0^\infty f(x)(\alpha\beta)(dx) = \int_0^\infty \int_0^\infty f(xy)\alpha(dx)\beta(dy) \tag{3.14}$$

for all pairs $\alpha, \beta \in V$ and all bounded Borel functions on $\boldsymbol{R}_+$. From Proposition 1.1 one can easily get the relation $\alpha\beta \in V_0$ for all $\alpha \in V_0$ and $\beta \in V$.

PROPOSITION 3.2. *Let* $\sigma \in S_p \cap Q$ $(0 < p < \infty)$. *Then the set* $\{\sigma\alpha\colon \alpha \in V\}$ *is dense in* V_0.

Proof. Since the set $\{\sigma\alpha\colon \alpha \in V\}$ is closed under linear combinations it suffices to prove that each linear continuous functional l on V_0 vanishing on this set vanishes identically on V_0. By (3.1)

$$l(\beta) = \int_0^\infty f(y)\beta(dy) \qquad (\beta \in V_0)$$

for some $f \in L_\infty(m_0)$. Thus $l(\sigma\delta_x) = (U_\sigma f)(x)$ $(x \in \boldsymbol{R}_+)$, which yields $U_\sigma f = 0$ on $\boldsymbol{R}_+$. Applying Proposition 3.1 we infer that $f = 0$ m_0-almost everywhere, which yields our assertion.

LEMMA 3.11. *Let* $\sigma \in S_p \cap Q$ $(0 < p < \infty)$. *Then* $\lim_{t\to\infty} h(T_t\alpha) = \alpha(\{0\})$ *for all* $h \in H_\sigma$ *and* $\alpha \in V_0$.

Proof. By Corollary 3.2 we have the formula $h(T_t\sigma) = \exp(ct^p)$ $(t \in \boldsymbol{R}_+)$ for a complex constant c with $\operatorname{Re} c < 0$. Let k be the kernel corresponding to h. Then, by (3.14),

$$h(T_t\sigma\beta) = \int_0^\infty k(tx)(\sigma\beta)(dx) = \int_0^\infty \exp(ct^p y^p)\beta(dy)$$

for any $\beta \in V$. The right-hand side of the above formula tends to $\beta(\{0\})$ as $t \to \infty$. But $\beta(\{0\}) = (\sigma\beta)(\{0\})$ because σ has no atom at the origin. Hence it follows that our assertion is true on the set $\{\sigma\beta\colon \beta \in V\}$. Taking into account the boundedness of the norm $\|h(T_t\cdot)\| \leqslant 1$ and Proposition 3.2 we get, by virtue of the Banach–Steinhaus Theorem ([4], Theorem 2.12.1), the assertion of the lemma.

PROPOSITION 3.3. *Let* $\sigma \in S_p \cap Q$ $(0 < p < \infty)$. *Then* $H^s \cap H_\sigma \neq \emptyset$.

Proof. By Lemma 3.2 the set H_σ is nonvoid. Let h be an arbitrary homomorphism from H_σ. By Corollary 3.2 we have the formula $h(T_t\sigma) = \exp(ct^p)$ for a complex constant c with $\operatorname{Re} c < 0$. Moreover, by (3.14)

$$h(T_t\sigma\alpha) = \int_0^\infty \exp(ct^p x^p)\alpha(dx) \qquad (t \in \boldsymbol{R}_+) \tag{3.15}$$

for all $\alpha \in V$.

First consider the case $\operatorname{Im} c = 0$. Then, by (3.15),

$$h(\overline{\sigma\alpha}) = h(\sigma\bar{\alpha}) = \overline{h(\sigma\alpha)} \qquad (\alpha \in V),$$

which, by Proposition 3.2, shows that $h \in H^s$.

Suppose now that $\operatorname{Im} c \neq 0$. We note that the equality $\sigma\alpha = \sigma\beta$ $(\alpha, \beta \in V)$ yields $\alpha = \beta$. In fact, setting

$$f_\alpha(x) = \int_0^\infty f(xy)\alpha(dy), \qquad f_\beta(x) = \int_0^\infty f(xy)\beta(dy)$$

for any continuous bounded function f we have, by (3.14), $U_\sigma f_\alpha = U_\sigma f_\beta$, which, by Proposition 3.1, yields $f_\alpha = f_\beta$. Since the function f was chosen arbitrarily, we get the equality $\alpha = \beta$. This property enables us to define linear functionals on $\{\sigma\alpha\colon \alpha \in V\}$ by means of the formula

$$l(\sigma\alpha, z) = \int_0^\infty \exp(zx^p)\alpha(dx) \qquad (\alpha \in V)$$

for all complex numbers z with $\operatorname{Re} z < 0$. For any $\alpha \in V$ the function $l(\sigma\alpha, \cdot)$ is analytic and bounded by $\|\alpha\|$ in the half-plane $\operatorname{Re} z < 0$. It is also continuous on the line $\operatorname{Re} z = 0$. Moreover, by Proposition 2.1 and formula (2.7), the set $\{\sigma\alpha\colon \alpha \in V\}$ is closed under the convolution $\circ$ and

$$l(\sigma\alpha \circ \sigma\beta, z) = l(\sigma\alpha, z)\, l(\sigma\beta, z) \tag{3.16}$$

for all $\alpha, \beta \in V$ in the half-plane $\operatorname{Re} z < 0$. By (3.15) we have the formulas

$$l(\sigma\alpha, ct) = h(T_{t^{1/p}}\sigma\alpha), \qquad l(\sigma\alpha, \bar{c}t) = \overline{h(T_{t^{1/p}}\sigma\bar{\alpha})}$$

for all $t \in \boldsymbol{R}_+$, which, by (3.4) and (3.5), imply the inequalities

$$|l(\sigma\alpha, ct)| \leqslant \|\sigma\alpha\|, \tag{3.17}$$

$$|l(\sigma\alpha, \bar{c}t)| \leqslant \|\sigma\bar{\alpha}\| = \|\sigma\alpha\| \tag{3.18}$$

for all $t \in \boldsymbol{R}_+$. Put

$$l_1 = \{ct\colon t \in \boldsymbol{R}_+\}, \qquad l_2 = \{\bar{c}t\colon t \in \boldsymbol{R}_+\}.$$

Since $\operatorname{Re} c < 0$, the angle between the half-lines l_1 and l_2 is less than π. Let D be the angular domain contained in the half-plane $\operatorname{Re} z < 0$ with the boundary $I_1 \cup I_2$. The function $l(\sigma\alpha, \cdot)$ is bounded on D. On the half-lines I_1 and I_2 we have the estimates (3.17) and (3.18). Applying the Phragmen–Lindelöf Theorem ([7], 5.61) we get the inequality

$$|l(\sigma\alpha, z)| \leqslant \|\sigma\alpha\|$$

for all $\alpha \in V$ and $z \in D$. We note that $-1 \in D$. Setting $g(\sigma\alpha) = l(\sigma\alpha, -1)$ we get, by (3.16), a linear and multiplicative functional on $\{\sigma\alpha: \alpha \in V\}$ satisfying the conditions $|g(\sigma\alpha)| \leqslant \|\sigma\alpha\|$ and $g(\sigma\alpha) = \overline{g(\sigma\alpha)}$ $(\alpha \in V)$. By Proposition 3.2 it can be extended to a symmetric homomorphism g of V_0 onto the field of complex numbers. Since $g(T_t\sigma) = \exp(-t^p)$, we infer that $g \in H_\sigma$, which completes the proof.

