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On isomorphisms of anisotropic Sobolev spaces with "classical Banach spaces" and a Sobolev type embedding theorem

by

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Abstract. Let W denote a finite nonempty set of partial derivatives in d variables which is identified with a subset of d-tuples of nonnegative integers. Assume that if $(a_j) \in W$ then $(b_j) \in W$ whenever $0 \leqslant b_j \leqslant a_j$ for $j = 1, 2, \ldots, d$.

THEOREM B. *The space C_W of all scalar-valued functions on the d-dimensional torus continuous with the partial derivatives belonging to W is isomorphic as a Banach space to the space C of all scalar-valued continuous functions on an interval iff there is (a_j^0) such that*

$$W = \{(a_j) \in Z_+^d : a_j \leqslant a_j^0 \text{ for } 1 \leqslant j \leqslant d\},$$

i.e. iff W is a "parallelepiped".

Analogous result holds for the Sobolev spaces L_W^1 and L_W^∞. For $1 < p < \infty$ the Sobolev spaces L_W^p are always isomorphic to classical L^p-spaces.

The proof of Theorem B bases upon the following generalization of the two-dimensional Sobolev embedding theorem:

THEOREM A. *Given positive integers n, m there exists a numerical constant $C = C(n, m)$ such that for every infinitely differentiable complex-valued function u on the plane $\boldsymbol{R}^2$ with compact support,*

$$\iint_{\boldsymbol{R}^2} |\xi|^{n-1} |\eta|^{m-1} |\hat{u}(\xi, \eta)|^2 \, d\xi \, d\eta \leqslant C \iint_{\boldsymbol{R}^2} |D_x^n u(x, y)| \, dx \, dy \iint_{\boldsymbol{R}^2} |D_y^m u(x, y)| \, dx \, dy$$

where $\hat{u}$ denotes the Fourier Transform of u.

Introduction. The present paper deals with anisotropic Sobolev spaces C_W and L_W^p for $1 \leqslant p \leqslant \infty$ defined either on the Euclidean space $\boldsymbol{R}^d$ or on the d-dimensional torus $\boldsymbol{T}^d$ (cf. Section 0.3 for precise description). These Sobolev spaces are determined by a nonempty finite set W (called "smoothness") of partial derivatives. For a set to be a smoothness we assume that if a partial derivative, say D, belongs to the set then every partial derivative which requires no more derivations with respect to each variable than D also belongs to the set. It is convenient to identify partial derivatives in $\boldsymbol{R}^d$ with d-tuples of nonnegative integers, i.e. with elements of the Cartesian product $\boldsymbol{Z}_+^d$.

We consider a typical question in structural theory of Banach spaces: under what condition on a smoothness W the Sobolev spaces C_W, L_W^p for

$1 \leqslant p \leqslant \infty$ are isomorphic to corresponding classical Banach spaces C and L^p. In view of the isomorphic classification of classical Banach spaces (cf. [B], pp. 259–274), without loss of generality one may assume that C is the space of all continuous scalar-valued functions on $[0, 1]$, and L^p is the space of p-absolutely Lebesgue integrable (resp. essentially bounded) scalar-valued measurable functions on $[0, 1]$.

The answer to our question is given by the following:

THEOREM B. *For a smoothness $W \subset \boldsymbol{Z}_+^d$ the following conditions are equivalent*:

(i) *each of the spaces $C_W(\boldsymbol{R}^d)$ and $C_W(\boldsymbol{T}^d)$ is isomorphic to C,*

(ii) *each of the spaces $L_W^1(\boldsymbol{R}^d)$ and $L_W^1(\boldsymbol{T}^d)$ is isomorphic to L^1,*

(iii) *each of the spaces $L_W^\infty(\boldsymbol{R}^d)$ and $L_W^\infty(\boldsymbol{T}^d)$ is isomorphic to L^∞,*

(iv) *W is an interval, i.e. there is a sequence $(a_j)_{1 \leqslant j \leqslant d} \in \boldsymbol{Z}_+^d$ such that*

$$W = \{(b_j) \in \boldsymbol{Z}_+^d:\ 0 \leqslant b_j \leqslant a_j \text{ for } j = 1, 2, \ldots, d\}.$$

Moreover in conditions (i)–(iii) *one can replace "isomorphic" by "isomorphic to a complemented subspace" or by the appropriate boundedness of the canonical projection* (cf. Section 6).

THEOREM C. *For every smoothness $W \subset \boldsymbol{Z}_+^d$ the space $L_W^p(\boldsymbol{R}^d)$ as well as $L_W^p(\boldsymbol{T}^d)$ is isomorphic to L^p for $1 < p < \infty$.*

Applying Mityagin's technique (cf. [MI], pp. 79–80) it is not difficult to extend Theorems B and C to anisotropic Sobolev spaces defined on regular domains of $\boldsymbol{R}^d$.

Theorem B generalizes some previously known results. First consider the one-dimensional case. If W is a smoothness in $\boldsymbol{Z}_+$, then W is an interval determined by a nonnegative integer k; $W = \{j:\ 0 \leqslant j \leqslant k\} \stackrel{\text{df}}{=} (k)$. Thus a special case of Theorem B is an old result due to Borsuk (cf. [BO]; [B], p. 168) that $C_{(k)}(\boldsymbol{T})$ is isomorphic to C. Next for $d = 2, 3, \ldots$ and $k = 1, 2, \ldots$ put

$$(k)_d = \{(a_j) \in \boldsymbol{Z}_+^d:\ \textstyle\sum a_j \leqslant k\},$$

$$[k]_d = \{(a_j) \in \boldsymbol{Z}_+^d:\ \max_j a_j \leqslant k\}.$$

Clearly $[k]_d$ is an interval while $(k)_d$ is not. Thus Theorem B yields (a) $C_{[k]_d}(\boldsymbol{T}^d)$ is isomorphic to C, while (b) $C_{(k)_d}(\boldsymbol{T}^d)$ is isomorphic to no complemented subspace of C. These results are due to Grothendieck. In [GRa] Grothendieck proved in fact (a) while erroneously claiming that he proved that $C_{(k)_d}(\boldsymbol{T}^d)$ is isomorphic to C. Later in [GRb] he corrected the mistake and anounced (b) giving also some hints how to prove it. The first complete proof of (b) was published by Henkin [HE]. An alternative proof of (b) was later discovered by Kisliakov [KI] who used the theory of absolutely summing operators. Further strengthening of (b), for instance that $C_{(k)_d}(\boldsymbol{T}^d)$ fails to have local unconditional structure has been obtained

independently in [KI] and [KW–P]; this property is related to the weak type 1-1 of the canonical projection for the smoothnesses $(k)_d$.

Our proof of the "nonisomorphic" part of Theorem B follows Kisliakov's approach from [KI]. It bases on the observation that the classical Sobolev embedding theorem in the 2-dimensional case for $p = 1$ can be interpreted as the existence of a bounded non-absolutely summing operator from $L^1_{(1)_2}(\boldsymbol{T}^2)$ into a Hilbert space. The technical difficulty we have to overcome is the right generalization of the embedding theorem. This is done in Theorem A (cf. Section 1 for precise formulation) and in Theorem 4.2 where the periodic case is treated. Theorem A, roughly speaking, says that if a function in $\boldsymbol{R}^2$ has absolutely integrable "pure" derivatives of orders n and m with respect to x and y respectively then it has square-integrable mixed fractional derivative $D_x^{(n-1)/2} D_y^{(m-1)/2}$. The proof of Theorem A is rather complicated. It is presented with some strengthening in Sections 1–4. Sections 5–6 contain the proofs of Theorems B and C.

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0. Preliminaries.

0.1. Finite subsets of $\boldsymbol{Z}^d$; sets of smoothness. By $\boldsymbol{Z}^d$ (resp. $\boldsymbol{Z}_+^d$) we denote the Cartesian product of d copies ($d = 1, 2, \ldots$) of the set $\boldsymbol{Z}$ of integers (resp. $\boldsymbol{Z}_+$ – the nonnegative integers). $\boldsymbol{Z}^d$ is a group with respect to coordinatewise addition, $0 = (0, 0, \ldots, 0)$ denotes the neutral element of the group.

In this subsection we state some particular properties of finite subsets of $\boldsymbol{Z}^d$. We begin with a lemma on divergent series which generalizes the fact that $\sum_{(p,q) \in \boldsymbol{Z}^2} (p^2+q^2+1)^{-1} = \infty$.

LEMMA 0.1. *Let $W \subset \boldsymbol{Z}_+^2$ be a finite set which contains two points (n_1, m_1) and (n_2, m_2) such that*

$$\alpha = \frac{m_2 - m_1}{n_1 - n_2} > 0, \tag{0.1}$$

$$n\alpha + m \leqslant n_1 \alpha + m_1 \quad \textit{for every } (n, m) \in W. \tag{0.2}$$

Then

$$\sum_{(p,q) \in \boldsymbol{Z}^2} |p|^{n_1+n_2-1} |q|^{m_1+m_2-1} \Big(\sum_{(n,m) \in W} p^{2n} q^{2m} \Big)^{-1} = \infty. \tag{0.3}$$

Proof. Denote the left-hand side of (0.3) by S and by k the number of elements of W. Then

$$\begin{aligned} S &\geqslant \sum_{q>1}\sum_{1\leqslant p<q^{\alpha}} p^{n_1+n_2-1} q^{m_1+m_2-1}\Big(\sum_{(n,m)\in W} p^{2n}q^{2m}\Big)^{-1} \\ &\geqslant \sum_{q>1}\sum_{1\leqslant p<q^{\alpha}} p^{n_1+n_2-1} q^{m_1+m_2-1}\Big(\sum_{(n,m)\in W} q^{2(n\alpha+m)}\Big)^{-1} \\ &\geqslant \sum_{q>1}\sum_{1\leqslant p<q^{\alpha}} p^{n_1+n_2-1} q^{m_1+m_2-1}\cdot k^{-1} q^{-2(n_1\alpha+m_1)} \\ &= k^{-1}\sum_{q>1} q^{m_2-m_1-1-2n_1\alpha}\sum_{1\leqslant p<q^{\alpha}} p^{n_1+n_2-1}. \end{aligned}$$

Note that for $q\geqslant 2$ one has

$$\begin{aligned} \sum_{1\leqslant p<q^{\alpha}} p^{n_1+n_2-1} &\geqslant \int_0^{q^{\alpha}-1} x^{n_1+n_2-1}\,dx = (n_1+n_2)^{-1}(q^{\alpha}-1)^{n_1+n_2} \\ &\geqslant (cq)^{\alpha(n_1+n_2)} \end{aligned}$$

where $c = 1-2^{-\alpha}$. Thus, taking into account (0.1),

$$\begin{aligned} S &\geqslant k^{-1}c\sum_{q>1} q^{(m_2-m_1)-\alpha(n_1-n_2)-1} \\ &= k^{-1}c\sum_{q>1} q^{-1} = +\infty. \end{aligned}$$

Next we pass to sets of smoothness.

In the Cartesian product Z^d we have the natural partial ordering "$\leqslant$" defined as follows: given $A=(a_j)_{1\leqslant j\leqslant d}$, $B=(b_j)_{1\leqslant j\leqslant d}$, we write $A\leqslant B$ provided $a_j\leqslant b_j$ for $j=1,2,\ldots,d$. The set

$$I(A;B) = \{C\in Z^d:\ A\leqslant C\leqslant B\}$$

is called an *interval*.

DEFINITION 0.1. A nonempty finite set $W\subset Z_+^d$ is called a *set of smoothness*, shortly "smoothness", provided

(0.4) if $A\in W$ then $I(0;A)\subset W$.

The natural character of condition (0.4) and the reason why we use the word "smoothness" will be explained in the next subsection. Here we derive two formal consequences of (0.4).

First observe that we can apply Lemma 0.1 for 2-dimensional sets of smoothness which are not intervals via the following

LEMMA 0.2. *If $W\subset Z_+^2$ is a smoothness which is not an interval, then there exist (n_1, m_1) and (n_2, m_2) in W which satisfy conditions* (0.1) *and* (0.2) *of Lemma* 0.1.

Proof. Let $E_W = \{A\in W:$ if $B\in W$, $B\geqslant A$ then $B=A\}$. Observe that E_W has at least two elements, and two different elements of E_W have different both the first and second coordinates. Pick $(n_1, m_1)\in E_W$ so that $n_1 = \sup\{n':\ (n', m')\in E_W\}$. Clearly if $(n', m')\in E_W\setminus\{(n_1, m_1)\}$ then $n'<n_1$ and $m'>m_1$ (otherwise $(n_1, m_1)\neq(n', m')\leqslant(n_1, m_1)$). Thus $\alpha(n', m') = (m'-m_1)(n_1-n')^{-1}>0$. Pick $(n_2, m_2)\in E_W\setminus\{(n_1, m_1)\}$ so that

$$\alpha(n_2, m_2) = \sup\{\alpha(n', m'):\ (n', m')\in E_W\setminus\{(n_1, m_1)\}\}.$$

Put $\alpha=\alpha(n_2, m_2)$. Then α satisfies (0.1). Assume that (0.2) were violated. Then there would exist $(n, m)\in W$ with $(*)$ $n\alpha+m>n_1\alpha+m_1$. Pick $(n', m')\in E_W$ so that $(n, m)\leqslant(n', m')$. Clearly $n'\alpha+m'\geqslant n\alpha+m$. Hence $(*)$ would yield $n'\alpha+m'>n_1\alpha+m_1$, equivalently $m'-m_1>(n_1-n')\alpha$. Since $n_1-n'>0$, we would get $\alpha(n', m')>\alpha$; a contradiction.

To state the next result we need more terminology.

Let $d\geqslant 2$. A smoothness $V\subset Z_+^{d-1}$ is said to be *simply generated* by a smoothness $W\subset Z_+^d$ provided there exists a rearrangement of coordinates such that $V=\varphi(W)$ where $\varphi\colon Z_+^d\to Z_+^{d-1}$ is in the new arrangement defined by

(0.5) $\varphi(A) = (a_1, a_2, \ldots, a_{d-2}, a_{d-1}+a_d)$ for $A=(a_j)\in Z_+^d$.

Let $1\leqslant r<d$. A smoothness $V\subset Z_+^{d-r}$ is said to be *generated* by a smoothness $W\subset Z_+^d$ provided there exist smoothnesses $W=V_0, V_1, \ldots, V_r=V$ such that V_k is simply generated by V_{k-1} for $k=1,2,\ldots,r$.

In the sequel we shall need the following fact:

LEMMA 0.3. *Let $d\geqslant 3$. Let $W\subset Z_+^d$ be a smoothness which is not an interval. Then W simply generates a smoothness $V\subset Z_+^{d-1}$ which is not an interval.*

Proof. Pick an $A^0\in W$ so that

$$s(A^0) = \max\{s(A):\ A\in W\}$$

where $s(A)$ denotes the sum of the coordinates of A. Clearly A^0 is a maximal element in W, i.e., if $A\in W$ and $A^0\leqslant A$ then $A=A^0$. Since W is not an interval, there exists in W an element B such that $B\nleqslant A^0$. Thus after a suitable rearrangement of coordinates we may assume that $B=(b_j)$, $A^0=(a_j^0)$ and $a_1^0<b_1$.

Put $V=\varphi(W)$ where φ is in the new arrangement defined by (0.5). Observe that the quantity $s(A)$ does not depend on the particular arrangement of coordinates, and

$$s(\varphi(A)) = s(A) \quad \text{for } A\in W.$$

Thus

$$s(\varphi(A^0)) = \max\{s(C):\ C \in V\}.$$

Hence $\varphi(A^0)$ is a maximal element in V.

Let $\varphi(A)_1$ denote the first coordinate of $\varphi(A)$ (in the new arrangement). Taking into account that $d \geqslant 3$ we then have

$$\varphi(A^0)_1 = a_1^0 < b_1 = \varphi(B)_1.$$

Thus $\varphi(B)$ is an element of V such that $\varphi(B) \nleqslant \varphi(A^0)$. Hence V is not an interval.

COROLLARY 0.1. *Let $d \geqslant 3$. A smoothness $W \subset \mathbf{Z}_+^d$ is an interval if and only if every smoothness in $\mathbf{Z}_+^2$ which is generated by W is an interval.*

Proof. Clearly every smoothness generated by an interval is an interval. To prove the converse apply Lemma 0.3 and use the induction with respect to the number of coordinates.

0.2. Differentiable functions. Throughout this paper $\mathbf{R}$ stands for the real line, $\mathbf{C}$ for the complex plane, $\mathbf{R}^d$ for the Cartesian product of d copies of $\mathbf{R}$ $(d = 1, 2, \ldots)$. For $X = (x_j) \in \mathbf{R}^d$, $Y = (y_j) \in \mathbf{R}^d$ we put

$$(X, Y) = \sum_{j=1}^{d} x_j y_j \quad \text{and} \quad \|X\| = \Big(\sum_{j=1}^{d} x_j^2\Big)^{1/2}.$$

A function is understood to be complex-valued unless otherwise stated; $\operatorname{supp} f$ denotes the support of an $f\colon \mathbf{R}^d \to \mathbf{C}$, i.e. the closure of the set $\{X \in \mathbf{R}^d:\ |f(X)| > 0\}$. $C_0^\infty(\mathbf{R}^d)$ stands for the space of all infinitely differentiable functions with compact support; $\mathscr{S}(\mathbf{R}^d)$ — for the space of all infinitely differentiable functions on $\mathbf{R}^d$ rapidly decreasing together with all their derivatives (cf. [SCH]). The symbols $\langle f, g\rangle$ and $f * g$ stand for the scalar product and the convolution of functions on $\mathbf{R}^d$. For $f\colon \mathbf{R}^d \to \mathbf{C}$ we denote (if they exist) by $\hat{f}$ and $\check{f}$ the Fourier Transform and the inverse Fourier Transform defined by

$$\hat{f}(\Xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^d} f(X) e^{-i(X,\Xi)}\, dX,$$
$$\check{f}(X) = (2\pi)^{-n/2} \int_{\mathbf{R}^d} f(\Xi) e^{i(\Xi,X)}\, d\Xi.$$

Given an $A = (a_j) \in \mathbf{Z}_+^d$ by $D^A = D_{x_1}^{a_1} \ldots D_{x_d}^{a_d}$ we denote the operator of partial derivative; we always assume that the mixed derivatives do not depend on the order of differentiation with respect to different variables. It is also natural to assume that if for some $u\colon \mathbf{R}^d \to \mathbf{C}$ the derivative $D^A u$ exists then there do exist all derivatives $D^B u$ which require no more differentiation with respect to any variable than D^A. In other words we assume that if $D^A u$ exists then $D^B u$ exists for every B with $0 \leqslant B \leqslant A$. This is exactly condition (0.4) appearing in Definition 0.1 of sets of smoothness.