Propositions 2.3, 3.3 and Corollary 3.3 imply the following statement:

COROLLARY 3.4. $H^s_+ \neq \emptyset$.

The next proposition states an important property of H^s_+.

PROPOSITION 3.4. *Let h be an arbitrary homomorphism from H^s_+. Then*

$$H^s_+ = \{h(T_a \cdot): 0 < a < \infty\} \tag{3.19}$$

and the correspondence

$$(0, \infty) \ni a \to h(T_a \cdot) \in H^s_+$$

is one-to-one.

Proof. Let k and k' be the kernels corresponding to homomorphisms h and h' from H^s_+ respectively. By Proposition 2.3 there exists an index p $(0 < p < \infty)$ such that $S_p \cap Q \neq \emptyset$. Let $\sigma \in S_p \cap Q$. Then, by Corollaries 3.2 and 3.3, $(U_\sigma k)(t) = \exp(-ct^p)$ and $(U_\sigma k')(t) = \exp(-c' t^p)$ $(t \in \boldsymbol{R}_+)$ for some positive constants c and c'. Setting $a = (c'/c)^{1/p}$ and $k_a(x) = k(ax)$, we have the equality

$$(U_\sigma k_a)(t) = \exp(-c' t^p) = (U_\sigma k')(t) \quad (t \in \boldsymbol{R}_+),$$

which, by Proposition 3.1, yields $k_a = k'$ m_0-almost everywhere. In other words $h' = h(T_a \cdot)$. Formula (3.19) is thus proved. Suppose now that $h(T_a \cdot) = h(T_b \cdot)$. Then $\exp(-ca^p) = h(T_a\sigma) = h(T_b\sigma) = \exp(-cb^p)$ and consequently $a = b$, which completes the proof.

In the sequel we shall use the notation $h(T_0\alpha) = h_0(\alpha)$ and $h(T_\infty\alpha) = h_\infty(\alpha)$ for all $\alpha \in V_0$ and $h \in H^s_+$. As a consequence of Corollary 3.3, Proposition 3.4 and Lemmas 3.4 and 3.11 we get the following corollary.

COROLLARY 3.5. *Let h be an arbitrary homomorphism from H^s_+. Then*

$$H^s = \{h(T_a \cdot): a \in \bar{\boldsymbol{R}}_+\}$$

and the correspondence

$$\bar{\boldsymbol{R}}_+ \ni a \to h(T_a \cdot) \in H^s$$

is one-to-one. Moreover, for any $\alpha \in V_0$ the mapping $\bar{\boldsymbol{R}}_+ \ni a \to h(T_a\alpha)$ is continuous.

A continuous linear functional l on V_0 is said to be *positive* if $l(\alpha \circ \bar{\alpha}) \geqslant 0$ for all $\alpha \in V_0$.

PROPOSITION 3.5. *Each positive continuous linear functional l on V_0 has for any $h \in H^s_+$ the representation*

$$l(\alpha) = \int_{\bar{\boldsymbol{R}}_+} h(T_t\alpha)\,\varrho(dt)$$

where ϱ is a bounded Borel measure on $\bar{\boldsymbol{R}}_+$.

Proof. The space H^s of symmetric homomorphisms of the Banach algebra V_0 can be identified with the space of all symmetric maximal ideals of V_0 ([6], Corollaries I and II, pp. 228, 229). Let $h \in H^s_+$. Then, by Corollary 3.5, the mapping

$$\bar{\boldsymbol{R}}_+ \ni t \to h(T_t \cdot) \in H^s$$

generates a topology on H^s such that H^s is compact and for each $\alpha \in V_0$ the functions $h(T_t\alpha)$ $(t \in \bar{\boldsymbol{R}}_+)$ are continuous. The space H^s with this topology is homeomorphic to $\bar{\boldsymbol{R}}_+$. Hence it follows, by Theorem 3 in [6], p. 234, that the space of all symmetric maximal ideals of V_0 with the natural topology is also homeomorphic to $\bar{\boldsymbol{R}}_+$. Applying the representation theorem for positive functionals ([6], Theorem 3, p. 323) and Corollary 3.5 we get our statement.

4. Weak characteristic functions. The concept of characteristic function for generalized convolutions has been introduced in [8]. We say that the generalized convolution $\circ$ *admits a characteristic function* if there exists a one-to-one correspondence $\mu \leftrightarrow \tilde{\mu}$ between measures μ from P and real-valued bounded continuous functions $\tilde{\mu}$ on $\boldsymbol{R}_+$ such that, for all $\mu, \nu \in P$, $(c\mu + (1-c)\nu)^\sim = c\tilde{\mu} + (1-c)\tilde{\nu}$ $(0 \leqslant c \leqslant 1)$, $(T_a\mu)^\sim(t) = \tilde{\mu}(at)$ $(a > 0)$, $(\mu \circ \nu)^\sim = \tilde{\mu}\tilde{\nu}$ and the uniform convergence $\tilde{\mu}_n \to \tilde{\mu}$ on every compact subset of $\boldsymbol{R}_+$ is equivalent to the convergence $\mu_n \to \mu$. The function $\mu \to \tilde{\mu}$ is called a *characteristic function* for the generalized convolution $\circ$. It has been proved in [8], Theorem 3 that $\circ$ admits a characteristic function if and only if it is regular, i.e. there exists a nonconstant continuous homomorphism from the generalized convolution algebra in question into the algebra of real numbers with operations of convex combinations and multiplication. The generalized convolution $*_\infty$ is not regular ([8], p. 219). Another example of nonregular generalized convolutions is given in [10].

Our aim is to introduce a substitute of characteristic functions for all generalized convolutions.

We say that the generalized convolution $\circ$ *admits a weak characteristic function* if there exists a one-to-one correspondence $\mu \leftrightarrow \tilde{\mu}$ between measures

μ from P and real-valued functions $\hat{\mu}$ from $L_\infty(m_0)$ such that the functions $\hat{\mu}$ are continuous for $\mu \in P_0$, and

$$(4.1)\qquad (c\mu+(1-c)\nu)\hat{} = c\hat{\mu}+(1-c)\hat{\nu} \quad (0 \leqslant c \leqslant 1),$$

$$(4.2)\qquad (T_a\mu(t))\hat{} = \hat{\mu}(at) \quad (a > 0),$$

$$(4.3)\qquad (\mu \circ \nu)\hat{} = \hat{\mu}\hat{\nu}$$

for all μ, $\nu \in P$. Here the equalities between functions from $L_\infty(m_0)$ are taken m_0-almost everywhere. We also assume that for every $\mu \in P$

$$(4.4)\qquad (\omega_h \mu)\hat{} \to \hat{\mu}$$

m_0-almost everywhere as $h \to 0$ where ω_h is the uniform distribution on the interval $[1, 1+h]$ and, moreover, the pointwise convergence $(\omega_h \mu_n)\hat{} \to (\omega_h \mu)\hat{}$ on R_+ for all $h > 0$ is equivalent to the convergence $\mu_n \to \mu$. The function $\mu \to \hat{\mu}$ is called *a weak characteristic function* for the generalized convolution $\circ$.