Next we introduce a concept of fractional derivatives which we shall use.

Given $\alpha \geqslant 0$ and an index j with $1 \leqslant j \leqslant d$ we define operators $\mathring{D}_{x_j}^\alpha\colon \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}(\mathbf{R}^d)$ and $|D_{x_j}^\alpha|\colon \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}(\mathbf{R}^d)$ as follows: for a $u \in \mathscr{S}(\mathbf{R}^d)$, $\mathring{D}_{x_j}^\alpha u$ is the unique $V \in \mathscr{S}(\mathbf{R}^d)$ such that

$$\hat{V}(\Xi) = i^\alpha \operatorname{sign} \xi_j |\xi_j|^\alpha \hat{u}(\Xi) \qquad \text{for } \Xi = (\xi_1, \xi_2, \ldots, \xi_d);$$

similarly $|D_{x_j}^\alpha| u$ is the unique $W \in \mathscr{S}(\mathbf{R}^d)$ such that $\hat{W}(\Xi) = |\xi_j|^\alpha \hat{u}(\Xi)$. Let $\partial_{x_j}^\alpha$ denote either $\mathring{D}_{x_j}^\alpha$ or $|D_{x_j}^\alpha|$. The composition $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_d}^{\alpha_d}$ is called an *operator of fractional derivative* and its value for a $u \in \mathscr{S}(\mathbf{R}^d)$ a *fractional derivative* of u. In particular, if all $\partial_{x_j}^{\alpha_j}$ are the operators $|D_{x_j}^{\alpha_j}|$ we write $|D_{x_1}^{\alpha_1} \ldots D_{x_d}^{\alpha_d}|$ and we call the function $|D_{x_1}^{\alpha_1} \ldots D_{x_d}^{\alpha_d}| u$ an *absolute fractional derivative* of u.

0.3. Anisotropic Sobolev spaces. In this paper we study spaces of smooth functions for an arbitrary smoothness W. On the other hand we restrict ourselves to the case where the domain of the functions is either the whole space $\mathbf{R}^d$ or the torus $\mathbf{T}^d$.

Given a measurable $f\colon \mathbf{R}^d \to \mathbf{C}$ we denote as usual

$$(0.6)\qquad \begin{aligned} \|f\|_p &= \Big(\int_{\mathbf{R}^d} |f(X)|^p\, dX\Big)^{1/p} && \text{for } 0 < p < \infty,\\ \|f\|_\infty &= \operatorname{ess\,sup}\{|f(X)|:\ X \in \mathbf{R}^d\} && \text{for } p = \infty. \end{aligned}$$

Similarly for $f\colon [-\pi, \pi]^d \to \mathbf{C}$ we put

$$(0.7)\qquad \begin{aligned} \|f\|_p &= \Big((2\pi)^{-d} \int_{[-\pi,\pi]^d} |f(X)|^p\, dX\Big)^{1/p},\\ \|f\|_\infty &= \operatorname{ess\,sup}\{|f(X)|:\ X \in [-\pi, \pi]^d\}. \end{aligned}$$

Next for a fixed smoothness $W \subset \mathbf{Z}_+^d$ we put

$$(0.8)\qquad \|f\|_{p,W} = \Big(\sum_{A\in W} \|D^A f\|_p^p\Big)^{1/p}, \qquad \|f\|_{\infty,W} = \max_{A \in W} \|D^A f\|_\infty.$$

By $L_W^p(\mathbf{R}^d)$ we denote the completion of $C_0^\infty(\mathbf{R}^d)$ under the norm $\|\cdot\|_{p,W}$ and by $C_W(\mathbf{R}^d)$ the completion of $C_0^\infty(\mathbf{R}^d)$ under the norm $\|\cdot\|_{\infty,W}$. Note that for $0 < p < \infty$ the usual spaces $L^p(\mathbf{R}^d)$ coincide with $L_{\{0\}}^p(\mathbf{R}^d)$ (for the one point smoothness $\{0\}$) while $C_{\{0\}}(\mathbf{R}^d)$ coincides with the space of all continuous functions on $\mathbf{R}^d$ vanishing at infinity. The definition of the space $L^{\infty}_W(\mathbf{R}^d)$ is slightly more complicated. Consider the Banach space $E = \bigoplus_{A\in W} L^1(\mathbf{R}^d)$ of all sequences $(g_A)_{A\in W}$ with $g_A \in L^1(\mathbf{R}^d)$ for $A\in W$ equipped with the norm

$$\|(g_A)_{A\in W}\| = \sum_{A\in W} \|g_A\|_1 .$$

Clearly $C_W(\mathbf{R}^d)$ can be identified with the subspace of the dual E^* of E consisting of the functionals Φ_f^* with $f\in C_W(\mathbf{R}^d)$ where

$$\Phi_f^*((g_A)_{A\in W}) = \sum_{A\in W} \int_{\mathbf{R}^d} (D^A f)(X) g_A(X)\, dX \quad \text{ for } (g_A)_{A\in W} \in E.$$

Now we define $L^{\infty}_W(\mathbf{R}^d)$ to be the closure of this subspace in the weak star topology of E^*. It is not hard to show that $L^{\infty}_W(\mathbf{R}^d)$ can be identified with the space of functions $f\colon \mathbf{R}^d \to \mathbf{C}$ having essentially bounded generalized derivatives (cf. [AD] for definition) $D^A f$ for $A\in W$ with the norm $\|f\|_{\infty,W}$.

The definition of the spaces $C_W(\mathbf{T}^d)$ and $L^p_W(\mathbf{T}^d)$ for $0 \leqslant p \leqslant \infty$ is almost the same. The role of the class $C_0^{\infty}(\mathbf{R}^d)$ of "test functions" is played by the space $\mathcal{T}(\mathbf{T}^d)$ of all trigonometric polynomials $f\colon [-\pi,\pi]^d \to \mathbf{C}$, i.e. of functions of the form

$$f(X) = \sum_{P\in \mathbf{Z}^d} \alpha_P e^{i(X,P)}$$

with $\{P\in \mathbf{Z}^d\colon \alpha_P \neq 0\}$ finite. We take completions of $\mathcal{T}(\mathbf{T}^d)$ under the norms defined by (0.7) and (0.8). Clearly the spaces $L^p_W(\mathbf{T}^d)$ and $C_W(\mathbf{T}^d)$ can be regarded as spaces of functions on the torus $\mathbf{T}^d$.

The spaces $L^p_W(\mathbf{R}^d)$ and $C_W(\mathbf{R}^d)$, $L^p_W(\mathbf{T}^d)$ and $C_W(\mathbf{T}^d)$ are called (*anisotropic*) *Sobolev spaces* of smoothness W. They are Banach spaces except the case where $p<1$.

To distinguish between functions on $\mathbf{T}^d$ and $\mathbf{R}^d$ we often use for the norms of functions on $\mathbf{T}^d$ the symbols $\|\cdot\|_{L^p_W(\mathbf{T}^d)}$ and $\|\cdot\|_{C_W(\mathbf{T}^d)}$ reserving the symbols $\|\cdot\|_{p,W}$ for functions on $\mathbf{R}^d$.

0.4. A lemma on oscillating integrals. The result of this section is probably known. (For similar results cf. [DIE], IV.4). We include it for self-sufficiency of the paper.

LEMMA 0.4. *Let g and φ be functions on a finite closed interval $[a, b]$. Assume that g is a complex-valued absolutely continuous function, in particular*

$$\|g'\|_1 = \int_a^b |g'(t)|\, dt < +\infty .$$

Furthermore assume that φ is real-valued, three times continuously differentiable, φ' has finitely many zeros and $\varphi''(t)\neq 0$ whenever $\varphi'(t)=0$.

Then there exists a constant $C(\varphi)=C$ depending only on φ such that for all $\varrho>0$,

$$(0.9)\qquad \Big|\int_a^b g(t) e^{i\varrho\varphi(t)}\, dt\Big| \leqslant C\min(1, \varrho^{-1/2})(\|g\|_\infty + \|g'\|_1).$$

Moreover, if φ is $(b-a)$-periodic together with its derivatives, then $C(\varphi)=C(\varphi_{t_0})$ for every $t_0\in[a,b]$ where $\varphi_{t_0}(t)=\varphi(t-t_0)$.

Proof. If $\varrho\leqslant 1$ then we use the trivial estimate

$$\Big|\int_a^b g(t) e^{i\varrho\varphi(t)}\, dt\Big| \leqslant \|g\|_\infty (b-a).$$

In the sequel we assume that $\varrho>1$. Dividing the original interval into a finite number of intervals determined by the zeros of φ' and replacing accordingly φ by $\varphi_{\varepsilon,\eta,c}$ where

$$\varphi_{\varepsilon,\eta,c}(t) = \varepsilon\varphi(\eta t + c) - \varepsilon\varphi(c), \quad \varepsilon=\pm 1,\ \eta=\pm 1,\ c\in\mathbf{R}$$

we may assume without loss of generality that

$$(0.10)\qquad a=0,\quad \varphi(0)=0,\quad \varphi'(t)>0 \quad \text{for } 0<t\leqslant b.$$

First we consider two special cases:

(i) $\varphi(t)=t$. Then integration by parts gives

$$J_1(g) = \int_0^b g(t) e^{i\varrho t}\, dt = -\int_0^b g'(t)(i\varrho)^{-1} e^{i\varrho t}\, dt + (i\varrho)^{-1}[g(b)e^{i\varrho b} - g(0)].$$

Hence $|J_1(g)| \leqslant \varrho^{-1}(\|g'\|_1 + 2\|g\|_\infty)$.

(ii) $\varphi(t)=t^2$. Put $F_\varrho(t)=\int_t^\infty e^{i\varrho\tau^2}\, d\tau$. It is well known (cf. [VAL], p. 160) that this integral called the Fresnel integral exists and obviously $M_\varrho = \sup\{|F_\varrho(t)|\colon\ 0\leqslant t<+\infty\} < +\infty$. Substituting $\sigma=\varrho^{1/2}\tau$, we get $F_\varrho(t)=\varrho^{-1/2}F_1(\varrho^{1/2}t)$. Hence $M_\varrho\leqslant \varrho^{-1/2}M_1$.

Now we integrate our oscillating integral by parts,

$$J_2(g) = \int_0^b g(t) e^{i\varrho t^2}\, dt = \int_0^b g'(t)F_\varrho(t)\, dt + g(0)F_\varrho(0) - g(b)F_\varrho(b).$$

Hence

$$|J_2(g)| \leqslant (\|g'\|_1 + 2\|g\|_\infty) M_\varrho \leqslant 2\varrho^{-1/2} M_1 (\|g'\|_1 + \|g\|_\infty).$$

Now we pass to the general case. Again it splits into two cases:

1°. $\varphi'(0)\neq 0$. Hence, by (0.10), there is $\delta=\delta_\varphi>0$ such that $\varphi'(t)>\delta$ for

$0 \leqslant t \leqslant b$. We substitute $t = \varphi^{-1}(s)$. Then

$$\int_0^b g(t)\, e^{i\varrho\varphi(t)}\, dt = \int_0^{\varphi(b)} f(s)\, e^{i\varrho s}\, ds = J_1(f)$$

where

$$f(s) = g(\varphi^{-1}(s))\left[\varphi'(\varphi^{-1}(s))\right]^{-1}.$$

Clearly $\|f\|_\infty \leqslant \delta^{-1}\|g\|_\infty$ while

$$f'(s) = g'(\varphi^{-1}(s))\left[\varphi'(\varphi^{-1}(s))\right]^{-2} + \varphi''(\varphi^{-1}(s))\left[\varphi'(\varphi^{-1}(s))\right]^{-1} g(\varphi^{-1}(s)).$$

Thus

$$\begin{aligned}\|f'\|_1 &\leqslant \delta^{-1}\int_0^{\varphi(b)} |g'(\varphi^{-1}(s))|\left[\varphi'(\varphi^{-1}(s))\right]^{-1} ds \\ &\quad + \|\varphi''\|_\infty \int_0^{\varphi(b)} g(\varphi^{-1}(s))\left[\varphi'(\varphi^{-1}(s))\right]^{-1} ds \\ &= \delta^{-1}\|g'\|_1 + \|\varphi''\|_\infty \|g\|_1 \\ &\leqslant \delta^{-1}\|g'\|_1 + b\|\varphi''\|_\infty \|g\|_\infty.\end{aligned}$$

Combining these estimates with the estimates for $J_1(f)$ given in (i), we get (0.9). Note that b depends only on the splitting of the original interval into subintervals which was determined by the original function φ.

2°. $\varphi'(0) = 0$. By (0.10), φ is one-to-one and nonnegative. Thus $t(s) = \varphi^{-1}(s^2)$ is a well-defined increasing function for $0 \leqslant s \leqslant \sqrt{\varphi(b)}$. We check first that t' is continuous on $[0, \sqrt{\varphi(b)}]$. Since $\varphi'(t) > 0$ for $t > 0$, it is enough to check it at the point $s = 0$. Note that $t(0) = 0$. Thus

$$t'(0) = \lim_{s=0}\frac{t(s)-t(0)}{s} = \lim_{s=0}\frac{t(s)}{s} = \lim_{t=0}\frac{t}{\sqrt{\varphi(t)}} = \sqrt{\frac{2}{\varphi''(0)}}$$

(because the condition $0 = \varphi(0) = \varphi'(0)$ yields $\lim_{t=0}(\varphi(t)/t^2) = \varphi''(0)/2$; moreover $\varphi''(0) \neq 0$ because $\varphi'(0) = 0$).

For $s > 0$ we have $t'(s) = 2s/\varphi'(\varphi^{-1}(s^2))$. Thus

$$\begin{aligned}\lim_{s=0} t'(s) &= \lim_{t=0}\frac{2\sqrt{\varphi(t)}}{\varphi'(t)} = \lim_{t=0} 2\frac{\sqrt{\varphi(t)}}{t}\cdot\frac{t}{\varphi'(t)} \\ &= 2\sqrt{\frac{\varphi''(0)}{2}}\cdot\frac{1}{\varphi''(0)} = t'(0).\end{aligned}$$

This proves the continuity of t' at 0 and therefore in the whole interval $[0, \sqrt{\varphi(b)}]$.

Now, to estimate our oscillating integral, we substitute $t = t(s)$. We then get

$$\int_0^b g(t)\, e^{i\varrho\varphi(t)}\, dt = \int_0^{\sqrt{\varphi(b)}} f(s)\, e^{i\varrho s^2}\, ds = J_2(f)$$

where $f(s) = g(t(s))\, t'(s)$. Clearly $\|f\|_\infty \leqslant \|g\|_\infty \|t'\|_\infty = C_1(\varphi)\|g\|_\infty$. Next we have, for $s > 0$,

$$f'(s) = g'(t(s))\,(t'(s))^2 + g(t(s))\, t''(s).$$

For the first summand, remembering that $t' > 0$, we have the estimate

$$\begin{aligned}\int_0^{\sqrt{\varphi(b)}} |g'(t(s))|\,[t'(s)]^2\, ds &\leqslant \|t'\|_\infty \int_0^{\sqrt{\varphi(b)}} |g'(t(s))|\, t'(s)\, ds \\ &= \|t'\|_\infty \|g'\|_1.\end{aligned}$$

To estimate the second summand observe that, for $s > 0$,

$$\begin{aligned}t''(s) &= \frac{d}{ds}\left(\frac{2s}{\varphi'(\varphi^{-1}(s^2))}\right) = 2\frac{[\varphi'(\varphi^{-1}(s^2))]^2 - 2s^2\varphi''(\varphi^{-1}(s^2))}{[\varphi'(\varphi^{-1}(s^2))]^3} \\ &= 2\frac{[\varphi'(t)]^2 - 2\varphi(t)\varphi''(t)}{[\varphi'(t)]^3}.\end{aligned}$$

Thus, using for instance de l'Hopital's rule, we get

$$\begin{aligned}\lim_{s=0} t''(s) &= \lim_{t=0} 2\frac{[\varphi'(t)]^2 - 2\varphi(t)\varphi''(t)}{t^3}\lim_{t=0}\left(\frac{t}{\varphi'(t)}\right)^3 \\ &= \frac{-2\varphi''(0)\varphi'''(0)}{3}[\varphi''(0)]^{-3} = -\frac{2\varphi'''(0)}{3[\varphi''(0)]^2}.\end{aligned}$$

This shows that t'' is essentially bounded (in fact continuous) on $[0, \sqrt{\varphi(b)}]$. Thus for the second summand we have the estimate

$$\int_0^{\sqrt{\varphi(b)}} |g(t(s))|\,|t''(s)|\, ds \leqslant \|g\|_\infty \|t''\|_\infty \sqrt{\varphi(b)} = C_2(\varphi)\|g\|_\infty.$$

Combining these estimates with the estimate for $J_2(f)$ given in (ii) we get (0.9).

The "moreover" part follows from an easy analysis of the dependence of $C(\varphi)$ on φ.

We shall use the following simple consequence of Lemma 0.4.

COROLLARY 0.2. *Let g: $[0, 2\pi] \to \boldsymbol{C}$ satisfy the assumptions of Lemma* 0.1, *let $c > 0$, $d > 0$, $X = (x, y) \in \boldsymbol{R}^2$ with $\|X\| = (x^2+y^2)^{1/2} > 0$. Then there is a* *numerical constant C independent of X, c, d such that for $r > 0$*

$$(0.11)\qquad \Big|\int_0^{2\pi} g(t)\, e^{i(xr^c\cos t + yr^d \sin t)}\, dt\Big| \leqslant C \min(1, r^{-\gamma})(\|g\|_\infty + \|g'\|_1)\, \|X\|^{-1/2}$$

where $2\gamma = \min(c, d)$.

Proof. Put $\varrho = (x^2 r^{2c} + y^2 r^{2d})^{1/2}$ and define $t_0 \in [0, 2\pi]$ so that $xr^c = \varrho\cos t_0$, $yr^d = \varrho \sin t_0$. Then $xr^c \cos t + yr^d \sin t = \varrho\cos(t - t_0)$. Thus applying Lemma 0.4 for translates of the function $\varphi(t) = \cos t$ and using the obvious estimate $\varrho \geqslant r^{2\gamma}\|X\|$ we get (0.11) with $C = C(\cos t)$.

1. A 2-dimensional Sobolev type inequality; elementary case. Our aim is the following inequality.

THEOREM A. *Let n, m be positive integers. Then there exists an absolute constant $C(n, m)$ such that for every $u \in C_0^\infty(\boldsymbol{R}^2)$*

$$(1.1)\qquad \| |D_x^{(n-1)/2} D_y^{(m-1)/2}| u\|_2^2 \leqslant C(n, m)\, \|D_x^n u\|_1\, \|D_y^m u\|_1 .$$

In this section we prove (1.1) in the case when n and m are both odd positive integers. This is the elementary case when the absolute fractional derivative $|D_x^{(n-1)/2} D_y^{(m-1)/2}| u$ in (1.1) can be replaced by the ordinary partial derivative $D_x^k D_y^l u$ where $n = 2k+1$, $m = 2l+1$. In particular, for $k = l = 0$ we get the classical Sobolev inequality.

Note that the quantity $\|\partial_x^\alpha \partial_y^\beta u\|_2^2$ is the same for all four fractional derivatives. Moreover, if either α or β or both are nonnegative integers then the quantity does not change if the partial fractional derivative is replaced by the ordinary partial derivative.

We begin with establishing a simple identity (1.2) which yields the classical Sobolev inequality (1.3).

Let $\boldsymbol{R}^+ = \{a \in \boldsymbol{R}\colon a > 0\}$, $\boldsymbol{R}^- = -\boldsymbol{R}^+$. Put

$$\varphi = \chi_{\boldsymbol{R}^+ \times \boldsymbol{R}^-} + \chi_{\boldsymbol{R}^- \times \boldsymbol{R}^+} - \chi_{\boldsymbol{R}^+ \times \boldsymbol{R}^+} - \chi_{\boldsymbol{R}^- \times \boldsymbol{R}^-}$$

where χ_A denotes the characteristic function of a set A.

LEMMA 1.1. *For $u, v \in C_0^\infty(\boldsymbol{R}^2)$ one has*

$$(1.2)\qquad \langle 4^{-1} \varphi * D_x w, D_y v\rangle = \langle w, v\rangle,$$

$$(1.3)\qquad |\langle w, v\rangle| \leqslant 4^{-1}\, \|D_x w\|_1\, \|D_y v\|_1 .$$

Proof. (1.2) is an immediate consequence of the identity

$$\langle \chi_{\varepsilon_1 \boldsymbol{R}^+ \times \varepsilon_2 \boldsymbol{R}^+} * D_x w, D_y v\rangle = -\varepsilon_1 \varepsilon_2 \langle w, v\rangle$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. The latter identity follows directly by applying the Fubini Theorem combined with the integration by parts formula (note that for $u \in C_0^\infty(\boldsymbol{R}^2)$, $u(\pm\infty, \cdot) = u(\cdot, \pm\infty) = 0$).

To obtain (1.3) note that $\|\varphi\|_\infty = 1$ and use (1.2) and the standard inequalities $|\langle f, g\rangle| \leqslant \|f\|_\infty \|g\|_1$ and $\|f * g\|_\infty \leqslant \|f\|_\infty \|g\|_1$; indeed,

$$|\langle \varphi * D_x w, D_y v\rangle| \leqslant \|\varphi * D_x w\|_\infty \|D_y v\|_1 \leqslant \|\varphi\|_\infty \|D_x w\|_1 \|D_y v\|_1 = \|D_x w\|_1 \|D_y v\|_1 .$$

Now we are ready for:

PROPOSITION 1.1. *If n and m are odd natural numbers, then for every $u \in C_0^\infty(\boldsymbol{R}^2)$ one has*

$$(1.4)\qquad \|D_x^{(n-1)/2} D_y^{(m-1)/2} u\|_2^2 \leqslant 4^{-1} \|D_x^n u\|_1 \|D_y^m u\|_1 .$$

Proof. We apply (1.3) for $w = D_x^{n-1} u$ and $v = D_y^{m-1} u$. Then

$$|\langle D_x^{n-1} u, D_y^{m-1} u\rangle| \leqslant 4^{-1} \|D_x^n u\|_1 \|D_y^m u\|_1 .$$

On the other hand, applying twice the Plancherel identity and taking into account that $n-1$ and $m-1$ are even integers, we get

$$\begin{aligned}\langle D_x^{n-1} u, D_y^{m-1} u\rangle &= \langle (D_x^{n-1} u)\hat{}\,, (D_y^{m-1} v)\hat{}\,\rangle \\ &= \iint_{\boldsymbol{R}^2} \xi^{n-1} \eta^{m-1} |\hat u(\xi, \eta)|^2\, d\xi\, d\eta \\ &= \iint_{\boldsymbol{R}^2} |\xi^{(n-1)/2} \eta^{(m-1)/2}\, \hat u(\xi, \eta)|^2\, d\xi\, d\eta \\ &= \iint_{\boldsymbol{R}^2} |(D_x^{(n-1)/2} D_y^{(m-1)/2} u)\hat{}\,|^2\, d\xi\, d\eta \\ &= \|D_x^{(n-1)/2} D_y^{(m-1)/2} u\|_2^2 .\end{aligned}$$

2. The proof of Theorem A in the cases $n = m = 2k$ and $\min(n, m) = 1$. The results of this section depend upon Corollary 2.1 which gives an estimate for the sup-norm of an arbitrary smooth function of two variables by the L^1-norms of its pure derivatives of the second order. It is a special case of both Theorem 3.1 and Proposition 3.2 proved in the next section by more advanced methods. We begin with a known identity (2.1) in the theory of wave equation which immediately yields Corollary 2.1.

For $\tau > 0$ we put

$$W_\tau = \{(\alpha, \beta) \in \boldsymbol{R}^2\colon\ 0 < |\beta| < \tau\alpha\},$$

$$W_\tau' = \{(\alpha, \beta) \in \boldsymbol{R}^2\colon\ 0 < |\alpha| < \tau^{-1}\beta\},$$

$$\psi_\tau = \chi_{W_\tau} + \chi_{-W_\tau} - \chi_{W_\tau'} - \chi_{-W_\tau'} .$$

LEMMA 2.1. *For $u \in C_0^\infty(\boldsymbol{R}^2)$ and $\tau > 0$ one has*

$$(2.1)\qquad u = (8\tau)^{-1} (D_x^2 u - \tau^2 D_y^2 u) * \psi_\tau .$$

Proof. We shall verify that for $\tau > 0$

$$u = (2\tau)^{-1}(D_x^2 u - \tau^2 D_y^2 u) * \chi_{W_\tau}.$$

The proof of analogous identities for $-W_\tau$ and $\pm W_\tau'$ is the same.

Fix $X_0 = (x_0, y_0) \in \boldsymbol{R}^2$ and put $s = (x_0 + \tau^{-1} y_0)/2$, $t = (x_0 - \tau^{-1} y_0)/2$. Define $v \in C_0^\infty(\boldsymbol{R}^2)$ by $v(a, b) = u(a+b, \tau(a-b))$. Observe that

$$D_x^1 D_y^1 v(a, b) = (D_x^2 u - \tau^2 D_y^2 u)(a+b, \tau(a-b)), \tag{2.2}$$

$$\chi_{W_\tau}(a+b, \tau(a-b)) = \chi_{\boldsymbol{R}^+ \times \boldsymbol{R}^+}(a, b). \tag{2.3}$$

Next we put

$$I(X_0) = \big((2\tau)^{-1}(D_x^2 u - \tau^2 D_y^2 u) * \chi_{W_\tau}\big)(X_0)$$
$$= (2\tau)^{-1} \iint_{\boldsymbol{R}^2} (D_x^2 u - \tau^2 D_y^2 u)(x_0 - \xi, y_0 - \eta)\, \chi_{W_\tau}(\xi, \eta)\, d\xi\, d\eta.$$

We substitute $\xi = a+b$, $\eta = \tau(a-b)$. Taking into account (2.2), (2.3) and that the jacobian $J = \partial(\xi, \eta)/\partial(a, b) = -2\tau$ we get

$$I(X_0) = (2\tau)^{-1} \iint_{\boldsymbol{R}^2} (D_x^1 D_y^1 v)(s-a, t-b)(|J|\, \chi_{\boldsymbol{R}^+ \times \boldsymbol{R}^+})(a, b)\, da\, db$$
$$= \int_0^\infty \Big(\int_0^\infty D_x^1 D_y^1 v(s-a, t-b)\, da\Big)\, db$$
$$= v(s, t) = u(X_0).$$

Corollary 2.1. *For every $u \in C_0^\infty(\boldsymbol{R}^2)$ one has*

$$\|u\|_\infty \leqslant 4^{-1}(\|D_x^2 u\|_1 \|D_y^2 u\|_1)^{1/2}, \tag{2.4}$$

$$\|u\|_\infty \leqslant 8^{-1}(\|D_x^2 u\|_1 + \|D_y^2 u\|_1). \tag{2.5}$$

Proof. Note that $\|\psi_\tau\|_\infty = 1$ for every $\tau > 0$. Thus, combining (2.1) with the Young inequality we get

$$\|u\|_\infty \leqslant \inf\{(8\tau)^{-1}\|D_x^2 u - \tau^2 D_y^2 u\|_1 \colon \tau > 0\}$$
$$\leqslant \inf\{(8\tau)^{-1}(\|D_x^2 u\|_1 + \tau^2 \|D_y^2 u\|_1) \colon \tau > 0\}$$
$$= 4^{-1}(\|D_x^2 u\|_1 \|D_y^2 u\|_1)^{1/2}.$$

This proves (2.4). Combining the inequality between the arithmetic and geometric means with (2.4) we get (2.5).

Our next result gives a proof of Theorem A for $n = m = 2k$.

Proposition 2.1. *Let $n = 2k$ be a positive even integer. Then for every $u \in C_0^\infty(\boldsymbol{R}^2)$ one has*

$$\||D_x^{(n-1)/2} D_y^{(n-1)/2}| u\|_2^2 \leqslant (2n)^{-1} \|D_x^n u\|_1 \|D_y^n u\|_1. \tag{2.6}$$

Proof. Consider the differential operators depending on the parameter $\tau > 0$

$$\tilde{Q}_\tau^{(n)} = \tau^{-k} D_x^n - \tau^k D_y^n, \qquad \tilde{P}_\tau^{(n)} = \sum_{j=0}^{k-1} \tau^{k-2j-1} D_x^{2j} D_y^{2(k-j-1)}$$

which correspond to the polynomials

$$Q_\tau^{(n)}(\xi, \eta) = \tau^{-k} \xi^n - \tau^k \eta^n, \qquad P_\tau^{(n)}(\xi, \eta) = \sum_{j=0}^{k-1} \tau^{k-2j-1} \xi^{2j} \eta^{2(k-j-1)}$$

respectively. Clearly we have the identity

$$\tau^{-1}(\xi^2 - \tau^2 \eta^2) P_\tau^{(n)}(\xi, \eta) = Q_\tau^{(n)}(\xi, \eta)$$

which yields the identity

$$\tau^{-1}(D_x^2 - \tau^2 D_y^2) \tilde{P}_\tau^{(n)} = \tilde{Q}_\tau^n.$$

The latter identity combined with (2.1) gives

$$8^{-1} \tilde{Q}_\tau^{(n)} u * \psi_\tau = (8\tau)^{-1}(D_x^2 - \tau^2 D_y^2)(\tilde{P}_\tau^{(n)} u) * \psi_\tau = \tilde{P}_\tau^{(n)} u.$$

Hence

$$\|\tilde{P}_\tau^{(n)} u\|_\infty \leqslant 8^{-1} \|\psi_\tau\|_\infty \|\tilde{Q}_\tau^{(n)} u\|_1 \leqslant 8^{-1}(\tau^{-k}\|D_x^n u\|_1 + \tau^k \|D_y^n u\|_1).$$

Thus

$$|\langle \tau^{-k} D_x^n u + \tau^k D_y^n u, \tilde{P}_\tau^{(n)} u\rangle| \leqslant 8^{-1}(\tau^{-k}\|D_x^n u\|_1 + \tau^k \|D_y^n u\|_1)^2. \tag{2.7}$$

Next note that $(\tau^{-k}\xi^n + \tau^k \eta^n) P_\tau^{(n)}(\xi, \eta) = W_\tau(\xi^2, \eta^2)$ where W_τ is a polynomial (in ξ^2 and η^2) with nonnegative coefficients. Therefore, using the inequality between arithmetic mean and geometric mean, we get

$$W_\tau(\xi^2, \eta^2) \geqslant n|\xi\eta|^{n-1}. \tag{2.8}$$

Combining (2.8) with the Plancherel identity (used twice) we get

$$\langle \tau^{-k} D_x^n u + \tau^k D_y^n u, \tilde{P}_\tau^{(n)} u\rangle = \langle (\tau^{-k} D_x^n u + \tau^k D_y^n u)\hat{\ }, (\tilde{P}_\tau^{(n)} u)\hat{\ }\rangle$$
$$= \iint_{\boldsymbol{R}^2} W_\tau(\xi^2, \eta^2) |\hat{u}(\xi, \eta)|^2\, d\xi\, d\eta$$
$$\geqslant \iint_{\boldsymbol{R}^2} n\big(|\xi\eta|^{(n-1)/2} |\hat{u}(\xi, \eta)|\big)^2\, d\xi\, d\eta$$
$$= n\|D_x^{(n-1)/2} D_y^{(n-1)/2} u\|_2^2.$$

Combining the latter inequality with (2.7) and minimizing over all $\tau > 0$ we get

$$\|D_x^{(n-1)/2} D_y^{(n-1)/2} u\|_2^2 \leqslant (8n)^{-1} \inf\{(\tau^{-k}\|D_x^n u\|_1 + \tau^k \|D_y^n u\|_1)^2 \colon \tau > 0\}$$
$$= (2n)^{-1} \|D_x^n u\|_1 \|D_y^n u\|_1.$$

For the proof of Theorem A in the case $\min(n, m) = 1$, we need the following one-dimensional inequality which is probably well known but which also is a consequence of Propositions 1.1 and 2.1.

LEMMA 2.2. *Let $f \in C_0^\infty(\boldsymbol{R})$. Then, for every positive integer n,*

$$|||D^{(n-1)/2}| f\|_2^2 \leqslant 2^{-1} \|f\|_1 \|D^n f\|_1. \tag{2.9}$$

Proof. Consider $u \in C_0^\infty(\boldsymbol{R}^2)$ defined by $u(x, y) = f(x)f(y)$. Note that

$$\||D_x^{(n-1)/2} D_y^{(n-1)/2}| u\|_2^2 = \||D^{(n-1)/2}| f\|_2^4;$$

$$\|D_x^n u\|_1 = \|D_y^n u\|_1 = \int_{\boldsymbol{R}} |f(x)|\, dx \cdot \int_{\boldsymbol{R}} |D^n f(y)|\, dy = \|f\|_1 \|D^n f\|_1.$$

Thus, using inequality (1.4) for n odd, and (2.6) for n even, we get

$$\|D^{(n-1)/2} f\|_2^4 \leqslant C(n) \|f\|_1^2 \|D^n f\|_1^2$$

where $C(n) = 4^{-1}$ for n odd and $C(n) = (2n)^{-1}$ for n even. The latter inequality clearly yields (2.9) because $\sqrt{C(n)} \leqslant 2^{-1}$ for all n.

Now we are ready to establish Theorem A in the case $\min(n, m) = 1$. This is an immediate consequence of the next

PROPOSITION 2.2. *Let $u \in C_0^\infty(\boldsymbol{R}^2)$. Then*

$$\||D_x^{(n-1)/2}| u\|_2^2 \leqslant \|D_x^n u\|_1 \|D_y u\|_1. \tag{2.10}$$

Proof. Assume first that u is real and satisfies the condition:

(*) the set of zeros of u is a union of finitely many points and intervals of straight lines and the exterior of a square $\{\max(|x|, |y|) < a\}$ for some $a > 0$.

It can be easily seen that if a real u satisfies (*) then $F(y) = \int_{\boldsymbol{R}} |u(x, y)|\, dx$ is differentiable everywhere except may be a finite set and $\lim_{y=\infty} F(y) = 0$; the function $|u(x, y)|$ is differentiable with respect to y almost everywhere and we have the equality

$$\int_{\boldsymbol{R}} \big|D_y |u(x, y)|\big|\, dx = \int_{\boldsymbol{R}} |D_y u(x, y)|\, dx \qquad y\text{-a.e.} \tag{2.11}$$

Thus, applying (2.9) for the function $u(\cdot, y)$, y fixed, we get

$$\int_{\boldsymbol{R}} \||D_x^{(n-1)/2}| u(x, y)|^2\, dx \leqslant 2^{-1} \int_{\boldsymbol{R}} |u(x, y|\, dx \int_{\boldsymbol{R}} |D_x^n u(x, y)|\, dy.$$

Integrating this inequality against dy and making use of properties of real functions satisfying (*) we get:

$$\begin{aligned} \||D_x^{(n-1)/2}| u\|_2^2 &\leqslant 2^{-1} \int_{\boldsymbol{R}} \Big(\int_{\boldsymbol{R}} |u(x, y)|\, dx \int_{\boldsymbol{R}} |D_x^n u(x, y)|\, dx\Big) dy \\ &\leqslant 2^{-1} \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} |D_x^n u(x, y)|\, dx\, dy \max_{y \in \boldsymbol{R}} F(y) \\ &\leqslant 2^{-1} \|D_x^n u\|_1 \int_{\boldsymbol{R}} |F'(y)|\, dy. \end{aligned}$$

Next using (2.11) we have

$$\begin{aligned} \int_{\boldsymbol{R}} |F'(y)|\, dy &= \int_{\boldsymbol{R}} \left| \frac{d}{dy} \int_{\boldsymbol{R}} |u(x, y)|\, dx \right| dy = \int_{\boldsymbol{R}} \left| \int_{\boldsymbol{R}} D_y |u(x, y)|\, dx \right| dy \\ &\leqslant \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} \big|D_y |u(x, y)|\big|\, dx\, dy = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} |D_y u(x, y)|\, dx\, dy \\ &= \|D_y u\|_1. \end{aligned}$$

Thus for a real u satisfying (*) we get

$$\||D_x^{(n-1)/2}| u\|_2^2 \leqslant 2^{-1} \|D_x^n u\|_1 \|D_y u\|_1. \tag{2.12}$$

The set of u satisfying (*) is dense in the space of all real functions in $C_0^\infty(\boldsymbol{R}^2)$ in the topology of quasi-uniform convergence of functions with all their derivatives. Thus (2.12) extends to all real $u \in C_0^\infty(\boldsymbol{R}^2)$. Finally, applying (2.12) for the real and imaginary parts of an arbitrary $u \in C_0^\infty(\boldsymbol{R}^2)$, we get (2.10). (Note that (2.10) differs from (2.12) only by the "size" of a constant.)