We start with proving some simple properties of weak characteristic functions.

LEMMA 4.1. *The equality $\hat{\delta}_0(t) = 1$ holds for all $t \in R_+$.*

Proof. Since $\delta_0 \in P_0$, the function $\hat{\delta}_0$ is continuous and, by (4.3), $\hat{\delta}_0\hat{\delta}_0 = \hat{\delta}_0$ everywhere on R_+. Thus either $\hat{\delta}_0(t) = 1$ for all $t \in R_+$ or $\hat{\delta}_0(t) = 0$ for all $t \in R_+$. The last case is impossible because then, by (4.3), $\hat{\mu} = \hat{\mu}\hat{\delta}_0 = 0$ m_0-almost everywhere, which contradicts the assumption that the correspondence $\mu \leftrightarrow \hat{\mu}$ is one-to-one. This completes the proof.

LEMMA 4.2. *For all $\mu \in P$ and $h > 0$ the formula*

$$(\omega_h \mu)\hat{} = U_\mu \hat{\omega}_h$$

is true everywhere on R_+.

Proof. First we note that $\omega_h \in P_0$ and consequently, by Proposition 1.1, $\omega_h \mu \in P_0$ for all $\mu \in P$. Thus both functions $\hat{\omega}_h$ and $(\omega_h \mu)\hat{}$ are continuous, which yields the continuity of $U_\mu \hat{\omega}_h$.

For measures μ concentrated on a finite set our assertion is obvious. In fact, if $\mu = \sum_{j=1}^{m} \delta_{a_j} p_j$ where $a_j, p_j \in R_+$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{m} p_j = 1$, then, by (1.3) and (1.4),

$$\omega_h \mu = \sum_{j=1}^{m} \omega_h \delta_{a_j} p_j = \sum_{j=1}^{m} T_{a_j} \omega_h p_j.$$

Taking into account properties (4.1), (4.2) and the continuity of $(\omega_h \mu)\hat{}$ and $U_\mu \hat{\omega}_h$ we have

$$(\omega_h \mu)\hat{}(t) = \sum_{j=1}^{m} \hat{\omega}_h(a_j t) p_j = (U_\mu \hat{\omega}_h)(t)$$

for all $t \in R_+$. Approximating an arbitrary measure from P by measures concentrated on finite sets we get, by Corollary 1.1, the assertion of the lemma.

LEMMA 4.3. *For any $\mu \in P$ we have $\hat{\mu}(0) = 1$.*

Proof. For every $\mu \in P$, by (4.4),

$$(4.5)\qquad (\omega_h \mu)\hat{}(0) \to \hat{\mu}(0)$$

as $h \to 0$ because the measure m_0 has a positive mass at the origin. Further, by Lemma 4.1,

$$(4.6)\qquad (\omega_h \mu)\hat{}(0) = \hat{\omega}_h(0)$$

for every $\mu \in P$. In particular, $\hat{\delta}_0(0) = (\omega_h \delta_0)\hat{}(0) = \hat{\omega}_h(0)$, which, by Lemma 4.1, gives the formula $\hat{\omega}_h(0) = 1$ $(h > 0)$. Now our assertion follows directly from (4.5) and (4.6).

LEMMA 4.4. *For any $\mu \in P$ we have $\|\hat{\mu}\|_\infty = 1$.*

Proof. For any $\mu \in P$ we have, by Lemma 4.3, the inequality $\|\hat{\mu}\|_\infty \geqslant 1$. Suppose that $c = \|\hat{\nu}\|_\infty > 1$ for a measure $\nu \in P$. Then

$$(4.7)\qquad \left\| \frac{\hat{\nu}^2}{c^2 - \hat{\nu}^2} \right\|_\infty = \infty.$$

Put

$$\lambda = (c^2-1) \sum_{n=1}^{\infty} \frac{\nu^{\circ n}}{c^{2n}}.$$

Of course, $\lambda \in P$ and

$$\lambda = \nu^{\circ 2} \circ \left(\frac{c^2-1}{c^2} \delta_0 + \frac{1}{c^2} \lambda \right),$$

which, by (4.1), (4.2) and Lemma 4.1, yields the equality

$$\hat{\lambda} = \hat{\nu}^2 \left(\frac{c^2-1}{c^2} + \frac{1}{c^2} \hat{\lambda} \right)$$

m_0-almost everywhere. Consequently $\hat{\lambda}(c^2 - \hat{\nu}^2) = (c^2-1)\hat{\nu}^2$ m_0-almost everywhere. This shows that if $c^2 - \hat{\nu}^2 = 0$ on a set A, then $\hat{\nu}^2 = 0$ m_0-almost everywhere on A, which yields $m_0(A) = 0$. Thus $c^2 - \hat{\nu}^2 > 0$ m_0-almost everywhere on R_+ and consequently

$$\hat{\lambda} = (c^2-1) \frac{\hat{\nu}^2}{c^2 - \hat{\nu}^2}$$

m_0-almost everywhere. But this contradicts formula (4.7) because $\hat{\lambda} \in L_\infty(m_0)$. The lemma is thus proved.

PROPOSITION 4.1. *For every* $\mu \in P$ *the formula*

$$\hat{\mu} = U_\mu \hat{\delta}_1$$

is true m_0*-almost everywhere.*

Proof. By (4.4), $\hat{\omega}_u = (\omega_u \delta_1)\hat{} \to \hat{\delta}_1$ m_0-almost everywhere as $u \to 0$. Since, by Lemma 4.2, $(\omega_h \omega_u)\hat{} = U_{\omega_h} \hat{\omega}_u$ and, by Lemma 4.4, $\|\omega_u\|_\infty = 1$, we have $(\omega_h \omega_u)\hat{} \to U_{\omega_h} \hat{\delta}_1$ as $u \to 0$ because $\omega_h \in P_0$. On the other hand, by (4.4), $(\omega_h \omega_u)\hat{} \to \hat{\omega}_h$ m_0-almost everywhere as $u \to 0$. Thus $\hat{\omega}_h = U_{\omega_h} \hat{\delta}_1$ m_0-almost everywhere on $\boldsymbol{R}_+$. Applying (1.9) and Lemma 4.2 we obtain for any $\mu \in P$ the equality

$$(\omega_h \mu)\hat{} = U_\mu (U_{\omega_h} \hat{\delta}_1) = U_{\omega_h \mu} \hat{\delta}_1$$

m_0-almost everywhere. By Proposition 1.2 the right-hand side of the above formula tends to $U_\mu \hat{\delta}_1$ m_0-almost everywhere as $h \to 0$. By (4.4) the left-hand side tends to $\hat{\mu}$ m_0-almost everywhere as $h \to 0$, which yields the assertion of the lemma.