3. Estimates of sup-norms of partial derivatives of a smooth function of two variables by the L^1-norms of its derivatives of higher orders. Theorem 3.1 of the present section gives the essential ingredient which makes it possible to complete the proof of Theorem A in the remaining cases. In fact it provides more information than is needed for Theorem A. Our proof of Theorem 3.1 bases upon Proposition 3.1 where the crucial analytic difficulties are overcome.

For the sake of brevity we denote in the sequel by $D(\alpha, \beta)$ each of the four derivatives $\mathring{D}_x^\alpha \mathring{D}_y^\beta$, $|D_x^\alpha| \mathring{D}_y^\beta$, $\mathring{D}_x^\alpha |D_y^\beta|$, $|D_x^\alpha D_y^\beta|$.

THEOREM 3.1. *Let $n \geqslant 2$, $m \geqslant 2$ be integers such that at least one of them is even. Let α, β be nonnegative numbers such that*

$$\alpha n^{-1} + \beta m^{-1} = 1 - n^{-1} - m^{-1}. \tag{3.1}$$

Then there exists a constant $C^* = C^*(\alpha, \beta, n, m)$ such that for every $u \in C_0^\infty(\boldsymbol{R}^2)$

$$\|D(\alpha, \beta) u\|_\infty \leqslant C^* (\|D_x^n u\|_1 + \|D_y^m u\|_1). \tag{3.2}$$

Theorem 3.1 is an easy consequence of the next proposition. Define $\varepsilon\colon \mathbf{Z}\to\{1,-1\}$ by

(3.3) $\varepsilon(r) = -1$ for $r \equiv 2 \bmod 4$ and $\varepsilon(r) = 1$ otherwise.

PROPOSITION 3.1. *Let n, m, α, β be as in Theorem 3.1. Then, for each of the four derivatives $D(\alpha,\beta)$, there exists a locally absolutely integrable function $E\colon \mathbf{R}^2\to\mathbf{C}$ such that*

(3.4) $$E*(D_x^n+\varepsilon D_y^m)u = D(\alpha,\beta)u \quad \textit{for every } u\in C_0^\infty(\mathbf{R}^2),$$

(3.5) $$E = C_1\log(|x|^m+|y|^n)+f$$

where $f\in L^\infty(\mathbf{R}^2)$ and $C_1 = C_1(\alpha,\beta,n,m)$ is a numerical constant and ε stands for $\varepsilon(m-n)$.

First we show how Proposition 3.1 implies Theorem 3.1.

Proof of Theorem 3.1. Assume the existence of an $E\colon \mathbf{R}^2\to\mathbf{C}$ satisfying (3.4) and (3.5). For $\tau>0$ put $E_\tau(x,y) = E(x,\tau^{-1}y)$. Then

(3.6) $$E_\tau-E_\sigma\in L^\infty(\mathbf{R}^2) \quad \text{for } 0<\sigma<\tau<\infty.$$

Indeed, by (3.5),

$$(E_\tau-E_\sigma)(x,y) = C_1\log\frac{|x|^m+\tau^{-n}|y|^n}{|x|^m+\sigma^{-n}|y|^n}+f\left(x,\frac{y}{\tau}\right)-f\left(x,\frac{y}{\sigma}\right).$$

Note that $n\log(\sigma/\tau)\leqslant \log\dfrac{|x|^m+\tau^{-n}|y|^n}{|x|^m+\sigma^{-n}|y|^n}\leqslant 0$ because of the identity

$$\frac{|x|^m-\tau^{-n}|y|^n}{|x|^m+\sigma^{-n}|y|^n} = 1-\frac{1-(\sigma\tau^{-1})^n}{1+|x|^m(\sigma|y|^{-1})^n}.$$

Thus $\|E_\tau-E_\sigma\|_\infty\leqslant n|\log(\sigma/\tau)|+2\|f\|_\infty$.

Next observe that (3.4) yields

(3.7) $$\tau^{\beta+1}D(\alpha,\beta)u = E_\tau*D_x^n u+\varepsilon\tau^m E_\tau*D_y^m u \quad \text{for } u\in C_0^\infty(\mathbf{R}^2) \text{ and } \tau>0.$$

To this end put $v(x,y) = u(x,\tau y)$. Then $D(\alpha,\beta)v(a,b) = \tau^\beta D(\alpha,\beta)u(a,\tau b)$. Hence using (3.4) with u replaced by v for arbitrary $(a,b)\in\mathbf{R}^2$ we obtain

$$\begin{aligned}\tau^\beta D(\alpha,\beta)u(a,\tau b) &= \big(E*(D_x^n+\varepsilon\tau^m D_y^m)v\big)(a,b) = \iint_{\mathbf{R}^2} E(a-x,b-y)(D_x^n+\varepsilon\tau^m D_y^m)u(x,\tau y)\,dx\,dy\\ &= \tau^{-1}\iint_{\mathbf{R}^2} E\left(a-x,\frac{\tau b-\tau y}{\tau}\right)(D_x^n+\varepsilon\tau^m D_y^m)u(x,\tau y)\,dx\,d(\tau y)\\ &= \tau^{-1}\big(E_\tau*(D_x^n+\varepsilon\tau^m D_y^m)u\big)(a,\tau b).\end{aligned}$$

The latter identity is obviously equivalent to (3.7).

Now fix $1<\sigma<\tau$. Put in (3.7) 1 in place of τ and substract "by sides" the new identity from the original one. Then

(3.8) $$(\tau^{\beta+1}-1)D(\alpha,\beta)u = (E_\tau-E_1)*D_x^n u+\varepsilon[(\tau^m-1)E_\tau+(E_\tau-E_1)]*D_y^m u.$$

Divide both sides of (3.8) by τ^m-1. Put in the obtained identity σ in place of τ and substract from (3.8) divided by τ^m-1. Then one gets

(3.9) $$\left[\frac{\tau^{\beta+1}-1}{\tau^m-1}-\frac{\sigma^{\beta+1}-1}{\sigma^m-1}\right]D(\alpha,\beta)u$$
$$= \left[\frac{E_\tau-E_1}{\tau^m-1}-\frac{E_\sigma-E_1}{\sigma^m-1}\right]*D_x^n u+\varepsilon\left[(E_\tau-E_\sigma)+\frac{E_\tau-E_1}{\tau^m-1}-\frac{E_\sigma-E_1}{\sigma^m-1}\right]*D_y^m u.$$

Clearly the function $\tau\to(\tau^{\beta+1}-1)/(\tau^m-1)$ is not constant because $\alpha\geqslant 0$, $\beta\geqslant 0$ and (3.1) imply that $\beta+1\neq m$. Thus we can fix $1<\sigma<\tau$ so that

$$\frac{\tau^{\beta+1}-1}{\tau^m-1}-\frac{\sigma^{\beta+1}-1}{\sigma^m-1}\neq 0.$$

(In fact a simple calculus argument shows that for $1<\tau$ the function $\tau\to(\tau^{\beta+1}-1)/(\tau^m-1)$ is monotone; thus every choice of σ and τ with $1<\sigma<\tau$ is good.) Dividing both sides of (3.9) by $\dfrac{\tau^{\beta+1}-1}{\tau^m-1}-\dfrac{\sigma^{\beta+1}-1}{\sigma^m-1}\neq 0$ we see that

$$D(\alpha,\beta)u = g_1*D_x^n u+g_2*D_y^m u$$

where $g_1\in L^\infty(\mathbf{R}^2)$ and $g_2\in L^\infty(\mathbf{R}^2)$ are independent of $u\in C_0^\infty(\mathbf{R}^2)$. Thus

$$\|D(\alpha,\beta)u\|_\infty\leqslant\|g_1\|_\infty\|D_x^n u\|_1+\|g_2\|_\infty\|D_y^m u\|_1$$

which obviously yields (3.2).

Proof of Proposition 3.1. We define E to be a function which represents the distribution being the Fourier transform of a certain regularization of the function $K\colon \mathbf{R}^2\setminus\{0\}\to\mathbf{C}$ defined by

(3.10) $$K(\Xi) = k_{\alpha,\beta}(\Xi)\cdot(i^n\xi^n+\varepsilon i^m\eta^m)^{-1} \quad \text{for } \Xi = (\xi,\eta)$$

where ε stands for $\varepsilon(m-n)$ and $k_{\alpha,\beta}$ denotes the symbol of $D(\alpha,\beta)$, i.e. one of the functions

$$i^{\alpha+\beta}|\xi|^\alpha|\eta|^\beta\operatorname{sign}(\xi\eta),\quad |\xi|^\alpha i^\beta|\eta|^\beta\operatorname{sign}\eta,\quad i^\alpha|\xi|^\alpha\operatorname{sign}\xi\,|\eta|^\beta,\quad |\xi|^\alpha|\eta|^\beta.$$

First we define the tempered distribution V being a regularization of K. We put $V = V_1+V_2$ where for $\varphi\in\mathcal{S}(\mathbf{R}^2)$,

(3.11) $$\langle V_1|\varphi\rangle = \int_{\|\Xi\|<1}K(\Xi)[\varphi(\Xi)-\varphi(0)]\,d\Xi,$$
$$\langle V_2|\varphi\rangle = \int_{\|\Xi\|\geqslant 1}K(\Xi)\varphi(\Xi)\,d\Xi$$

(the symbol $\langle U|g\rangle$ stands for the evaluation of a distribution U at a function g). We check that the Fourier transform $(2\pi)^{-1}\hat{V}$ satisfies (3.4) in the distribution sense, i.e. given $u\in C_0^\infty(\boldsymbol{R}^2)$ we have

(3.12) $\quad (2\pi)^{-1}\langle \hat{V}*(D_x^n+\varepsilon D_y^m)u|\psi\rangle = \langle D(\alpha,\beta)u|\psi\rangle \quad$ for every $\psi\in\mathscr{S}(\boldsymbol{R}^2)$.

Indeed, by the definitions of the convolution of a distribution with a function and of the Fourier transform of a distribution (cf. [SCH]) we have

$$\begin{aligned}\langle \hat{V}*(D_x^n+\varepsilon D_y^m)u|\psi\rangle &= \langle \hat{V}|[(D_x^n+\varepsilon D_y^m)u]*\psi\rangle\\ &= \langle V|([(D_x^n+\varepsilon D_y^m)u]*\psi)^\wedge\rangle\\ &= 2\pi\,\langle V|[(D_x^n+\varepsilon D_y^m)u]^\wedge\cdot\hat{\psi}\rangle\end{aligned}$$

Note that if $n, m>1$, then $([(D_x^n+\varepsilon D_y^m)u]^\wedge\cdot\hat{\psi})(0)=0$. Hence

$$\begin{aligned}\langle V|[(D_x^n+\varepsilon D_y^m)u]^\wedge\cdot\hat{\psi}\rangle &= \int_{\boldsymbol{R}^2} K(\Xi)(i^n\xi^n+\varepsilon i^m\eta^m)(\hat{u}\cdot\hat{\psi})(\Xi)\,d\Xi\\ &= \int_{\boldsymbol{R}^2} k_{\alpha,\beta}(\Xi)\,\hat{u}(\Xi)\,\hat{\psi}(\Xi)\,d\Xi\\ &= \langle [D(\alpha,\beta)u]^\wedge, \bar{\psi}^\wedge\rangle\\ &= \langle D(\alpha,\beta)u, \bar{\psi}\rangle = \langle D(\alpha,\beta)u|\psi\rangle.\end{aligned}$$

This completes the proof of (3.12).

Next we shall show that $(2\pi)^{-1}\hat{V}$ is represented by a locally integrable function, say E, i.e. for every $\psi\in\mathscr{S}(\boldsymbol{R}^2)$,

(3.13) $$(2\pi)^{-1}\langle \hat{V}|\psi\rangle = \langle E, \bar{\psi}\rangle.$$

We shall show (which is slightly easier) that there exists an E such that it satisfies (3.13) for $\psi\in C_0^\infty(\boldsymbol{R}^2)$; next we shall show that this E satisfies (3.5). The two facts together yield that E satisfies (3.13) for all $\psi\in\mathscr{S}(\boldsymbol{R}^2)$. Clearly (3.12) and (3.13) imply (3.4).

First we consider V_1. Then for $\psi\in C_0^\infty(\boldsymbol{R}^2)$ in view of (3.11) we have

$$\begin{aligned}\langle \hat{V}_1|\psi\rangle &= \langle V_1|\hat{\psi}\rangle\\ &= \int_{\|\Xi\|<1}\Big(K(\Xi)(2\pi)^{-1}\int_{\boldsymbol{R}^2}\psi(X)(e^{-(\Xi,X)i}-1)\,dX\Big)\,d\Xi.\end{aligned}$$

Put $\xi=r\cos t$, $\eta=r^{n/m}\sin t$. Then the Jacobian $\dfrac{\partial(\xi,\eta)}{\partial(r,t)}=r^{n/m}\Big(\cos^2 t+\dfrac{n}{m}\sin^2 t\Big)$ and, in view of (3.10),

$$K(\xi(r,t),\eta(r,t)) = \frac{r^{\alpha+\beta\frac{n}{m}}k_{\alpha,\beta}(\cos t,\sin t)}{r^n(i^n(\cos t)^n+\varepsilon i^m(\sin t)^m)}.$$

Thus using the rule of changing variables in the integral and taking into account (3.1) we get

$$\langle \hat{V}_1|\psi\rangle = \int_0^1 r^{-1}\int_0^{2\pi} g(t)\int_{\boldsymbol{R}^2}(e^{ih(X,r,t)}-1)\,\psi(X)\,dX\,dt\,dr$$

where

(3.14) $$g(t) = (2\pi)^{-1}k_{\alpha,\beta}(\cos t,\sin t)\frac{\cos^2 t+\frac{n}{m}\sin^2 t}{i^n(\cos t)^n+\varepsilon i^m(\sin t)^m},$$

(3.15) $$h(X,r,t) = -(xr\cos t+yr^{n/m}\sin t).$$

Note that $|e^{hi}-1|\leqslant|h|\leqslant Cr^{\min(1,n/m)}$ for $0<r<1$ and for all $(t,X)\in[0,2\pi]\times\operatorname{supp}\psi$, where $C=2\sup\{\|X\|:\ X\in\operatorname{supp}\psi\}$. Thus the function $r^{-1}g(t)(e^{ih(X,r,t)}-1)\psi(X)$ is absolutely integrable in the strip $[0,1]\times[0,2\pi]\times\boldsymbol{R}^2$. Therefore one may change the order of integration. Hence

$$\langle \hat{V}_1|\psi\rangle = \int_{\boldsymbol{R}^2}\psi(X)\Big(\int_0^1\int_0^{2\pi} r^{-1}g(t)(e^{ih(X,r,t)}-1)\,dt\,dr\Big)\,dX.$$

Thus

$$(2\pi)^{-1}\langle \hat{V}_1|\psi\rangle = \langle E_1,\bar{\psi}\rangle \quad \text{for } \psi\in C_0^\infty(\boldsymbol{R}^2)$$

where the function $E_1\colon \boldsymbol{R}^2\to \boldsymbol{C}$ is defined by

(3.16) $$2\pi E_1(X) = \int_0^1\int_0^{2\pi} r^{-1}g(t)(e^{ih(X,r,t)}-1)\,dt\,dr \quad \text{for } X=(x,y)\in\boldsymbol{R}^2.$$

The case of V_2 is more subtle. Similarly as for V_1 we have

$$\begin{aligned}\langle \hat{V}_2|\psi\rangle &= \int_1^\infty\int_0^{2\pi}\Big(r^{-1}g(t)\int_{\boldsymbol{R}^2}e^{ih(X,r,t)}\psi(X)\,dX\Big)\,dt\,dr\\ &= \int_1^\infty r^{-1}\int_{\operatorname{supp}\psi}\psi(X)\int_0^{2\pi}g(t)\,e^{ih(X,r,t)}\,dt\,dX\,dr\end{aligned}$$

where g and h are defined by (3.14) and (3.15) respectively.

Using Corollary 0.2 we infer that

$$\Big|\int_0^{2\pi} g(t)\,e^{hi}\,dt\Big| \leqslant C_1\min(1,\|X\|^{-1/2}r^{-\gamma})$$

where $\gamma=\min(2^{-1}, n(2m)^{-1})$ and the constant C_1 depends only on α, β, n, m (via the function g).

Hence the function

$$(r,X)\to r^{-1}\psi(X)\int_0^{2\pi}g(t)\,e^{ih(X,r,t)}\,dt$$

is absolutely integrable in the strip

$$A = [1, \infty) \times \mathbf{R}^2 \supset [1, \infty) \times \operatorname{supp} \psi.$$

This is obvious for the set $A \cap \{(r, X)\colon 1 \geqslant \|X\|^{-1/2} r^{-\gamma}\}$ because the function $\|X\|^{-1/2} \psi(X) r^{-1-\gamma}$ is absolutely integrable in A. For the set

$$A_1 = A \cap \{(r, X)\colon 1 \leqslant \|X\|^{-1/2} r^{-\gamma}\}$$

it follows from the absolute integrability of the function $r^{-1}\psi(X)$ on this set. Indeed,

$$\begin{aligned}\int_{A_1} |r^{-1}\psi(X)|\, dX\, dr &\leqslant \|\psi\|_\infty \int_{r=1}^{\infty} r^{-1} \int_{\|X\| \leqslant r^{-2\gamma}} 1\, dX\, dr \\ &= \|\psi\|_\infty \int_{r=1}^{\infty} \pi r^{-1-4\gamma}\, dr \\ &= \pi (4\gamma)^{-1} \|\psi\|_\infty < \infty.\end{aligned}$$

Thus the changing of the order of integrals in the formula for $\langle \hat{V}_2 | \psi \rangle$ is rigorous and we get

$$\langle \hat{V}_2 | \psi \rangle = \int_{\mathbf{R}^2} \psi(X) \Big(\int_1^\infty \int_0^{2\pi} r^{-1} g(t)\, e^{ih(X,r,t)}\, dt\, dr \Big) dX.$$

Thus

$$(2\pi)^{-1} \langle \hat{V}_2 | \psi \rangle = \langle E_2, \psi \rangle \quad \text{for } \psi \in C_0^\infty(\mathbf{R}^2)$$

where the function $E_2\colon \mathbf{R}^2 \to \mathbf{C}$ is defined by

$$(3.17) \qquad 2\pi E_2(X) = \int_1^\infty \int_0^{2\pi} r^{-1} g(t)\, e^{ih(X,r,t)}\, dt\, dr \quad \text{for } X = (x, y) \in \mathbf{R}^2.$$

Thus $E = E_1 + E_2$ satisfies (3.13) for $\psi \in C_0^\infty(\mathbf{R}^2)$.

Next we show that E satisfies (3.5). To this end for fixed $X = (x, y) \neq 0$ consider the function $r(\varrho, X)$ defined for $\varrho > 0$ by

$$\varrho^2 = r^2 x^2 + r^{2n/m} y^2.$$

It can be easily verified that $r(\cdot, X)$ is increasing and differentiable and $r(\|X\|, X) = 1$. Clearly $h(X, r(\varrho), t) = \varrho \cos(t - t_0)$ where h is defined by (3.15) and $t_0 = t_0(X, r)$ is the unique number in $[0, 2\pi)$ such that $-\cos t_0 = r x \varrho^{-1}$ and $-\sin t_0 = r^{n/m} y \varrho^{-1}$. Thus substituting in the integrals (3.16) and (3.17) $r = r(\varrho)$ we get

$$\begin{aligned}2\pi E(X) &= \int_0^1 r^{-1} \int_0^{2\pi} g(t)(e^{hi} - 1)\, dt\, dr + \int_1^\infty r^{-1} \int_0^{2\pi} g(t)\, e^{hi}\, dt\, dr \\ &= \int_0^{\|X\|} r^{-1} \frac{dr}{d\varrho} \int_0^{2\pi} g(t)(e^{i\varrho\cos(t-t_0)} - 1)\, dt\, d\varrho \\ &\quad + \int_{\|X\|}^\infty r^{-1} \frac{dr}{d\varrho} \int_0^{2\pi} g(t)\, e^{i\varrho\cos(t-t_0)}\, dt\, d\varrho \\ &= f_1(X) + l(X)\end{aligned}$$

where

$$f_1(X) = \int_0^1 r^{-1} \frac{dr}{d\varrho} \int_0^{2\pi} g(t)(e^{i\varrho\cos(t-t_0)} - 1)\, dt\, d\varrho + \int_1^\infty r^{-1} \frac{dr}{d\varrho} \int_0^{2\pi} g(t)\, e^{i\varrho\cos(t-t_0)}\, dt\, d\varrho$$

and

$$l(X) = -\int_1^{\|X\|} r^{-1} \frac{dr}{d\varrho} \int_0^{2\pi} g(t)\, dt\, d\varrho.$$

First we show that $f_1 \in L^\infty(\mathbf{R}^2)$. We have

$$\frac{dr}{d\varrho} = \varrho \Big(r x^2 + \frac{n}{m} r^{2n/m - 1} \Big)^{-1}.$$

Thus

$$r^{-1} \frac{dr}{d\varrho} = \varrho \Big(r^2 x^2 + \frac{n}{m} r^{2n/m} y^2 \Big)^{-1} \leqslant \max(1, m/n)\, \varrho^{-1}.$$

Hence to estimate the first summand of f_1 observe that for $0 \leqslant \varrho \leqslant 1$,

$$\Big| \int_0^{2\pi} g(t)(e^{i\varrho\cos(t-t_0)} - 1)\, dt \Big| \leqslant 2\pi\varrho(e-1) \cdot \|g\|_\infty.$$

Thus

$$\Big| \int_0^1 r^{-1} \frac{dr}{d\varrho} \int_0^{2\pi} g(t)(e^{i\varrho\cos(t-t_0)} - 1)\, dt \Big| \leqslant 2\pi \|g\|_\infty (e-1) \max(1, m/n).$$

To estimate the second summand we use Lemma 0.4 which for $\varrho > 1$ yields

$$\Big| \int_0^{2\pi} g(t)\, e^{i\varrho\cos(t-t_0)}\, dt \Big| \leqslant C(\cos t)\, \varrho^{-1/2} (\|g\|_\infty + \|g'\|_1).$$

Therefore

$$\begin{aligned}&\Big| \int_1^\infty r^{-1} \frac{dr}{d\varrho} \int_0^{2\pi} g(t)\, e^{i\varrho\cos(t-t_0)}\, dt \Big| \\ &\qquad \leqslant C(\cos t)(\|g\|_\infty + \|g'\|_1) \max(1, m/n) \cdot \int_1^\infty \varrho^{-3/2}\, d\varrho < \infty.\end{aligned}$$

Finally we examine the integral $l(X)$. Put $C_1(g) = \int_0^{2\pi} g(t)\,dt$. Remembering that $r(\|X\|, X) = 1$ we get

$$l(X) = -C_1(g) \int_{r(1,X)}^{1} r^{-1}\,dr = C_1(g) \log r(1, X).$$

Now we use the identity

$$r(\varrho, X) = \lambda r(\varrho, (x\lambda, y\lambda^{-m/n}))$$

in the special case $\varrho = 1$ and $\lambda = (|x|^m + |y|^n)^{1/m}$. Then

$$\log(r(1, X)) = m^{-1} \log(|x|^m + |y|^n) + f_2(X)$$

where

$$f_2(X) = \log r\big(1, \big(x(|x|^m+|y|^n)^{-1/m}, y(|x|^m+|y|^n)^{-1/n}\big)\big).$$

Clearly the function $r(1, X)$ restricted to the compact set

$$\{X = (x, y) \in \boldsymbol{R}^2 \colon |x|^m + |y|^n = 1\}$$

is bounded and bounded from below by a positive number. Hence $f_2 \in L^\infty(\boldsymbol{R}^2)$. Thus

$$l(X) = m^{-1} C_1(g) \log(|x|^m + |y|^n) + C_1(g) f_2(X).$$

Hence

$$2\pi E(X) = m^{-1} C_1(g) \log(|x|^m + |y|^n) + f(X),$$

$$\text{where } f = f_1 + C_1(g) f_2 \in L^\infty(\boldsymbol{R}^2).$$

This completes the proof of the proposition.

Added in proof (May 1986). Proposition 3.1 can also be extended to the case where both n and m are odd integers $\geqslant 2$. The proof requires only a minor modification, namely the function $r \to \varepsilon(r)$ should be defined by $\varepsilon(r) = i^{r+1}$ for $r = 1, 2, \ldots$ (cf. A. Pełczyński and K. Senator, *Addendum to the paper "On isomorphisms of anisotropic Sobolev spaces with «classical Banach spaces» and a Sobolev type embedding theorem"*, this volume, pp. 217–218).

We end this section by presenting an alternative argument in the case $n = m = 1, 2, \ldots$

THEOREM 3.2. *Let $n \geqslant 2$ be a positive integer. Then for every $k = 0, 1, \ldots, n-2$ there are constants $C_{n,k}$ such that for every $u \in C_0^\infty(\boldsymbol{R}^2)$*

$$\|D_x^k D_y^{n-2-k} u\|_\infty \leqslant C_{n,k}(\|D_x^n u\|_1 + \|D_y^n u\|_1). \tag{3.18}$$

The proof of Theorem 3.2 requires some preparation.

Recall that the function $E = (4\pi)^{-1} \log(x^2 + y^2)$ is a fundamental solution for the Laplacian $\Delta = D_x^2 + D_y^2$, i.e.

$$E * \Delta u = u \quad \text{for } u \in C_0^\infty(\boldsymbol{R}^2). \tag{3.19}$$

Combining (3.19) with an appropriate affine transformation of coordinates we obtain

LEMMA 3.1. *Let $S(\xi, \eta) = a\xi^2 + 2b\xi\eta + c\eta^2$ be an elliptic polynomial, precisely $a > 0$ and the discriminant $b^2 - ac < 0$. Put*

$$E_S = (4\pi\sqrt{-(b^2 - ac)})^{-1} \log S(y, -x).$$

Then E_S is a fundamental solution for the differential operator $\tilde{S} = aD_x^2 + 2bD_x D_y + cD_y^2$, i.e.

$$E_S * \tilde{S} u = u \quad \text{for } u \in C_0^\infty(\boldsymbol{R}^2). \tag{3.20}$$

Note that the left-hand side of (3.20) is a convolution of a C^∞-function with bounded support with a locally absolutely integrable function; hence it is well defined.

We shall also use some algebraic properties of polynomials.

Fix an integer $n \geqslant 2$. Let $m = [n/2]$ for $\tau > 0$ and for $j = 1, 2, \ldots, m$ put

$$S_{\tau,j}(\xi, \eta) = \xi^2 + 2\tau\xi\eta \cos\frac{2j-1}{n}\pi + \tau^2\eta^2,$$

$$T_{\tau,j}(\xi, \eta) = (\xi^n + \tau^n\eta^n) \cdot [S_{\tau,j}(\xi, \eta)]^{-1}.$$

LEMMA 3.2. (i) *$S_{\tau,j}$ is an elliptic polynomial with discriminant $-\left(\tau \sin\frac{2j-1}{n}\pi\right)^2$;*

(ii) *$S_{\tau,j}$ is a factor of the polynomial $\xi^n + \tau^n\eta^n$, hence $T_{\tau,j}$ can be identified with the homogeneous polynomial of degree $n-2$ such that*

$$(S_{\tau,j} T_{\tau,j})(\xi, \eta) = \xi^n + \tau^n\eta^n; \tag{3.21}$$

(iii) *For each $k = 0, 1, \ldots, n-2$ there exist scalars $a_{j,k}$ and $b_{j,k}$ for $j = 1, 2, \ldots, m$ such that for $\tau > 0$*

$$\tau^{n-2-k}\xi^k\eta^{n-2-k} = \sum_{j=1}^m a_{j,k} T_{\tau,j}(\xi, \eta) + \sum_{j=1}^m b_{j,k} T_{2\tau,j}(\xi, \eta). \tag{3.22}$$

Proof. We omit the straightforward proof of (i) and (ii). By a homogeneity argument it is enough to prove (iii) for $\tau = 1$. First we show that the polynomials $T_{1,1}, T_{1,2}, \ldots, T_{1,m}$ are linearly independent. Let $0 = \sum_{j=1}^m c_j T_{1,j}$. Fix j_0 with $1 \leqslant j_0 \leqslant m$ and multiply both sides of the identity by S_{1,j_0}. Then

$$0 = c_{j_0}(\xi^n+\eta^n)+\sum_{j\neq j_0} S_{1,j_0} T_{1,j}(\xi,\eta) = c_{j_0}(\xi^n+\eta^n)+[S_{1,j_0}]^2(\xi,\eta)\cdot P(\xi,\eta)$$

for some homogeneous polynomial P, because if $j \neq j_0$ then $S_{1,j_0}|T_{1,j}$. Since the polynomial x^n+1 does not have multiple zeros, the latter identity implies that $c_{j_0} = 0$. Hence the $T_{1,j}$'s are linearly independent and the space spanned by them is m-dimensional. Therefore this space coincides with the space of all homogeneous polynomials

$$\sum_{k=0}^{n-2} a_k \xi^k \eta^{n-2-k}$$

such that $a_k = a_{n-2-k}$ for $k = 0, 1, \ldots, n-2$ (because the coefficients of each of the $T_{1,j}$'s satisfy this relation). Thus for each $k = 0, 1, \ldots, n-2$ there exists a linear combination, say X_k, of the polynomials $T_{1,1}, T_{1,2}, \ldots, T_{1,m}$ such that

$$\xi^k \eta^{n-2-k}+\xi^{n-2-k}\eta^k = X_k. \tag{3.23}$$

Similarly we infer that there is a linear combination, say Y_k, of the polynomials $T_{2,1}, T_{2,2}, \ldots, T_{2,m}$ such that

$$\xi^k \eta^{n-2-k}+2^{2k-n+2}\xi^{n-2-k}\eta^k = Y_k \tag{3.24}$$

Solving for $k \neq n/2-1$ the system of equations (3.23) and (3.24) we get

$$\xi^k \eta^{n-2-k} = (1-2^{2k-n+2})^{-1}(Y_k-2^{2k-n+2}X_k)$$

while for $k = n/2-1$ (this is possible only in the case where n is even) we get

$$\xi^{m-1}\eta^{m-1} = 2^{-1}X_{m-1}.$$

This completes the proof of Lemma 3.2.

In the sequel it is convenient to denote by $S^*_{\tau,j}$ the polynomial defined by $S^*_{\tau,j}(x, y) = S_{\tau,j}(y, -x)$ for $\tau > 0$ and $j = 1, 2, \ldots, m$.

Proof of Theorem 3.2. Fix $u\in C_0^\infty(\boldsymbol{R}^2)$. It follows from Lemmas 3.1 and 3.2 (i) and (ii) (more specifically, from the formulae (3.20), (3.21)) that for $\tau > 0$ and for $j = 1, 2, \ldots, m$ one has

$$\tau\tilde{T}_{\tau,j}u = \left[\left(4\pi\sin\frac{2j-1}{n}\pi\right)^{-1}\log S^*_{\tau,j}\right]*(D^n_x+\tau^n D^n_y)\,u,$$

$$\tau\tilde{T}_{2\tau,j}u = \left[\left(8\pi\sin\frac{2j-1}{n}\pi\right)^{-1}\log S^*_{2\tau,j}\right]*(D^n_x+2^n\tau^n D^n_y)\,u.$$

(Recall that for a polynomial T by $\tilde{T}$ we denote the corresponding differential operator.) Thus, by (3.22)

$$\tau^{n-1-k}D^k_x D^{n-2-k}_y u = f_{\tau,k}*D^n_x u+\tau^n g_{\tau,k}*D^n_y u \quad \text{for } k = 0, 1, \ldots, n-2 \tag{3.25}$$

where

$$f_{\tau,k} = \sum_{j=1}^m \left(4\pi\sin\frac{2j-1}{n}\pi\right)^{-1} a_{j,k}\log S^*_{\tau,j}+\sum_{j=1}^m \left(8\pi\sin\frac{2j-1}{n}\pi\right)^{-1} b_{j,k}\log S^*_{2\tau,j},$$

$$g_{\tau,k} = \sum_{j=1}^m \left(4\pi\sin\frac{2j-1}{n}\pi\right)^{-1} a_{j,k}\log S^*_{\tau,j}+2^n\sum_{j=1}^m \left(8\pi\sin\frac{2j-1}{n}\pi\right)^{-1} b_{j,k}\log S^*_{2\tau,j}.$$

Note that the formula (3.25) is an analogue of (3.7). From this point the proof of Theorem 3.2 is the same as the proof of Theorem 3.1, because we also have an analogue of (3.6), namely:

$$f_{\tau,k}-f_{\sigma,k}\in L^\infty(\boldsymbol{R}^2), \quad g_{\tau,k}-g_{\sigma,k}\in L^\infty(\boldsymbol{R}^2) \quad \text{for } 0<\sigma<\tau. \tag{3.26}$$

Indeed, $f_{\tau,k}-f_{\sigma,k}$ and $g_{\tau,k}-g_{\sigma,k}$ are linear combinations of the functions $\log[S^*_{\tau,j}(S^*_{\sigma,j})^{-1}]$ and $\log[S^*_{2\tau,j}(S^*_{2\sigma,j})^{-1}]$ for $j = 1, 2, \ldots, m$; the latter functions are uniformly bounded because the exponent of each of them is a quotient of two homogeneous polynomials of the same degree ($= 2$); each polynomial is positive and bounded away from zero.

Now by the same manipulation as in the proof of Theorem 3.1 we infer that for each $k = 0, 1, \ldots, n-2$

$$D^k_x D^{n-k-2}_y u = m_{1,k}*D^n_x u+m_{2,k}*D^n_y u$$

where $m_{1,k}\in L^\infty(\boldsymbol{R}^2)$ and $m_{2,k}\in L^\infty(\boldsymbol{R}^2)$ are functions independent of $u\in C_0^\infty(\boldsymbol{R}^2)$. This clearly implies (3.18).

4. Estimates of the L^2-norms of derivatives by the L^1-norms of derivatives of higher orders. In this section we complete the proof of Theorem A and at the same time obtain a more general result in the cases of pairs (n, m) discussed in the previous section.

THEOREM 4.1. *Let $n\geq 2$ and $m\geq 2$ be integers such that either at least one of them is even or $n = m$. Let a, b be nonnegative numbers such that*

$$an^{-1}+bm^{-1} = 1-(2n)^{-1}-(2m)^{-1}. \tag{4.1}$$

Then there exists a constant $C = C(a, b, n, m)$ such that for every $u\in C_0^\infty(\boldsymbol{R}^2)$

$$\||D^a_x|\,|D^b_y|\,u\|_2 \leq C(\|D^n_x u\|_1+\|D^m_y u\|_1). \tag{4.2}$$

Proof. *Case* 1: at least one of n and m is even. Fix $u\in C_0^\infty(\boldsymbol{R}^2)$. Put $A = n(1-(2n)^{-1}-(2m)^{-1})$. By the Plancherel identity

$$\||D^A_x|\,u\|_2^2 = \iint_{\boldsymbol{R}^2}|\xi|^{2A}|\hat{u}(\xi,\eta)|^2\,d\xi\,d\eta.$$

On the other hand

$$\iint_{\boldsymbol{R}^2}|\xi|^{2A}|\hat{u}(\xi,\eta)|^2\,d\xi\,d\eta = |\langle(D_x\{2A-n\}\,u)^\wedge, (D^n_x u)^\wedge\rangle|$$

where

$$D_x\{2A-n\} = \begin{cases} \mathring{D}_x^{2A-n} & \text{if } n \text{ is odd,} \\ |D_x^{2A-n}| & \text{if } n \text{ is even.} \end{cases}$$

Thus, using again the Plancherel identity

$$|||D_x^A|\, u||_2^2 = |\langle D_x\{2A-n\}\, u,\ D_x^n u\rangle| \leqslant ||D_x\{2A-n\}\, u||_\infty\, ||D_x^n u||_1 .$$

Next observe that the quadruple $(2A-n, 0, n, m)$ satisfies (3.1). Thus, by Theorem 3.1,

$$(4.3) \qquad |||D_x^A|\, u||_2^2 \leqslant C^* (||D_x^n u||_1 + ||D_y^m u||_1)\, ||D_x^n u||_1 .$$

Similarly, putting $B = m(1-(2n)^{-1}-(2m)^{-1})$, we obtain:

$$(4.4) \qquad |||D_y^B|\, u||_2^2 \leqslant C^* (||D_x^n u||_1 + ||D_y^m u||_1)\, ||D_y^m u||_1 .$$

Now fix nonnegative a and b so that the quadruple (a, b, n, m) satisfies (4.1). Then, for $t = aA^{-1}$ we have $a = tA$, $b = (1-t)B$ and $0 \leqslant t \leqslant 1$. Thus, combining (4.3) and (4.4) with the elementary inequality

$$|\xi|^{2t}\, |\eta|^{2(1-t)} \leqslant t\xi^2 + (1-t)\,\eta^2, \qquad \xi \in \boldsymbol{R},\ \eta \in \boldsymbol{R},$$

one gets

$$\begin{aligned} |||D_x^a D_y^b|\, u||_2^2 &= \iint_{\boldsymbol{R}^2} |\xi|^{2a}\, |\eta|^{2b}\, |\hat{u}(\xi, \eta)|^2\, d\xi\, d\eta \\ &\leqslant \frac{a}{A} \iint_{\boldsymbol{R}^2} |\xi|^{2A}\, |\hat{u}(\xi, \eta)|^2\, d\xi\, d\eta + \frac{b}{B} \iint_{\boldsymbol{R}^2} |\eta|^{2B}\, |\hat{u}(\xi, \eta)|^2\, d\xi\, d\eta \\ &= \frac{a}{A}\, |||D_x^A|\, u||_2^2 + \frac{b}{B}\, |||D_y^B|\, u||_2^2 \\ &\leqslant C^* (||D_x^n u||_1 + ||D_y^m u||_1)^2 . \end{aligned}$$

Hence we get (4.2) with $C = (C^*)^{1/2}$.