As a direct consequence of Proposition 4.1 and formula (1.9) we get the following statement.

COROLLARY 4.1. *For any pair* $\mu, \nu \in P$ *we have*

$$(\mu\nu)\hat{} = U_\mu \hat{\nu}$$

m_0*-almost everywhere.*

The above formula and the continuity of weak characteristic functions for measures belonging to P_0 yield, by Corollary 1.1, the following result.

COROLLARY 4.2. *If* $\mu, \mu_n \in P$, $\nu \in P_0$ *and* $\mu_n \to \mu$, *then* $(\nu\mu_n)\hat{} \to (\nu\mu)\hat{}$ *pointwise on* $\boldsymbol{R}_+$.

PROPOSITION 4.2. *For any* $\mu \in P$ *we have the relation*

$$\lim_{t \to 0} t^{-1} \int_0^t \hat{\mu}(u)\, du = 1.$$

Proof. Let ω be the uniform distribution on the unit interval [0, 1]. For any $\mu \in P$ we have, by Corollary 4.1,

$$(\omega\mu)\hat{}(t) = (U_\omega \hat{\mu})(t) = t^{-1} \int_0^t \hat{\mu}(u)\, du$$

m_0-almost everywhere for $t > 0$. The right-hand side of the above formula is continuous for $t > 0$. Since $\omega \in P_0$ and consequently, by Proposition 1.1, $\omega\mu \in P_0$, the left-hand side is continuous for $t \geqslant 0$ and, by Lemma 4.3, $(\omega\mu)\hat{}(0) = 1$, which yields our assertion.

Our next aim is to establish a relationship between weak characteristic functions and homomorphisms from H_+^s.

PROPOSITION 4.3. *Each weak characteristic function* $\hat{\mu}$ *induces a homomorphism* h *from* H_+^s *by means of the formula*

$$h(\alpha) = \int_0^\infty \hat{\delta}_1(x)\, \alpha(dx) \quad (\alpha \in V_0). \tag{4.8}$$

Proof. Since $\hat{\delta}_1 \in L_\infty(m_0)$, formula (4.8) defines a continuous linear functional on V_0. Using condition (4.3), the continuity of $\hat{\mu}$ for $\mu \in P_0$ and Proposition 4.1 we get the multiplicativity of h. Moreover, by Lemma 4.1, $h(\delta_0) = 1$. Since the function $\hat{\delta}_1$ is real-valued, we have $h(\bar{\alpha}) = \overline{h(\alpha)}$ for all $\alpha \in V_0$. Thus $h \in H^s$. Observing that $\hat{\mu}(t) = h(T_t \mu)$ $(t > 0,\ \mu \in P_0)$ and taking into account that the correspondence $\mu \leftrightarrow \hat{\mu}$ is one-to-one we obtain $h \neq h_0$ and $h \neq h_\infty$. Consequently $h \in H_+^s$, which completes the proof.

THEOREM 4.1. *Each generalized convolution admits a weak characteristic function. The kernel* k *corresponding to a homomorphism from* H_+^s *defines a weak characteristic function by means of the formula*

$$\hat{\mu} = U_\mu k \quad (\mu \in P). \tag{4.9}$$

Proof. We have, by Corollary 3.4, $H_+^s \neq \emptyset$. Let k be the kernel corresponding to a homomorphism from H_+^s. Defining for any $\mu \in P$ the function $\hat{\mu}$ by formula (4.9), we infer that $\hat{\mu} \in L_\infty(m_0)$ and, by Corollary 3.3 and Lemma 3.5, that $\hat{\mu}$ is continuous on $\boldsymbol{R}_+$ for $\mu \in P_0$. Conditions (4.1) and (4.2) are evident. Condition (4.3) follows immediately from Corollary 3.3 and Lemma 3.6. Condition (4.4) is a consequence of Proposition 1.3. Since $\omega_h \in P_0$, and consequently $\hat{\omega}_h$ is continuous, we infer by virtue of Corollary 1.1 that the convergence $\mu_n \to \mu$ $(\mu, \mu_n \in P)$ yields the pointwise convergence $(\omega_h \mu_n)\hat{} \to (\omega_h \mu)\hat{}$ on $\boldsymbol{R}_+$ for all $h > 0$.

By Proposition 2.3 there exists a measure σ belonging to $S_p \cap Q$ for some p $(0 < p < \infty)$. By Corollaries 3.2 and 3.3, $\hat{\sigma}(t) = \exp(-bt^p)$ for a positive constant b. Suppose now that $\mu, \mu_n \in P$ and $(\omega_h \mu_n)\hat{} \to (\omega_h \mu)\hat{}$ pointwise on $\boldsymbol{R}_+$ for all $h > 0$. By (3.6), $\|(\omega_h \mu_n)\hat{}\|_\infty \leqslant 1$ $(n = 1, 2, \ldots)$. Thus $U_\sigma(\omega_h \mu_n)\hat{} \to U_\sigma(\omega_h \mu)\hat{}$ pointwise on $\boldsymbol{R}_+$ for all $h > 0$, which, by formula (1.9), yields the pointwise convergence $U_{\omega_h \mu_n} \hat{\sigma} \to U_{\omega_h \mu} \hat{\sigma}$ for all $h > 0$. The last result can be written as follows:

$$\int_0^\infty \exp(-bt^p x^p)(\omega_h \mu_n)(dx) \to \int_0^\infty \exp(-bt^p x^p)(\omega_h \mu)(dx)$$

for all $t \in \boldsymbol{R}_+$ and $h > 0$. This yields, by the well-known properties of the Laplace transform, the convergence $\omega_h \mu_n \to \omega_h \mu$ for all $h > 0$. Hence it follows that the sequence μ_n is conditionally compact in P and each its limit point λ fulfils the equality $\omega_h \lambda = \omega_h \mu$ for all $h > 0$ ([9], Proposition 1.1 and 1.2). Since $\omega_h \to \delta_1$ as $h \to 0$, we finally get the equality $\lambda = \mu$, which shows that $\mu_n \to \mu$.

It remains to prove that the correspondence $\mu \leftrightarrow \hat{\mu}$ is one-to-one. Suppose that $\hat{\mu} = \hat{\nu}$ m_0-almost everywhere. Then $U_\sigma \hat{\mu} = U_\sigma \hat{\nu}$, which, by (1.9) and (4.9), implies the equality $U_\mu \hat{\sigma} = U_\nu \hat{\sigma}$. This equality can be written in the form

$$\int_0^\infty \exp(-bt^p x^p)\,\mu(dx) = \int_0^\infty \exp(-bt^p x^p)\,\nu(dx) \qquad (t \in \boldsymbol{R}_+),$$

which, by the Uniqueness Theorem for the Laplace transform, yields $\mu = \nu$. The theorem is thus proved.

By Proposition 4.1 every weak characteristic function $\mu \leftrightarrow \hat{\mu}$ is uniquely determined by its value $\hat{\delta}_1$. From Proposition 4.3 and Theorem 4.1 we get the following corollary.