Case 2: $n = m \geqslant 2$. The argument is essentially the same as in Case 1. Combining the Plancherel identity with Theorem 3.2, we infer that

$$||D_x^{n-1} u||_2^2 \leqslant C_{n,0} (||D_x^n u||_1 + ||D_y^n u||_1)\, ||D_x^n u||_1 ,$$

$$||D_y^{n-1} u||_2^2 \leqslant C_{n,n-2} (||D_x^n u||_1 + ||D_y^n u||_1)\, ||D_y^n u||_1 .$$

Now "interpolating between the points $(n-1, 0)$ and $(0, n-1)$" one gets for (a, b, n, m) satisfying (4.1)

$$|||D_x^a D_y^b|\, u||_2^2 \leqslant C_{n,0} (||D_x^n u||_1 + ||D_y^n u||_1)^2 .$$

Thus we get (4.2) with $C = (C_{n,0})^{1/2}$. (Note that by the symmetry reason the constants $C_{n,0}$ and $C_{n,n-2}$ appearing in (3.18) are equal.)

The next result is a slight improvement of Theorem A.

COROLLARY 4.1. *For all positive integers n and m there is a constant $C = C(n, m)$ such that for every $u \in C_0^\infty(\boldsymbol{R}^2)$*

$$(4.5) \qquad |||D_x^{(n-1)/2} D_y^{(m-1)/2}|\, u||_2^2 \leqslant C\, ||D_x^n u||_1\, ||D_y^m u||_1 ,$$

$$(4.6) \qquad |||D_x^{(n-1)/2} D_y^{(m-1)/2}|\, u||_2 \leqslant 2^{-1} \sqrt{C}\, (||D_x^n u||_1 + ||D_y^m u||_1).$$

Proof. For n, m both odd we obtain (4.5) using Proposition 1.1. For $\min(n, m) = 1$ we also obtain (4.5) using Proposition 2.2. In the remaining cases Theorem 4.1 yields (4.6). Clearly (4.5) implies (4.6) because

$$(||D_x^n u||_1 + ||D_y^m u||_1)^2 \geqslant 4\, ||D_x^n u||_1\, ||D_y^m u||_1 .$$

Finally assume that (4.6) holds for all $u \in C_0^\infty(\boldsymbol{R}^2)$. Fix a $u \in C_0^\infty(\boldsymbol{R}^2)$ and for $\tau > 0$ define $u_\tau \in C_0^\infty(\boldsymbol{R}^2)$ by $u_\tau(x, y) = u(\tau x, \tau^{-1} y)$. A simple computation gives

$$|||D_x^{(n-1)/2} D_y^{(m-1)/2}|\, u_\tau||_2 = \tau^{(n-m)/2}\, |||D_x^{(n-1)/2} D_y^{(m-1)/2}|\, u||_2 ;$$

$$||D_x^n u_\tau||_1 = \tau^n\, ||D_x^n u||_1 ; \qquad ||D_y^m u_\tau||_1 = \tau^{-m}\, ||D_y^m u||_1 .$$

Thus writing (4.6) for u_τ we get

$$\tau^{(n-m)/2}\, |||D_x^{(n-1)/2} D_y^{(m-1)/2}|\, u||_2 \leqslant 2^{-1} \sqrt{C}\, (\tau^n\, ||D_x^n u||_1 + \tau^{-m}\, ||D_y^m u||_1).$$

Thus

$$\begin{aligned} |||D_x^{(n-1)/2} D_y^{(m-1)/2}|\, u||_2 &\leqslant 2^{-1} \sqrt{C} \inf_{\tau > 0} (\tau^{(n+m)/2}\, ||D_x^n u||_1 + \tau^{-(n+m)/2}\, ||D_y^m u||_1) \\ &= \sqrt{C}\, (||D_x^n u||_1\, ||D_y^m u||_1)^{1/2}. \end{aligned}$$

This completes the proof of Corollary 4.1 and of Theorem A.

It is convenient for us to state explicitly the following trivial consequence of Corollary 4.1.

COROLLARY 4.2. *Assume that we are given two pairs of positive integers (n_1, m_1) and (n_2, m_2) such that $(n_1-n_2)(m_1-m_2) < 0$. Then, for every $u \in C_0^\infty(\boldsymbol{R}^2)$,*

$$(4.7) \qquad |||D_x^{(n_1+m_1-1)/2} D_y^{(n_2+m_2-1)/2}|\, u||_2^2 \leqslant K\, ||D_x^{n_1} D_y^{m_1} u||_1\, ||D_x^{n_2} D_y^{m_2} u||_1 ,$$

$$(4.8) \qquad |||D_x^{(n_1+m_1-1)/2} D_y^{(n_2+m_2-1)/2}|\, u||_2 \leqslant 2^{-1} \sqrt{K}\, (||D_x^{n_1} D_y^{m_1} u||_1 + ||D_x^{n_2} D_y^{m_2} u||_1)$$

where $K = C(|n_1-n_2|, |m_1-m_2|)$ and $C(\cdot, \cdot)$ is the constant appearing in Corollary 4.1.

Proof. Put $v = D_x^{\min(n_1, n_2)} D_y^{\min(m_1, m_2)} u$ and apply Corollary 4.1 for v and for $n = |n_1-n_2|$, $m = |m_1-m_2|$.

Our next result is a version of Theorem A and Corollary 4.2 for periodic functions.

THEOREM 4.2. *Let $W \subset \boldsymbol{Z}_+^2$ be a smoothness, $(n_1, m_1) \in W$, $(n_2, m_2) \in W$ and $(n_1, m_1) \neq (n_2, m_2)$. Then there exists a constant $C = C(n_1, n_2, m_1, m_2)$ such that for every trigonometric polynomial $f = \sum_{p,q} \alpha_{p,q} e^{ipx+iqy}$*

$$\sum_{(p,q)\in \boldsymbol{Z}^2} |p|^{n_1+n_2-1} |q|^{m_1+m_2-1} |\alpha_{p,q}|^2 \leqslant C^2 \|f\|^2_{L^1_W(\boldsymbol{T}^2)}. \tag{4.9}$$

Proof. Let $\varphi(x, y) = (2\pi)^{-1}\exp\left(-\frac{1}{2}(x^2+y^2)\right)$. Regarding f as a $(2\pi, 2\pi)$-periodic function on $\boldsymbol{R}^2$ we consider the function $\varphi f\colon \boldsymbol{R}^2 \to \boldsymbol{C}$. We note first that there is a numerical constant K_1 independent of f such that

$$\|\varphi f\|_{1,W} \leqslant K_1 \|f\|_{L^1_W(\boldsymbol{T}^2)}. \tag{4.10}$$

To prove (4.10) fix $A \in W$ and use the identity

$$\|D^A(\varphi f)\|_1 = \sum_{(p,q)\in \boldsymbol{Z}^2} \iint_{[-\pi,\pi]^2} |D^A(f\varphi_{p,q})(x, y)|\, dx\, dy$$

where $\varphi_{p,q}(x, y) = \varphi(x+2\pi p, y+2\pi q)$. Now we fix $(p, q) \in \boldsymbol{Z}^2$. Using the Leibniz formula for the derivative of the product of functions we infer that

$$D^A(f\varphi_{p,q})(x, y) = \varphi_{p,q}(x, y) \sum_{0\leqslant B\leqslant A} P_B(x+2\pi p, y+2\pi q)(D^B f)(x, y)$$

where P_B is a polynomial in two variables which is independent of f and the pair (p, q). Thus

$$\begin{aligned}\iint_{[-\pi,\pi]^2} &|D^A(f\varphi_{p,q})(x, y)|\, dx\, dy \\ &\leqslant \sum_{0\leqslant B\leqslant A} \sup_{(x,y)\in[-\pi,\pi]^2} |P_B(x+2\pi p, y+2\pi q)\varphi_{p,q}(x, y)|\, \|D^B f\|_{L^1(\boldsymbol{T}^2)} \\ &\leqslant C_A (N+1)^{d_A} e^{-N^2/2} \|f\|_{L^1_W(\boldsymbol{T}^2)}\end{aligned}$$

where $N = \max(|p|, |q|)$, d_A is the maximum of the degrees of the polynomials P_B, and C_A is a numerical constant depending only on A via the polynomials P_B. Next observe that for fixed $N \in \boldsymbol{Z}_+$ there is at most $8N+1$ pairs $(p, q) \in \boldsymbol{Z}^2$ such that $\max(|p|, |q|) = N$. Thus

$$\|D^A(\varphi f)\|_1 \leqslant C_A \sum_{N=0}^{\infty} (N+1)^{d_A} (8N+1) e^{-N^2/2} \|f\|_{L^1_W(\boldsymbol{T}^2)}.$$

Hence we obtain (4.10) with

$$K_1 = \sum_{A\in W} C_A \sum_{N=0}^{\infty} (N+1)^{d_A} (8N+1) e^{-N^2/2} < +\infty.$$

Let $G_{a,b} = |D_x^{n/2} D_y^{m/2}|(\varphi f_{a,b})$ where $n = n_1+n_2-1$, $m = m_1+m_2-1$, $f_{a,b}(x, y) = f(x+a, y+b)$. Combining Corollary 4.2 with (4.10) and the identity $\|f_{a,b}\|_{L^1_W(\boldsymbol{T}^2)} = \|f\|_{L^1(\boldsymbol{T}^2)}$ we get

$$\|G_{a,b}\|_2^2 \leqslant 4^{-1} K \|\varphi f_{a,b}\|^2_{1,W} \leqslant K_2^2 \|f\|^2_{L^1_W(\boldsymbol{T}^2)} \tag{4.11}$$

where K is the constant appearing in (4.7) and (4.8) and $K_2 = 2^{-1}\sqrt{K}\, K_1$.

Next we compute $\|G_{a,b}\|_2^2$. By the Plancherel formula we have

$$\|G_{a,b}\|_2^2 = \iint_{\boldsymbol{R}^2} |\xi|^n |\eta|^m |(\varphi f_{a,b})\hat{\ }(\xi, \eta)|^2\, d\xi\, d\eta.$$

Put $e_{p,q}(x, y) = \exp(ipx+iqy)$, for $(p, q) \in \boldsymbol{Z}^2$. A straightforward computation gives

$$(\varphi e_{p,q})\hat{\ }(\xi, \eta) = \varphi(\xi-p, \eta-q).$$

Thus

$$(\varphi f_{a,b})\hat{\ }(\xi, \eta) = \sum_{p,q} \alpha_{p,q} e^{iap+ibq} \varphi(\xi-p, \eta-q).$$

Hence

$$\|G_{a,b}\|_2^2 = \iint_{\boldsymbol{R}^2} |\xi|^n |\eta|^m \Big|\sum_{p,q} \alpha_{p,q} e^{iap+ibq} \varphi(\xi-p, \eta-q)\Big|^2\, d\xi\, d\eta.$$

Integrating the latter identity against $da\, db$ over the square $[-\pi, \pi]^2$ and using the Fubini theorem we get

$$\begin{aligned}\iint_{[-\pi,\pi]^2} &\|G_{a,b}\|_2^2\, da\, db \\ &= \iint_{\boldsymbol{R}^2} |\xi|^n |\eta|^m \iint_{[-\pi,\pi]^2} \Big|\sum_{p,q} \alpha_{p,q} e^{iap+ibq} \varphi(\xi-p, \eta-q)\Big|^2\, da\, db\, d\xi\, d\eta.\end{aligned}$$

Since

$$\iint_{[-\pi,\pi]^2} \Big|\sum_{p,q} \alpha_{p,q} e^{iap+ibq} \varphi(\xi-p, \eta-q)\Big|^2\, da\, db = (2\pi)^2 \sum_{p,q} |\alpha_{p,q}|^2 \varphi(\xi-p, \eta-q)^2,$$

we obtain

$$\iint_{[-\pi,\pi]^2} \|G_{a,b}\|_2^2\, da\, db = \iint_{\boldsymbol{R}^2} (2\pi)^2 |\xi|^n |\eta|^m \sum_{p,q} |\alpha_{p,q}|^2 \varphi(\xi-p, \eta-q)^2\, d\xi\, d\eta.$$

On the other hand (4.11) yields

$$\iint_{[-\pi,\pi]^2} \|G_{a,b}\|_2^2\, da\, db \leqslant K_2^2\, 4\pi^2 \|f\|^2_{L^1_W(\boldsymbol{T}^2)}.$$

Thus

$$\sum_{p,q} |\alpha_{p,q}|^2 \iint_{\boldsymbol{R}^2} |\xi|^n |\eta|^m \varphi(\xi-p, \eta-q)^2\, d\xi\, d\eta \leqslant K_2^2 \|f\|^2_{L^1_W(\boldsymbol{T}^2)}. \tag{4.12}$$

Finally observe that

$$\begin{aligned}\iint_{\boldsymbol{R}^2} &|\xi|^n |\eta|^m \varphi(\xi-p, \eta-q)^2\, d\xi\, d\eta \\ &\geqslant (2\pi)^{-2} \int_{-1}^{1} \int_{-1}^{1} |\xi-p|^n |\eta-q|^m \exp(-\xi^2-\eta^2)\, d\xi\, d\eta \\ &\geqslant c\,|p|^n |q|^m\end{aligned} \tag{4.13}$$

where c is a positive numerical constant independent of p and q. Combining (4.12) with (4.13) we get (4.9).

5. Nonisomorphism of spaces of smooth functions of several variables with $C(K)$ and $L^1(\mu)$ spaces. We begin with recalling some concepts in Banach space theory.

Let E, F, G be Banach spaces and $T\colon E\to F$, $T_1\colon E\to G$, $T_2\colon G\to F$ (bounded) linear operators. We say that T *factors through* G if $T=T_2T_1$; the diagram

$$T\colon E\xrightarrow{T_1}G\xrightarrow{T_2}F$$

is called a *factorization*. Note that if the identity operator on a Banach space E factors through a Banach space G, then E is isomorphic (= linearly homeomorphic) to a complemented (= a range of an idempotent on G) subspace of G.

An operator $T\colon E\to F$ is *absolutely summing* provided there exists a $C>0$ such that for every finite sequence $(e_j)\subset E$

$$\sum_j\|Te_j\|\leqslant C\sup\Big\{\Big\|\sum_j\varepsilon_j e_j\Big\|\colon |\varepsilon_j|=1,\ j=1,2,\ldots\Big\}. \tag{5.1}$$

An operator $S\colon H_1\to H_2$ acting between Hilbert spaces is *Hilbert–Schmidt* provided $\sum_j\|Se_j\|^2<+\infty$ for every orthonormal family (e_j) in H_1 (equivalently for some complete orthonormal family). An operator $S\colon H_1\to H_2$ acting between Hilbert spaces is *nuclear* provided

$$\sum_j|\langle Sh_j,\delta_j\rangle|<+\infty$$

for all orthonormal families (h_j) in H_1 and (δ_j) in H_2.

Now we are ready to describe the isomorphic invariants essentially discovered by Grothendieck [GR] which play crucial role in this section.

A Banach space E is said to have the *Hilbert–Schmidt Factorization Property*, shortly E has HSFP, provided every operator between Hilbert spaces which factors through E is Hilbert–Schmidt.

A Banach space E has the *Nuclear Factorization Property*, shortly E has NFP, provided for every absolutely summing operator $T\colon E\to F$ and all bounded operators $S_1\colon H_1\to E$ and $S_2\colon F\to H_2$ the composition

$$S_2TS_1\colon H_1\xrightarrow{S_1}E\xrightarrow{T}F\xrightarrow{S_2}H_2$$

is nuclear for every Banach space F and Hilbert spaces H_1, H_2.

Note that if a complemented subspace of a Banach space E fails to have HSFP (resp. NFP) then so does E.

Grothendieck [GR] (cf. also [L–P]) discovered the important fact:

THEOREM GR. *Every $L^1(\mu)$ space has HSFP. Every $C(S)$ space has both HSFP and NFP.*

Now we are ready to prove the nonisomorphic part of Theorem B.

PROPOSITION 5.1. *If $W\subset \boldsymbol{Z}_+^d$ is a smoothness which is not an interval, then $L^1_W(\boldsymbol{T}^d)$ and $L^1_W(\boldsymbol{R}^d)$ do not have HSFP while $C_W(\boldsymbol{T}^d)$, $C_W(\boldsymbol{R}^d)$, $L^\infty_W(\boldsymbol{T}^d)$ and $L^\infty_W(\boldsymbol{R}^d)$ do not have NFP.*

We begin with the two-dimensional case.

LEMMA 5.1. *If $W\subset\boldsymbol{Z}_+^2$ is a smoothness which is not an interval, then $L^1_W(\boldsymbol{T}^2)$ and $L^1_W(\boldsymbol{R}^2)$ do not have HSFP.*

Proof. By Lemmas 0.1 and 0.2, there are points $(n_1,m_1)\in W$ and $(n_2,m_2)\in W$ such that

$$\sum_{p,q\in\boldsymbol{Z}^2}|p|^{n_1+n_2-1}|q|^{m_1+m_2-1}\Big(\sum_{(n,m)\in W}p^{2n}q^{2m}\Big)^{-1}=+\infty. \tag{5.2}$$

By Corollary 4.2 the operator which assigns to every $u\in C_0^\infty(\boldsymbol{R}^2)$ its derivative $|D_x^{(n_1+n_2-1)/2}D_y^{(m_1+m_2-1)/2}|\,u$ extends to the bounded linear operator, say Q, from $L^1_W(\boldsymbol{R}^2)$ into $L^2(\boldsymbol{R}^2)$. Now we define the operator $S\colon L^2_W(\boldsymbol{T}^2)\to L^2(\boldsymbol{R}^2)$ by the factorization

$$S\colon\quad L^2_W(\boldsymbol{T}^2)\xrightarrow{I}L^1_W(\boldsymbol{T}^2)\xrightarrow{M_\varphi}L^1_W(\boldsymbol{R}^2)\xrightarrow{Q}L^2(\boldsymbol{R}^2),$$

i.e. $S=QM_\varphi I$, where I is the natural injection (precisely for $f\in L^2_W(\boldsymbol{T}^2)$, If is the same function regarded as an element of $L^1_W(\boldsymbol{T}^2)$) and M_φ is the operator of multiplication by the function $\varphi(x,y)=(2\pi)^{-1}\exp(-\frac{1}{2}(x^2+y^2))$.

The boundedness of M_φ has been established in the proof of Theorem 4.2 (formula (4.10)). The boundedness of I is well known. In fact $\|I\|\leqslant\sqrt{k}$ where k is the number of elements of W. Indeed, the inequality $\|g\|_1\leqslant\|g\|_2$ for every measurable $g\colon\boldsymbol{T}^2\to\boldsymbol{C}$ combined with the Schwarz inequality yields

$$\|f\|_{L^1_W(\boldsymbol{T}^2)}=\sum_{A\in W}\|D^Af\|_1\leqslant\sqrt{k}\Big(\sum_{A\in W}\|D^Af\|_1^2\Big)^{1/2}$$
$$\leqslant\sqrt{k}\Big(\sum_{A\in W}\|D^Af\|_2^2\Big)^{1/2}=\sqrt{k}\,\|f\|_{L^2_W(\boldsymbol{T}^2)}.$$

Thus S is a bounded linear operator.

To complete the proof we shall show that S is not Hilbert–Schmidt. To this end observe first that the functions

$$\Big\{e_{p,q}\Big(\sum_{(n,m)\in W}p^{2n}q^{2m}\Big)^{-1/2}\Big\}_{(p,q)\in\boldsymbol{Z}^2}$$

form an orthonormal system in $L^2_W(\boldsymbol{T}^2)$, where $e_{p,q}(x,y)=\exp(ipx+iqy)$. Next, as in the proof of Theorem 4.2, we estimate from below the quantity

$$\|Se_{p,q}\|_2^2=\||D_x^{(n_1+n_2-1)/2}D_y^{(m_1+m_2-1)/2}|(\varphi e_{p,q})\|_2^2.$$

Since $(\varphi e_{p,q})\hat{}\,(\xi, \eta) = \varphi(p-\xi, q-\eta)$, using the Plancherel identity we obtain

$$\|Se_{p,q}\|_2^2 = \iint_{\mathbf{R}^2} |(\varphi e_{p,q})\hat{}\,(\xi, \eta)|^2 |\xi|^{n_1+n_2-1} |\eta|^{m_1+m_2-1}\, d\xi\, d\eta$$
$$= (2\pi)^{-2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\xi-p|^{n_1+n_2-1} |\eta-q|^{m_1+m_2-1} e^{-\xi^2-\eta^2}\, d\xi\, d\eta$$
$$\geqslant c\,|p|^{n_1+n_2-1} |q|^{m_1+m_2-1}$$

where c is a positive numerical constant independent of p and q. Thus

$$\sum_{(p,q)\in\mathbf{Z}^2} \Big\|S\Big(e_{p,q}\Big(\sum_{(n,m)\in W} p^{2n} q^{2m}\Big)^{-1/2}\Big)\Big\|_2^2$$
$$\geqslant c \sum_{(p,q)\in\mathbf{Z}^2} |p|^{n_1+n_2-1} |q|^{m_1+m_2-1} \Big(\sum_{(n,m)\in W} p^{2n} q^{2m}\Big)^{-1} = +\infty.$$

Therefore S is not a Hilbert–Schmidt operator.

The similar lemma for $C_W(\mathbf{T}^2)$, $C_W(\mathbf{R}^2)$, $L_W^\infty(\mathbf{T}^2)$ and $L_W^\infty(\mathbf{R}^2)$ is more complicated.

Lemma 5.2. *If $W \subset \mathbf{Z}_+^2$ is a smoothness which is not an interval, then $C_W(\mathbf{T}^2)$, $C_W(\mathbf{R}^2)$, $L_W^\infty(\mathbf{T}^2)$ and $L_W^\infty(\mathbf{R}^2)$ do not have NFP.*

Proof. We consider only the case of C_W; the argument for L_W^∞ is the same. Pick again (n_1, m_1) and (n_2, m_2) to satisfy (5.2). Next pick a sequence of positive numbers $(\lambda_{p,q})_{(p,q)\in\mathbf{Z}^2}$ so that

$$\sum_{(p,q)\in\mathbf{Z}^2} \lambda_{p,q}^2 = 1,$$

(5.3) $$\sum_{(p,q)\in\mathbf{Z}^2} \lambda_{p,q} |p|^{(n_1+n_2-1)/2} |q|^{(m_1+m_2-1)/2} \Big(\sum_{(n,m)\in W} p^{2n} q^{2m}\Big)^{-1/2} = +\infty.$$

We consider the operator $S\colon L_W^2(\mathbf{T}^2) \to l^2(\mathbf{Z}^2)$ defined by the factorization

$$S\colon L_W^2(\mathbf{T}^2) \xrightarrow{C_\Omega} C_W(\mathbf{T}^2) \xrightarrow{M_{\varphi^{1/2}}} C_W(\mathbf{R}^2)$$
$$\xrightarrow{T} L_W^1(\mathbf{R}^2) \xrightarrow{P} L_W^1(\mathbf{T}^2) \xrightarrow{Q_\pi} l^2(\mathbf{Z}^2)$$

where C_Ω is the operator of convolution with the function

$$\Omega = \sum_{(p,q)\in\mathbf{Z}^2} \lambda_{p,q} e_{p,q} \in L^2(\mathbf{T}^2),$$

$M_{\varphi^{1/2}}$ is the operator of multiplication by the function $\varphi^{1/2}(x, y) = (2\pi)^{-1/2}\exp(-\frac{1}{4}(x^2+y^2))$, T is again the operator of multiplication by the same function but regarded as acting between other spaces, P is the Poisson summation operator defined by

$$(Pf)(x, y) = \sum_{(p,q)\in\mathbf{Z}^2} f(x+2\pi p, y+2\pi q) \quad \text{for } (x, y) \in [-\pi, \pi]$$

and Q_π is the operator which assigns to each $f \in L_W^1(\mathbf{T}^2)$ the sequence

$$(\alpha_{p,q} |p|^{(n_1+n_2-1)/2} |q|^{(m_1+m_2-1)/2})_{(p,q)\in\mathbf{Z}^2}$$

where $\alpha_{p,q} = (2\pi)^{-2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\overline{e_{p,q}(x, y)}\, dx\, dy$ is the (p, q)-th Fourier coefficient of f. It follows from Theorem 4.2 that Q_π is bounded. The boundedness of P follows from [S–W], Chap. VII, § 2, Theorem 2.4 where it is proved that for every $g \in L^1(\mathbf{R}^2)$, $\|Pg\|_1 \leqslant \|g\|_1$, in particular, $\|P(D^A f)\|_1 \leqslant \|D^A f\|_1$ for every $A \in W$ and every $f \in L_W^1(\mathbf{R}^2)$. The argument for the boundedness of $M_{\varphi^{1/2}}\colon C_W(\mathbf{T}^2) \to C_W(\mathbf{R}^2)$ is similar to that for the boundedness of $M_\varphi\colon L_W^1(\mathbf{T}^2) \to L_W^1(\mathbf{R}^2)$ presented in the proof of Theorem 4.2. The fact that $T\colon C_W(\mathbf{R}^2) \to L_W^1(\mathbf{R}^2)$ is absolutely summing (and therefore bounded) is easy. To this end note first that if a finite sequence $(f_j) \subset C_W(\mathbf{R}^2)$ satisfies

$$\Big\|\sum_j \varepsilon_j f_j\Big\|_{C_W(\mathbf{R}^2)} \leqslant 1$$

for every sequence (ε_j) with $|\varepsilon_j| = 1$ for $j = 1, 2, \ldots$, then for every $B \in W$, $\big\|\sum_j |D^B f_j|\big\|_\infty \leqslant 1$ where $|D^B f_j|(x, y) = |(D^B f_j)(x, y)|$ for $(x, y) \in \mathbf{R}^2$. Next fix $A \in W$ and observe that the Leibniz formula for the derivative of the product of functions yields

$$D^A(f\varphi^{1/2}) = \sum_{0\leqslant B\leqslant A} k(B, A) D^{A-B}(\varphi^{1/2}) D^B f$$

where $k(B, A)$ are real numbers independent of f. Thus

$$\sum_j \|D^A(f_j\varphi^{1/2})\|_1 = \iint_{\mathbf{R}^2} \sum_j |D^A(f_j\varphi^{1/2})|\, dx\, dy$$
$$\leqslant \sum_{0\leqslant B\leqslant A} |k(B, A)| \iint_{\mathbf{R}^2} |D^{A-B}(\varphi^{1/2})| \sum_j |D^B f_j|\, dx\, dy$$
$$\leqslant \sum_{0\leqslant B\leqslant A} |k(B, A)|\, \|D^{A-B}(\varphi^{1/2})\|_1 \Big\|\sum_j |D^B f_j|\Big\|_\infty$$
$$\leqslant k(A),$$

where $k(A) = \sum_{0\leqslant B\leqslant A} |k(B, A)|\, \|D^{A-B}(\varphi^{1/2})\|_1$ is a numerical constant independent of the choice of the sequence (f_j). Hence T satisfies (5.1) with $C = \max\{k(A)\colon A \in W\}$.

To complete the proof of the lemma we have to show that S is not nuclear. To this end, remembering that

$$\{e_{p,q}(\sum_{(n,m)\in W} p^{2n}q^{2m})^{-1/2}\}_{(p,q)\in \mathbf{Z}^2}$$

is an orthonormal basis in $L^2_W(\mathbf{T}^2)$, it is enough to show

$$\sum_{(p,q)\in \mathbf{Z}^2} \left|\left\langle S\left(e_{p,q}(\sum_{(n,m)\in W} p^{2n}q^{2m})^{-1/2}\right), \delta_{p,q}\right\rangle\right| = +\infty \tag{5.4}$$

where $(\delta_{p,q})_{(p,q)\in \mathbf{Z}^2}$ is the unit vector basis in $l^2(\mathbf{Z}^2)$.

We compute the scalar product $\langle S(e_{p,q}), \delta_{p,q}\rangle$. First note that

$$TM_{\varphi^{1/2}}C_\Omega(e_{p,q}) = \lambda_{p,q}\varphi e_{p,q};$$

next note that for $g\in L^1_W(\mathbf{T}^2)$,

$$\langle Q_\pi(g), \delta_{p,q}\rangle = \alpha_{p,q}|p|^{(n_1+n_2-1)/2}|q|^{(m_1+m_2-1)/2}$$

where $\alpha_{p,q}$ is the (p, q)-th Fourier coefficient of g.

By Theorem 2.4 of [S–W], Chap. VII, § 2, if $g = Pf$, then $\alpha_{p,q} = \hat{f}(p, q)$. Thus remembering that $(\varphi e_{p,q})\hat{}\,(\xi, \eta) = \varphi(\xi-p, \eta-q)$, we get

$$\begin{aligned}\langle Se_{p,q}, \delta_{p,q}\rangle &= \lambda_{p,q}(\varphi e_{p,q})\hat{}\,(p, q)|p|^{(n_1+n_2-1)/2}|q|^{(m_1+m_2-1)/2}\\ &= \lambda_{p,q}\varphi(0, 0)|p|^{(n_1+n_2-1)/2}|q|^{(m_1+m_2-1)/2}\\ &= (2\pi)^{-1}\lambda_{p,q}|p|^{(n_1+n_2-1)/2}|q|^{(m_1+m_2-1)/2}.\end{aligned}$$

The latter formula together with (5.3) yields (5.4). This completes the proof of Lemma 5.2.

Now, Proposition 5.1 follows from the next lemma which reduces the general d-dimensional case to the two-dimensional one.

LEMMA 5.3. *If $W\subset \mathbf{Z}^d_+$ is a smoothness which is not an interval, then there exists a smoothness $V\subset \mathbf{Z}^2_+$ which is not an interval and such that the space E_V is isomorphic to a complemented subspace of E_W.*

Here E_W stands for one of the spaces either $C_W(\mathbf{R}^d)$ or $C_W(\mathbf{T}^d)$ or $L^p_W(\mathbf{R}^d)$ or $L^p_W(\mathbf{T}^d)$ for fixed $p\in[1, \infty]$ and E_V stands for the corresponding space for the smoothness V.

Proof (suggested by S. V. Kisliakov and N. G. Sidorenko). The relation of being isomorphic to a complemented subspace is transitive. Thus, in view of Lemma 0.3 and Corollary 0.1, it suffices to show that if $d\geqslant 2$ and $V\subset \mathbf{Z}^{d-1}_+$ is simply generated by a smoothness $W\subset \mathbf{Z}^d_+$ then E_V is isomorphic to a complemented subspace of E_W. We consider the case of $\mathbf{R}^d$; the argument for $\mathbf{T}^d$ is similar.

Order the coordinates of $\mathbf{R}^d$ so that $V = \varphi(W)$ where φ is given by (0.5).

Let $x_{d-1} = u$, $x_d = v$, $y_{d-1} = w$; denote by ${}^{d-2}z$ the first $d-2$ coordinates of a vector z. Fix $h\in C_0^\infty(\mathbf{R})$ with

$$\int_{\mathbf{R}} h^2(t)\,dt = 1. \tag{+}$$

For $f\in E_W$ and $g\in E_V$ put

$$(Jg)({}^{d-2}x, u, v) = g({}^{d-2}x, u+v)h((u-v)/2) \quad \text{for } x = (x_j)\in \mathbf{R}^d,$$

$$(Pf)({}^{d-2}y, w) = \int_{\mathbf{R}} f({}^{d-2}y, t, w-t)h(w/2+t)\,dt \quad \text{for } y = (y_j)\in \mathbf{R}^{d-1}.$$

Clearly, by (+), $PJg = g$ for $g\in C_0^\infty(\mathbf{R}^{d-1})$. It remains to establish the boundedness of the operators $J\colon E_V\to E_W$ and $P\colon E_W\to E_V$.

Fix $A = (C, a, b)\in W$; $C\in \mathbf{Z}^{d-2}_+$. Put $r = a+b$, $D^m_w g = g^{(m)}$. Then

$$\begin{aligned}(D^A Jg)&({}^{d-2}x, u, v)\\ &= \sum_{n=0}^{a}\sum_{m=0}^{b}\binom{a}{n}\binom{b}{m}2^{n+m-r}(-1)^{b-m}D^C g^{(n+m)}({}^{d-2}x, u+v)\\ &\qquad\times h^{(r-n-m)}\left(\frac{u-v}{2}\right).\end{aligned}$$

Thus for $1\leqslant p\leqslant\infty$, by the triangle inequality,

$$\begin{aligned}\|D^A Jg\|_{L^p(\mathbf{R}^d)} \leqslant \sum_{q=0}^{r}\sum_{n+m=q}\binom{a}{n}\binom{b}{m}\Big(\int_{\mathbf{R}^{d-2}}\int_{\mathbf{R}^2}&|D^C g^{(q)}({}^{d-2}x, u+v)\\ &\times h^{(r-q)}((u-v)/2)|^p\,du\,dv\,d({}^{d-2}x)\Big)^{1/p}\end{aligned}$$

Since

$$\int_{\mathbf{R}^2} k(u+v)h((u-v)/2)\,du\,dv = \int_{\mathbf{R}} k(s)\,ds\int_{\mathbf{R}} h(t)\,dt$$

for $k, h\in C_0^\infty(\mathbf{R})$, we get

$$\|D^A Jg\|_{L^p(\mathbf{R}^d)} \leqslant 2^r\sum_{q=0}^{r}\|D^C g^{(q)}\|_{L^p(\mathbf{R}^{d-1})}\|h^{(r-q)}\|_{L^p(\mathbf{R})}.$$

The latter inequality is also valid for $p = \infty$ (by letting $p\to\infty$). This proves the boundedness of J because if $A\in W$ then $(C, q)\in V$ for $0\leqslant q\leqslant r$.

Fix $f\in C_0(\mathbf{R}^d)$ and $B = (C, r)\in V$. Then $B = \varphi(C, a, b)$ for some $(C, a, b)\in W$. Put $G = D^C Pf = P(D^C f)$. Differentiating the integrand of G b-times with respect to the variable w, then substituting $t = w-s$ and again differentiating the integrand a-times with respect to w we get

$$\begin{aligned}(D^r_w G)({}^{d-2}y, w) = \sum_{n=0}^{a}\sum_{m=0}^{b}&\binom{a}{n}\binom{b}{m}2^{m+n-r}3^{a-n}\\ &\times\int_{\mathbf{R}} D^{(C,n,m)}f({}^{d-2}y, w-s, s)h^{(r-n-m)}(\tfrac{3}{2}w-s)\,ds.\end{aligned}$$

Thus for fixed p with $1 \leqslant p < \infty$, by the triangle inequality, we get

$$\|D^B Pf\|_{L^p(\mathbf{R}^{d-1})} \leqslant 2^r 3^a \sum_{m=0}^{b} \sum_{n=0}^{a} \Big(\int_{\mathbf{R}^{d-2}} \int_{\mathbf{R}} \Big| \int_{\mathbf{R}} D^{(C,n,m)} f({}^{d-2}y,\, w-s,\, s) \times h^{(r-n-m)}(\tfrac{3}{2}w - s)\, ds \Big|^p dw\, d({}^{d-2}y)\Big)^{1/p}.$$

Estimating by the Hölder inequality the integral $|\int_{\mathbf{R}} \ldots ds|$ we get

$$\|D^B Pf\|_{L^p(\mathbf{R}^{d-1})} \leqslant 2^r 3^a \sum_{m=0}^{b} \sum_{n=0}^{a} \|D^{(C,n,m)} f\|_{L^p(\mathbf{R}^d)} \|h^{(r-n-m)}\|_{L^{p'}(\mathbf{R})}.$$

Letting $p \to \infty$ we infer the validity of the latter inequality also for $p = \infty$. This shows the boundedness of P because if $B \in V$ then $(C, n, m) \in W$ for $0 \leqslant n \leqslant a$, $0 \leqslant m \leqslant b$.