COROLLARY 4.3. *The set of all weak characteristic functions $\hat{\delta}_1$ coincides with the set of kernels corresponding to homomorphisms from H^s_+.*

Let $\mu \to \hat{\mu}$ be a weak characteristic function. It is evident that, for any $c > 0$, $\mu \to (T_c \mu)\hat{}$ is also a weak characteristic function. Two weak characteristic functions $\mu \to \hat{\mu}$ and $\mu \to \hat{\mu}'$ are said to be *similar* if there exists a positive number c such that $\hat{\mu}' = (T_c \mu)\hat{}$ m_0-almost everywhere for any $\mu \in P$.

As a direct consequence of Proposition 3.4 and Corollary 4.3 we get the following result.

COROLLARY 4.4. *All weak characteristic functions of a generalized convolution are similar.*

We proceed now to a description of $\circ$-stable measures in terms of weak characteristic functions.

THEOREM 4.2. *Suppose that $\varkappa(\circ) < \infty$. Then a probability measure λ is $\circ$-stable if and only if $\hat{\lambda}(t) = \exp(-ct^p)$ m_0-almost everywhere for some positive constants c and p such that $p \leqslant \varkappa(\circ)$.*

Proof. Sufficiency. Suppose that $\hat{\lambda}(t) = \exp(-ct^p)$ m_0-almost everywhere for some positive constants c and p. Using formulas (4.2) and (4.3) we get equality (2.3) which, by inclusion (2.5), shows that the measure λ is $\circ$-stable.

Necessity. Suppose that λ is $\circ$-stable. Then, by Propositions 2.2 and 2.3, $\lambda \in S_p$ for an index p fulfilling the inequality $0 < p \leqslant \varkappa(\circ)$. By Lemma 2.5 for every q satisfying the condition $0 < q < p$ there exists a measure $\lambda_q \in S_q \cap Q$ such that $\lambda_q \to \lambda$ as $q \to p$. From Corollaries 3.2, 3.3 and 4.3 we get the formula

$$\hat{\lambda}_q(t) = \exp(-c_q t^q) \qquad (t \in \boldsymbol{R}_+)$$

for some positive constants c_q. Thus, by Corollary 4.1,

$$(\omega_h \lambda_q)\hat{}(t) = h^{-1} \int_1^{1+h} \exp(-c_q t^q x^q)\,dx \qquad (t \in \boldsymbol{R}_+)$$

for all $h > 0$. Since $(\omega_h \lambda_q)\hat{} \to (\omega_h \lambda)\hat{}$ as $q \to p$ and the function $(\omega_h \lambda)\hat{}$ is continuous because $\omega_h \lambda \in P_0$, we conclude that the c_q tend to a finite nonnegative limit c and

$$(\omega_h \lambda)\hat{}(t) = h^{-1} \int_1^{1+h} \exp(-ct^p x^p)\,dx \qquad (t \in \boldsymbol{R}_+)$$

for all $h > 0$. Now applying (4.4) we obtain the equality

$$\hat{\lambda}(t) = \exp(-ct^p)$$

m_0-almost everywhere. The case $c = 0$ is impossible because $\lambda \neq \delta_0$ and, by Lemma 4.1, $\hat{\delta}_0(t) = 1$ $(t \in \boldsymbol{R}_+)$. The theorem is thus proved.

Remark 4.1. The assertion of Theorem 4.2 remains true in the case $\varkappa(\circ) = \infty$, i.e. for $\circ = *_\infty$, if we restrict ourselves to measures $\lambda \in S \setminus S_\infty$.

Two measures μ and ν from P are said to be *similar* if $\mu = T_c \nu$ for a positive number c.

PROPOSITION 4.4. *For any p $(0 < p \leqslant \infty)$ all measures belonging to S_p are similar.*

Proof. By Lemma 2.1 our assertion is obvious for $p = \infty$. Consider the case $p < \infty$. Let $\mu, \nu \in S_p$. Then, by Theorem 4.2 and Remark 4.1,

$$\hat{\mu}(t) = \exp(-at^p), \qquad \hat{\nu}(t) = \exp(-bt^p)$$

m_0-almost everywhere, where a and b are positive constants. Setting $c = (a/b)^{1/p}$, we have, by (4.2), $(T_c \nu)\hat{}(t) = \exp(-at^p)$ m_0-almost everywhere, which yields $\mu = T_c \nu$. This completes the proof.

From Propositions 2.3 and 4.4 we get the following property of $\circ$-stable measures.

COROLLARY 4.5. *All measures belonging to $S \setminus S_\varkappa$ are equivalent to the Lebesgue measure on $\boldsymbol{R}_+$.*

Each measure γ appearing in condition (1.1) for the generalized convolution will be called a *characteristic measure*. By Lemma 1.1 all characteristic measures are similar.

PROPOSITION 4.5. *The set $S_\varkappa$ consists of characteristic measures.*

Proof. Let γ be defined by condition (1.1). Of course, it is $\circ$-stable and, by Propositions 2.2 and 2.3, belongs to a set S_p with $0 < p \leqslant \varkappa$. To prove our statement it suffices, by Proposition 4.4, to show that $\gamma \in S_\varkappa$.

If $\varkappa = \infty$, then, by Lemma 2.1, $\circ = *_\infty$ and $S_\infty = \{\delta_a\colon a > 0\}$. One can easily check that in this case $\gamma = \delta_b$ for a positive number b, and consequently $\gamma \in S_\varkappa$.

Now consider the case $\varkappa < \infty$ and suppose the contrary: $p < \varkappa$. Let c_n be the norming sequence appearing in (1.1). By Lemma 2.6 we may assume without loss of generality that the sequence c_n is monotone nonincreasing. Let q be an arbitrary number satisfying the condition $p < q < \varkappa$. Then, by Corollary 2.1,

$$n^{1/q} c_n \to 0. \tag{4.10}$$

By Proposition 2.3, $S_q \cap Q \neq \emptyset$. Taking a measure λ_q from $S_q \cap Q$, we have

$$T_{n^{-1/q}} \lambda_q^{\circ n} = \lambda_q \quad (n = 1, 2, \ldots). \tag{4.11}$$

Further, by Corollary 4.2, we have the pointwise convergence

$$(\lambda_q T_{c_n} \delta_1^{\circ n})\hat{} \to (\lambda_q \gamma)\hat{},$$

which, by Theorem 4.2 and Corollary 4.1, gives

$$(\lambda_q T_{c_n} \delta_1^{\circ n})\hat{} \to \int_0^\infty \exp(-bt^p x^p) \lambda_q(dx) \tag{4.12}$$

where $\hat{\gamma}(t) = \exp(-bt^p)$ $(b > 0)$. Setting $b_n = (2n)^{1/q} c_n$ we have, by (4.10),

$$b_n \to 0 \tag{4.13}$$

and, by (4.11),

$$T_{c_{2n}} \lambda_q^{\circ 2n} = T_{b_n} \lambda_q \quad (n = 1, 2, \ldots)$$

The above equality, Corollary 4.1 and formulas (4.2), (4.3) imply the equality

$$\hat{\lambda}_q(b_n t) = (T_{c_{2n}} \lambda_q^{\circ 2n})\hat{}(t) = \Big(\int_0^\infty \hat{\delta}_1(c_{2n} tx) \lambda_q(dx)\Big)^{2n},$$

which together with the inequality

$$(\lambda_q T_{c_{2n}} \delta_1^{\circ 2n}(t))\hat{} = \int_0^\infty \hat{\delta}_1(c_{2n} tx)^{2n} \lambda_q(dx) \geq \Big(\int_0^\infty \hat{\delta}_1(c_{2n} tx) \lambda_q(dx)\Big)^{2n}$$

yields

$$(\lambda_q T_{c_{2n}} \delta_1^{\circ 2n})\hat{}(t) \geq \hat{\lambda}_q(b_n t) \quad (n = 1, 2, \ldots).$$

Passing to the limit as $n \to \infty$ we get, by (4.12), (4.13) and the continuity of $\hat{\lambda}_q$,

$$\int_0^\infty \exp(-bt^p x^p) \lambda_q(dx) \geq 1 \quad (t \in \boldsymbol{R}_+),$$

which contradicts the assumptions $b > 0$ and $\lambda_q \in Q$. The proposition is thus proved.

A generalized convolution is completely described by its characteristic exponent and characteristic measure. More precisely, we have the following theorem which for regular generalized convolutions has been proved in [9], Theorem 2.3.

THEOREM 4.3. *If $\varkappa(\circ) = \varkappa(\circ')$ and the characteristic measures of $\circ$ and $\circ'$ are similar, then $\circ = \circ'$.*

Proof. If $\varkappa(\circ) = \varkappa(\circ') = \infty$, then our statement follows immediately from Lemma 2.1. Consequently it suffices to consider the case $\varkappa(\circ) = \varkappa(\circ') = \varkappa < \infty$. Passing to similar measures if necessary, we may assume without loss of generality that γ is a characteristic measure for $\circ$ and $\circ'$ simultaneously. Moreover, for suitably chosen weak characteristic functions $\mu \to \hat{\mu}$ and $\mu \to \hat{\mu}'$ of $\circ$ and $\circ'$ respectively we have, by Theorem 4.2,

$$\hat{\gamma}(t) = \exp(-t^\varkappa) = \hat{\gamma}'(t)$$

m_0-almost everywhere. Let $p < \varkappa$. Setting $\sigma = \gamma\pi_{\varkappa,p}$ we infer, by Lemma 2.5, that σ is $\circ$-stable and $\circ'$-stable simultaneously and $\sigma \in Q$. Moreover, by Corollary 4.1, we have for any $\mu \in P$

$$U_\sigma \hat{\mu} = (\sigma\mu)\hat{} = U_{\mu\pi_{\varkappa,p}} \hat{\sigma} = U_{\mu\pi_{\varkappa,p}} \hat{\sigma}' = (\sigma\mu)\hat{}' = U_\sigma \hat{\mu}',$$

which, by Proposition 3.1, yields $\hat{\mu} = \hat{\mu}'$ m_0-almost everywhere. Hence and from (4.3) it follows that for any pair $\mu, \nu \in P$

$$(\mu \circ \nu)\hat{} = \hat{\mu}\hat{\nu} = \hat{\mu}'\hat{\nu}' = (\mu \circ' \nu)\hat{}' = (\mu \circ' \nu)\hat{}.$$

Consequently, $(\mu \circ \nu)\hat{} = (\mu \circ' \nu)\hat{}$, which completes the proof.

The similarity of all weak characteristic functions (Corollary 4.4) enables us to associate with every generalized convolution $\circ$ the subset $C(\circ)$ of $L_\infty(m_0)$ defined as follows: $f \in C(\circ)$ if and only if $f = \hat{\mu}$ m_0-almost everywhere for some $\mu \in P$. Of course, this set does not depend upon the choice of a weak characteristic function.

We proceed now to a description of the set $C(\circ)$.

THEOREM 4.4. *Let $f \in L_\infty(m_0)$. Then $f \in C(\circ)$ if and only if*

$$\lim_{t \to 0} t^{-1} \int_0^t f(u)\, du = f(0) = 1 \tag{4.14}$$

and for any pair $\mu, \nu \in P_0$ the inequality

$$\int_0^\infty f(x)(\mu \circ \mu)(dx) \int_0^\infty f(x)(\nu \circ \nu)(dx) \geq \Big(\int_0^\infty f(x)(\mu \circ \nu)(dx)\Big)^2 \tag{4.15}$$

holds.

Proof. Necessity. Suppose that $f = \hat{\lambda}$ m_0-almost everywhere for some $\lambda \in P$. Then condition (4.13) follows immediately from Lemma 4.3 and Proposition 4.2. Let $\mu, \nu \in P_0$. Taking into account the continuity of $\hat{\mu}$ and $\hat{\nu}$ we have, by (4.3), Corollary 4.1 and Proposition 1.1,

$$\int_0^\infty f(x)(\mu\circ\nu)(dx) = (U_{\mu\circ\nu}\hat{\lambda})(1) = \big(U_\lambda(\mu\circ\nu)^\frown\big)(1)$$

$$= (U_\lambda \hat{\mu}\hat{\nu})(1) = \int_0^\infty \hat{\mu}(x)\hat{\nu}(x)\lambda(dx).$$

Condition (4.15) is now a direct consequence of Schwarz's inequality.

Sufficiency. Suppose that $f \in L_\infty(m_0)$ and both conditions (4.14) and (4.15) are fulfilled. Put

$$(4.16)\qquad l(\alpha) = \int_0^\infty f(x)\alpha(dx) \quad (\alpha \in V_0).$$

By (4.14) we have the formula

$$(4.17)\qquad l(\delta_0) = 1.$$

Thus setting $\nu = \delta_0$ into (4.15) we get the inequality

$$l(\mu\circ\mu) \geqslant (l(\mu))^2 \geqslant 0$$

for any $\mu \in P_0$. The above inequality together with (4.15) yields $l(\alpha\circ\bar{\alpha}) \geqslant 0$ for all $\alpha \in V_0$. Consequently, l is a positive continuous linear functional on V_0. Let h be the homomorphism from H_+^s induced by a weak characteristic function by means of formula (4.8). Then $h(T_t\mu) = \hat{\mu}(t)$ $(t \in \boldsymbol{R}_+)$ and $h(T_\infty \mu) = h_\infty(\mu) = \mu(\{0\})$ for all $\mu \in P_0$. Thus applying Proposition 3.5 we have the formula

$$(4.18)\qquad l(\mu) = \int_{\bar{\boldsymbol{R}}_+} \hat{\mu}(t)\lambda(dt) + \mu(\{0\})\lambda(\{\infty\})$$

for a bounded Borel measure λ on $\bar{\boldsymbol{R}}_+$. Substituting $\mu = \delta_0$ into (4.18) we get, by virtue of (4.17) and Lemma 4.1,

$$(4.19)\qquad \lambda(\bar{\boldsymbol{R}}_+) = 1.$$

Let ν_h $(h > 0)$ denote the uniform distribution on the interval $[0, h]$. Then $l(\nu_h) = h^{-1}\int_0^h f(t)\,dt$ and consequently by (4.14)

$$(4.20)\qquad l(\nu_h) \to 1 \quad \text{as } h \to 0.$$

By Lemma 4.4 we have $\|\hat{\nu}_h\|_\infty = 1$. Since $\nu_h \in P_0$, the function $\hat{\nu}_h$ is continuous

and the last equality for the norm yields

$$(4.21)\qquad |\hat{\nu}_h(t)| \leqslant 1 \quad (t \in \boldsymbol{R}_+).$$

Substituting $\mu = \nu_h$ into (4.18) we get

$$l(\nu_h) = \int_{\boldsymbol{R}_+} \hat{\nu}_h(t)\lambda(dt).$$

Consequently, by (4.21),

$$|l(\nu_h)| \leqslant \lambda(\bar{\boldsymbol{R}}_+) \quad (h > 0),$$

which, by (4.20), yields $\lambda(\boldsymbol{R}_+) \geqslant 1$. Comparing this inequality with (4.19) we conclude that $\lambda(\boldsymbol{R}_+) = 1$ and $\lambda(\{\infty\}) = 0$. In other words, $\lambda \in P$ and representation (4.18) has the form

$$l(\mu) = \int_0^\infty \hat{\mu}(t)\lambda(dt) \quad (\mu \in P_0).$$

By Corollary 4.1 and the continuity of $\hat{\mu}$ the right-hand side of the above equality is equal to $\int_0^\infty \hat{\lambda}(t)\mu(dt)$. Consequently, by (4.16), we have the equality

$$\int_0^\infty f(t)\mu(dt) = \int_0^\infty \hat{\lambda}(t)\mu(dt)$$

for all $\mu \in P_0$. This yields $f = \hat{\lambda}$ m_0-almost everywhere, which completes the proof.

PROPOSITION 4.6. *Let $\mu_n \in P$ $(n = 1, 2, \ldots)$. If $\hat{\mu}_n \to f$ m_0-almost everywhere and*

$$\lim_{t\to 0} t^{-1}\int_0^t f(u)\,du = 1,$$

then there exists a measure $\mu \in P$ such that $\mu_n \to \mu$ and $f = \hat{\mu}$ m_0-almost everywhere.

Proof. We have, by Lemma 4.4, $\|f\|_\infty = 1$. Changing if necessary the function f on a set of the measure m_0 zero we may assume without loss of generality that f is a Borel function from $L_\infty(m_0)$. Since m_0 has an atom at the origin, we infer, by Lemma 4.3, that $f(0) = 1$. Consequently, f fulfils condition (4.14). By the dominated convergence theorem condition (4.15) is also fulfilled. Thus, by Theorem 4.4, $f = \hat{\mu}$ m_0-almost everywhere for a measure μ from P. By Corollary 4.1,

$$(\omega_h\mu_n)^\frown = U_{\omega_h}\hat{\mu}_n \to U_{\omega_h}\hat{\mu} = (\omega_h\mu)^\frown$$

for all $h > 0$, which yields $\mu_n \to \mu$. The proposition is thus proved.

THEOREM 4.5. *Let* $\sigma \in S_p$ $(0 < p < \varkappa(\circ))$. *A function* f *from* $L_\infty(m_0)$ *belongs to* $C(\circ)$ *if and only if*

$$\lim_{t \to 0} t^{-1} \int_0^t f(u)\,du = f(0) = 1 \tag{4.22}$$

and the function $\int_0^\infty f(t^{1/p}x)\sigma(dx)$ *is completely monotone on* $(0, \infty)$.

Proof. Passing if necessary to a similar weak characteristic function we may assume without loss of generality that $\hat{\sigma}(t) = \exp(-t^p)$ $(t \in \boldsymbol{R}_+)$.

Necessity. Suppose that $f = \hat{\lambda}$ m_0-almost everywhere for some $\lambda \in P$. We have already shown (4.22) in proving Theorem 4.4. Further, we note that, by Corollary 4.5, $\sigma \in Q$. Applying Corollary 4.1, we have

$$\int_0^\infty f(t^{1/p}x)\sigma(dx) = (U_\sigma \hat{\lambda})(t^{1/p}) = (U_\lambda \hat{\sigma})(t^{1/p})$$

$$= \int_0^\infty \exp(-tx^p)\lambda(dx) \qquad (t \in \boldsymbol{R}_+),$$

which yields the complete monotonicity of the function $\int_0^\infty f(t^{1/p}x)\sigma(dx)$ on $(0, \infty)$.

Sufficiency. Suppose that $f \in L_\infty(m_0)$, condition (4.22) holds and the function $\int_0^\infty f(t^{1/p}x)\sigma(dx)$ is completely monotone on $(0, \infty)$. Then, by the Bernstein Theorem,

$$\int_0^\infty f(t^{1/p}x)\sigma(dx) = \int_0^\infty \exp(-ty^p)\varrho(dy) \qquad (t \in \boldsymbol{R}_+) \tag{4.23}$$

for a Borel measure ϱ on $\boldsymbol{R}_+$.

By (4.22) we have

$$h^{-1}\int_0^h \int_0^\infty f(tx)\sigma(dx)\,dt \to 1$$

as $h \to 0$. Consequently, by (4.23),

$$h^{-1}\int_0^h \int_0^\infty \exp(-ty^p)\varrho(dy)\,dt \to 1$$

as $h \to 0$. On the other hand the left-hand side of the above formula tends to $\varrho(\boldsymbol{R}_+)$ as $h \to 0$. Thus $\varrho(\boldsymbol{R}_+) = 1$ and consequently $\varrho \in P$. Now equality (4.23) can be rewritten in the form $U_\sigma f = U_\varrho \hat{\sigma}$, which, by Corollary 4.1, yields $U_\sigma f = U_\sigma \hat{\varrho}$. Applying Proposition 3.1, we get $f = \hat{\varrho}$ m_0-almost everywhere, which completes the proof.

Now we shall discuss some criterions for the existence of characteristic functions. We begin with a simple lemma.

LEMMA 4.5. *If for every* $\mu \in P$ *the weak characteristic function* $\hat{\mu}$ *is equal* m_0-*almost everywhere to a continuous function* $\tilde{\mu}$, *then the correspondence* $\mu \to \tilde{\mu}$ *is a characteristic function for the generalized convolution in question.*

Proof. To prove this it suffices to show that the uniform convergence $\tilde{\mu}_n \to \tilde{\mu}$ on every compact subset of $\boldsymbol{R}_+$ is equivalent to the convergence $\mu_n \to \mu$. Suppose that $\mu_n \to \mu$ and $a_n \to a$ $(0 \leqslant a < \infty)$. Then we have $T_{a_n}\mu_n \to T_a \mu$ and consequently

$$\int_0^\infty \tilde{\delta}_1(a_n x)\mu_n(dx) \to \int_0^\infty \tilde{\delta}_1(ax)\mu(dx).$$

The above relation yields, by virtue of Proposition 4.1, the uniform convergence $\tilde{\mu}_n \to \tilde{\mu}$ on every compact subset of $\boldsymbol{R}_+$. Conversely, suppose that $\tilde{\mu}_n \to \tilde{\mu}$. By Lemma 4.4 the functions $\tilde{\mu}_n$ are bounded in common and, by Corollary 4.1,

$$(\omega_h \mu_n)^\wedge = U_{\omega_h}\tilde{\mu}_n \to U_{\omega_h}\tilde{\mu} = (\omega_h \mu)^\wedge,$$

which yields $\mu_n \to \mu$. This completes the proof.

THEOREM 4.6. *Suppose that* $\sigma \circ \delta_1 \in P_0$ *for some* $\circ$-*stable measure* σ. *Then the generalized convolution* $\circ$ *admits a characteristic function.*

Proof. We note that in this case $\sigma \notin S_\infty$. In fact, by Lemma 2.1, S_∞ consists of the measures δ_a $(a > 0)$ and $\circ = *_\infty$. Then we have, by (2.6), $\delta_a *_\infty \delta_1 = \delta_{g_\infty(a,1)} \notin P_0$. Thus, by Proposition 2.2, $\sigma \in S_p$ for a finite index p. Let $\mu \to \hat{\mu}$ be a weak characteristic function. By Theorem 4.2 and Remark 4.1, $\hat{\lambda}(t) = \exp(-ct^p)$ m_0-almost everywhere for a positive constant c. Since $\sigma \circ \delta_1 \in P_0$, the function $(\sigma \circ \delta_1)^\wedge$ is continuous, and consequently the function

$$\tilde{\delta}_1(t) = \exp(ct^p)(\sigma \circ \delta_1)^\wedge(t)$$

is continuous on $\boldsymbol{R}_+$. By (4.3) we have $\hat{\delta}_1 = \tilde{\delta}_1$ m_0-almost everywhere. Setting, for any $\mu \in P$, $\tilde{\mu} = U_\mu \tilde{\delta}_1$, we get continuous functions satisfying, by Proposition 4.1, the equality $\hat{\mu} = \tilde{\mu}$ m_0-almost everywhere. Applying Lemma 4.5 we get the assertion of the theorem.

A measure η from P is said to be $\circ$-*quasi-invariant* if for all $a > 0$ the measure $\eta \circ \delta_a$ is absolutely continuous with respect to η. This concept has been introduced by V. E. Vol'kovich in [12] and [13].

LEMMA 4.6. *Suppose that there exists a* $\circ$-*quasi-invariant measure in* P. *Then* $\sigma \circ \delta_1 \in P_0$ *for every* $\sigma \in S \cap Q$.

Proof. Let η be a $\circ$-quasi-invariant measure in P and $\sigma \in S \cap Q$. Then, by Lemma 1.3, we have the formula

$$\sigma\eta(E) = \int_0^\infty \eta(x^{-1}E)\,\sigma(dx) \tag{4.24}$$

for all Borel subsets E of $\boldsymbol{R}_+$. Further, by Lemmas 1.2 and 1.3,

$$\begin{aligned}\sigma\eta\circ\delta_1(E) &= \int_0^\infty \delta_x\circ\delta_1(E)\,\sigma\eta(dx)\\ &= \int_0^\infty\int_0^\infty \delta_{xy}\circ\delta_1(E)\,\sigma(dx)\,\eta(dy),\end{aligned} \tag{4.25}$$

which, by the equality

$$\delta_{xy}\circ\delta_1 = T_x(\delta_y\circ\delta_{x^{-1}}) \quad (x>0)$$

and Lemma 1.2 yields

$$\sigma\eta\circ\delta_1(E) = \int_0^\infty \eta\circ\delta_{x^{-1}}(x^{-1}E)\,\sigma(dx). \tag{4.26}$$

As a consequence of Lemma 1.2 we also have the formula

$$\sigma\circ\delta_1(E) = \int_0^\infty \delta_x\circ\delta_1(E)\,\sigma(dx). \tag{4.27}$$

Suppose that $m_0(E) = 0$. Since, by Proposition 1.1, $\sigma\eta\in P_0$, we then have $\sigma\eta(E) = 0$ and, by (4.24), $\eta(x^{-1}E) = 0$ for σ-almost all x. Since η is $\circ$-quasi-invariant, the last equality yields $\eta\circ\delta_{x^{-1}}(x^{-1}E) = 0$ for σ-almost all x. Now, by (4.26), we have $\sigma\eta\circ\delta_1(E) = 0$, which, by (4.25), implies $\delta_x\circ\delta_1(E) = 0$ for $\sigma\eta$-almost all x. Since, by Lemma 1.4, σ is absolutely continuous with respect to $\sigma\eta$, we also have $\delta_x\circ\delta_1(E) = 0$ for σ-almost all x and finally, by (4.27), $\sigma\circ\delta_1(E) = 0$. This shows that the measure $\sigma\circ\delta_1$ is absolutely continuous with respect to m_0, which completes the proof.

As an immediate consequence of Lemma 4.6 and Theorem 4.6 we get a criterion for the existence of characteristic functions. We note that some results of this type has been proved in [12] and [13].

THEOREM 4.7. *If there exists a $\circ$-quasi-invariant measure in P, then the generalized convolution $\circ$ admits a characteristic function.*

References

[1] R. Bojanic and E. Seneta, *A unified theory of regularly varying sequences*, Math. Z. 134 (1973), 91–106.

[2] F. Bohnenblust, *An axiomatic characterization of L_p-spaces*, Duke Math. J. 6 (1940), 627–640.

[3] P. R. Halmos, *Measure Theory*, Van Nostrand, New York 1950.

[4] E. Hille, *Functional Analysis and Semi-groups*, Amer. Math. Soc., New York 1948.

[5] I. A. Ibragimov and Yu. V. Linnik, *Independent and Stationarily Connected Random Variables*, Nauka, Moscow 1965 (in Russian).

[6] M. A. Naimark, *Normed Rings*, Nauka, Moscow 1968 (in Russian).

[7] E. C. Titchmarsh, *The Theory of Functions*, Oxford Univ. Press, London 1939.

[8] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), 217–245.

[9] —, *Generalized convolutions III*, ibid. 80 (1984), 167–189.

[10] —, *A counterexample on generalized convolutions*, Colloq. Math., in print.

[11] —, *Quasi-regular generalized convolutions*, ibid., in print.

[12] V. E. Vol'kovich, *On the normed rings generated by K. Urbanik's algebras*, Teor. Veroyatnost. i Primenen. 25 (1980), 432–433 (in Russian).

[13] —, *The normed rings generated by generalized convolutions*, in: Problems of Stability of Stochastic Models, Proc. Seminar, Moscow 1980, 12–18 (in Russian).

INSTYTUT MATEMATYKI UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY

Received December 10, 1984 (2019)