6. Sobolev spaces isomorphic to classical Banach spaces. In this section we complete the proofs of Theorems B and C stated in the Introduction.

We begin with a natural representation of Sobolev spaces as subspaces of certain L^p-spaces and spaces of continuous functions.

In the sequel by S we denote either $\mathbf{R}$ or $\mathbf{T}$. Given a smoothness $W \subset \mathbf{Z}^d_+$ and p with $1 \leqslant p \leqslant \infty$ we denote by $\sum_W L^p(S^d)$ the l^p-sum of w copies of $L^p(S^d)$ where w is the number of elements in W. Clearly the space $\sum_W L^p(S^d)$ can be naturally identified with an L^p-space on a measure space depending on W but independent of p. Next we define the isometric isomorphism

$$J_W\colon L^p_W(S^d) \to \textstyle\sum_W L^p(S^d)$$

by

$$J_W(f) = (D^A f)_{A \in W} \quad \text{for } f \in L^p_W(S^d).$$

The map J_W is called the *canonical isomorphism* of $L^p_W(S^d)$ and the range of J_W is called the *canonical image* of $L^p_W(S^d)$.

Similarly we define $\sum_W C(S^d)$, the canonical isomorphism and the canonical image of $C_W(S^d)$. Note that for $S = \mathbf{R}$, $C(\mathbf{R})$ denotes here the uniform closure of all scalar-valued continuous functions on $\mathbf{R}$ with compact supports.

The orthogonal projection

$$P_W\colon \textstyle\sum_W L^2(S^d) \xrightarrow[\text{onto}]{} J_W(L^2_W(S^d))$$

is called the *canonical projection*.

Next we introduce some notation.

Given $\Xi = (\xi_j) \in \mathbf{R}^d$ and $A = (a_j) \in \mathbf{Z}^d_+$ we put

$$\Xi^A = \prod_{j=1}^{d} \xi_j^{a_j} \quad \text{and} \quad |A| = \sum_{j=1}^{d} a_j.$$

For a nonempty finite set $W \subset \mathbf{Z}^d_+$ we define the function $Q(W)\colon \mathbf{R}^d \to \mathbf{R}_+$ by

$$Q(W)(\Xi) = \sum_{A \in W} \Xi^{2A}.$$

Now we are ready for

Proposition 6.1. *Let $W \subset \mathbf{Z}^d_+$ be a smoothness. Then the canonical projection $P_W\colon \sum_W L^2(\mathbf{R}^d) \to J_W(L^2_W(\mathbf{R}^d))$ is given by the formula*

$$P_W[(f_B)_{B\in W}] = \Big(\sum_{B \in W} T_{A,B}(f_B)\Big)_{A \in W}, \tag{6.1}$$

where $T_{A,B}\colon L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ is for $A, B \in W$ defined by

$$T_{A,B}(g) = \big(i^{|A|-|B|}\, \Xi^{A+B} Q(W)^{-1} \hat{g}\big)^{\check{}}. \tag{6.2}$$

Proof. If $(f_B)_{B\in W} = (D^B f)_{B\in W}$ for some $f \in \mathcal{S}(\mathbf{R}^d)$ then, for every $A \in W$,

$$\begin{aligned}\sum_{B\in W} T_{A,B}(D^B f) &= \sum_{B\in W} \big(i^{|A|-|B|}\, \Xi^{A+B} Q(W)^{-1} i^{|B|} \Xi^B \hat{f}\big)^{\check{}} \\ &= \Big(i^{|A|} \Xi^A \sum_{B\in W} \Xi^{2B} Q(W)^{-1} \hat{f}\Big)^{\check{}} \\ &= (i^{|A|} \Xi^A \hat{f})^{\check{}} = D^A f.\end{aligned}$$

Next pick $(f_B)_{B\in W}$ so that $f_B \in \mathcal{S}(\mathbf{R}^d)$ for all B in W. Then a similar calculation gives $P_W[(f_B)_{B\in W}] = J_W(f)$ for

$$f = \Big(\sum_{B\in W} i^{-|B|} \Xi^B Q(W)^{-1} \hat{f}_B\Big)^{\check{}}.$$

Thus P_W takes a dense subset of $\sum_W L^2(\mathbf{R}^d)$ into the canonical image of $L^2_W(\mathbf{R}^d)$ and it is the identity on a dense set of the canonical image. Hence P_W is a bounded projection from $\sum_W L^2(\mathbf{R}^d)$ onto $J_W(L^2_W(\mathbf{R}^d))$; the boundedness of P_W follows from the boundedness of each $T_{A,B}$; the operators $T_{A,B}$ are bounded because they are induced by the multipliers

$$i^{|A|-|B|}\, \Xi^{A+B} Q(W)^{-1} \in L^\infty(\mathbf{R}^d).$$

Finally the projection P_W is orthogonal because it is self-adjoint. Indeed, the adjoint P_W^* is given by

$$P_W^*[(f_B)_{B\in W}] = \Big(\sum_{B\in W} T^*_{B,A}(f_B)\Big)_{A\in W}$$

where $T^*_{B,A} = T_{A,B}$.

The analogue of Proposition 6.1 for T^d differs only by the form of the Fourier Transform for this group. Precisely we have:

PROPOSITION 6.1′. *The canonical projection*

$$P_W\colon \sum_W L^2(T^d) \to J_W(L^2_W(T^d))$$

is given by (6.1) *where the operators* $T_{A,B}$ *are defined by*

$$T_{A,B}(g) = \sum_{P\in Z^d} \alpha_P\, i^{|A|-|B|} P^{A+B} Q(W)^{-1}(P)\, e_P, \tag{6.2′}$$

where for $P \in Z^d$ *and for* $X \in [-\pi, \pi]^d$,

$$e_P(X) = e^{i\langle P,X\rangle}, \qquad \alpha_P = (2\pi)^{-d} \int_{[-\pi,\pi]^d} g(X)\, e_P(-X)\, dX.$$

Recall that an operator, say U, defined on some space of measurable functions on a measure space is called p-bounded for some p with $1 \leqslant p \leqslant \infty$ provided there is a positive constant $K = K_p$ such that $\|Ug\|_p \leqslant K\|g\|_p$ for every g in the intersection of the domain of the operator with the corresponding L^p-space.

PROPOSITION 6.2. *For an arbitrary smoothness* $W \subset Z^d_+$ *and for* $1 < p < \infty$ *the canonical projection* P_W *is p-bounded.*

Proof. By 2^d we denote the subset of Z^d_+ consisting of all characteristic functions of subsets of the set $\{1, 2, \ldots, d\}$ of indices; the characteristic function of the one-element set $\{j\}$ is denoted by $E^{(j)}$.

First we consider the case of R^d. In view of Proposition 6.1 it is enough to show that for every $A \in W$ and $B \in W$ the multiplier $m_{A,B} = \Xi^{A+B} Q(W)^{-1}$ induces via the Fourier Transform a bounded operator in $L^p(R^d)$. To this end it suffices to check that $m_{A,B}$ satisfies the hypothesis of the multidimensional Marcinkiewicz Multiplier Theorem (cf. [ST], Chap. IV, § 6, Theorem 6′). Since $|m_{A,B}| \leqslant 1$, we have to show that there exists a numerical constant $K = K(A, B, p)$ such that for every $0 \neq E \in 2^d$,

$$\sup_b \int_b |D^E m_{A,B}|\, d\xi_{j_1} d\xi_{j_2} \ldots d\xi_{j_k} \leqslant K, \tag{6.3}$$

where $j_1 < j_2 < \ldots < j_k$; E is the characteristic function of the set $\{j_1, j_2, \ldots, j_k\}$; the supremum is taken over all dyadic parallelepipeds in R^k. In fact the integral in (6.3) is a function depending on the variables whose indices belong to the complement of the set $\{j_1, j_2, \ldots, j_k\}$; inequality (6.3) can be regarded as an inequality between functions.

To verify (6.3) one may assume without loss of generality that E is the characteristic function of the set $\{1, 2, \ldots, k\}$ for some k with $1 \leqslant k \leqslant d$. Let us denote by $\Phi_k(A, B)$ the family of all sequences

$$\varphi = (E_0, E_1, \ldots, E_{s(\varphi)};\ A+B, A_1, A_2, \ldots, A_{s(\varphi)})$$

such that $E_t \in 2^d$ for $t = 0, 1, \ldots, s(\varphi)$; $0 \leqslant E_0 < E_1 < \ldots < E_{s(\varphi)} = E$; $A_t \leqslant A+B$; $E_t - E_{t-1} \leqslant A_t \in W$ for $t = 1, 2, \ldots, s(\varphi)$; $0 \leqslant s(\varphi) \leqslant k$. Let

$$C(\varphi) = A+B+\sum_{t=1}^{s(\varphi)} A_t - E.$$

Clearly $C(\varphi) \in Z^d_+$ because

$$C(\varphi) = A+B-E_0+\sum_{t=1}^{s(\varphi)} \big(A_t - (E_t - E_{t-1})\big) \geqslant 0.$$

We have

$$D^E m_{A,B} = \sum_{\varphi\in\Phi_k(A,B)} \beta_\varphi\, \Xi^{C(\varphi)} (Q(W))^{-s(\varphi)-1} \tag{6.4}$$

where β_φ are scalars.

Formula (6.4) can be easily verified by induction with respect to k (= the number of elements of the support of E) using the standard formulae of differentiation:

$$(PQ^{-1})_{\xi_j} = Q^{-1} P_{\xi_j} - PQ^{-2} Q_{\xi_j},$$

$$(Q(W))_{\xi_j} = \sum_{E^{(j)} \leqslant A \in W} 2a_j(A)\, \Xi^{2A-E^{(j)}},$$

where $a_j(A)$ denotes the jth coordinate of A.

In view of (6.4), to establish that $D^E m_{A,B}$ satisfies (6.3) it is enough to show

$$\int_b |\Xi^{C(\varphi)}| (Q(W))^{-s(\varphi)-1}\, d\xi_1 d\xi_2 \ldots d\xi_k \leqslant (\log 2)^k \tag{6.5}$$

for $\varphi \in \Phi_k(A, B)$.

Fix $\varphi = (E_0, E_1, \ldots, E_{s(\varphi)};\ A+B, A_1, A_2, \ldots, A_{s(\varphi)}) \in \Phi_k(A, B)$. Note that for every $A \in W$, $\Xi^{2A} \leqslant Q(W)$. Hence $|\Xi^{A+B}| \leqslant 2^{-1}(\Xi^{2A} + \Xi^{2B}) \leqslant Q(W)$. Thus

$$|\Xi^{C(\varphi)}| = |\Xi^{A+B}| \prod_{t=1}^{s(\varphi)} \Xi^{2A_t} |\Xi^E|^{-1} \leqslant |\Xi^E|^{-1} (Q(W))^{s(\varphi)+1}.$$

Therefore for $E \neq 0$,

$$|\Xi^{C(\varphi)}| (Q(W))^{-s(\varphi)-1} \leqslant |\Xi^E|^{-1} = \prod_{j=1}^{k} |\xi_j|^{-1}.$$

Hence, by the definition of a dyadic parallelepiped in R^k (cf. [ST] Chap. IV, § 5)

$$\int_b |\Xi^{C(\varphi)}| (Q(W))^{-s(\varphi)-1}\, d\xi_1 d\xi_2 \ldots d\xi_k \leqslant \int_b \prod_{j=1}^{k} |\xi_j|^{-1}\, d\xi_1 d\xi_2 \ldots d\xi_k = (\log 2)^k.$$

This completes the proof of the proposition for R^d.

The proof for T^d reduces to the case of R^d. In view of [S–W], Chap. VII, § 3, Theorem 3.8, the fact that the multiplier $m_{A,B}$ induces a p-bounded operator in $L^p(R^d)\cap L^2(R^d)$ implies that the sequence $(m_{A,B}(P))_{P\in Z^d}$ is a multiplier which induces a p-bounded operator in $L^p(T^d)\cap L^2(T^d)$. Thus in view of (6.2′) the canonical projection $P_W\colon \sum_W L^2(T^d)\to J_W(L^2_W(T^d))$ is p-bounded for $1<p<\infty$.

In contrast with Proposition 6.2 we have

THEOREM 6.1. *Let $W\subset Z^d_+$ be a smoothness. Then the following conditions are equivalent:*

(i) *W is an interval,*

(ii) *P_W is 1-bounded,*

(iii) *P_W is ∞-bounded,*

(iv) *P_W restricted to $\sum_W L^2(S^d)\cap\sum_W C(S^d)$ extends to a bounded operator from $\sum_W C(S^d)$ onto $J_W(C_W(S^d))$.*

Proof. Clearly (ii) ⇒ (iii) by a standard duality argument. (iii) ⇒ (iv) because

$$P_W\big((f_B)_{B\in W}\big)\subset J_W\big(\mathcal{S}(R^d)\big)\subset J_W\big(C_W(R^d)\big)$$

whenever $f_B\in\mathcal{S}(R^d)$ for all $B\in W$ (resp. $P_W\big((f_B)_{B\in W}\big)\subset J_W\big(C_W(T^d)\big)$ whenever all the f_B's are trigonometric polynomials); such sequences $(f_B)_{B\in W}$ are dense in $\sum_W C(R^d)$ (resp. in $\sum_W C(T^d)$). The implication (iv) ⇒ (i) follows from Proposition 5.1 combined with Theorem GR stated in Section 5 and with the fact that a complemented subspace of a space with NFP also has NFP. Thus to complete the proof of Theorem 6.1 we have to show that (i) ⇒ (ii). To this end we first prove

LEMMA 6.1. *For $0\leqslant k\leqslant 2n$ and for $n=0,1,\ldots$ put*

$$r_{k,n}(\xi)=\xi^k Q(n)^{-1};\qquad Q(n)(\xi)=\sum_{t=0}^{n}\xi^{2t}\qquad(\xi\in R).$$

Then $r_{k,n}$ is an (inverse) Fourier Transform of a finite Borel measure on R.

Proof. Note that if for some $f\in L^1(R)$, the derivatives f' and f'' belong to $L^1(R)$, then f is the Fourier Transform of a function belonging to $L^1(R)\cap L^2(R)$. Next observe that if P, Q are polynomials with $Q>0$ then $PQ^{-1}\in L^1(R)$ iff $\deg Q\geqslant\deg P+2$; thus $PQ^{-1}\in L^1(R)$ implies that all the derivatives of PQ^{-1} are in $L^1(R)$, hence PQ^{-1} is the Fourier Transform of a function belonging to $L^1(R)\cap L^2(R)$. Using this criterion we infer that $(r_{0,1})^{\check{}}\in L^1(R)$ and $(r_{k,n})^{\check{}}\in L^1(R)$ for $0\leqslant k\leqslant 2n-2$ and for $n\geqslant 2$. Clearly $(r_{0,0})^{\check{}}$ is the point mass at zero. A direct computation shows that $(r_{1,1})^{\check{}}$ $=\sqrt{\pi/2}\,i\,\mathrm{sign}\,x e^{-|x|}$. Thus $(r_{1,1})^{\check{}}\in L^1(R)$. The identity

$$r_{2n-1,n}-r_{1,1}=-\sum_{j=0}^{n-2}\xi^{2j+1}\big(Q(n)Q(1)\big)^{-1}\in L^1(R)$$

combined with the criterion mentioned at the beginning of the proof and with the formula for $(r_{1,1})^{\check{}}$ implies that $(r_{2n-1,n})^{\check{}}\in L^1(R)$ for $n\geqslant 2$. Finally the identity

$$r_{2n,n}=r_{0,0}-\sum_{j=0}^{n-1}r_{2j,n}$$

implies that $(r_{2n,n})^{\check{}}$ is also a finite Borel measure.

Now we complete the proof of the implication (i) ⇒ (ii). First we consider the case of R^d. Let $W\subset Z^d_+$ be an interval, say $W=I(0;C)$ with $C=(c_j)$. Pick $A=(a_j)\in W$ and $B=(b_j)\in W$. Clearly $a_j+b_j\leqslant 2c_j$ for $j=1,2,\ldots,d$ because $A\leqslant C$ and $B\leqslant C$. Combining Proposition 6.1 with [S–W], Chap. VII, § 3, Theorem 3.4, to prove that P_W is 1-bounded it suffices to show that the inverse Fourier Transform $\big(\Xi^{A+B}(Q(W))^{-1}\big)^{\check{}}$ is a finite Borel measure. Note that if $W=I(0;C)$ then

$$Q(W)(\Xi)=\prod_{j=1}^{d}Q(c_j)(\xi_j)\qquad\text{for }\Xi=(\xi_j)\in R^d.$$

Thus

$$\Xi^{A+B}(Q(W))^{-1}=\prod_{j=1}^{d}r_{a_j+b_j,c_j}(\xi_j).$$

Now using Lemma 6.1 we infer that the inverse Fourier Transform of the function $\Xi^{A+B}(Q(W))^{-1}$ is the product measure

$$\bigotimes_{j=1}^{d}(r_{a_j+b_j,c_j})^{\check{}}.$$

The case of T^d reduces to the previous one via [S–W], Chap. VII, § 3, Theorem 3.8.

To complete the proof of Theorems B and C we also need

PROPOSITION 6.3. *For an arbitrary smoothness $W\subset Z^d_+$ and for $1\leqslant p\leqslant\infty$ the space $L^p_W(S^d)$ (resp. $C_W(S^d)$) contains a complemented subspace isomorphic to $L^p(S)$ (resp. $C(S)$).*

Proof. Combine the next two essentially known results.

LEMMA 6.2. *Let $V\subset Z_+$ be a smoothness generated by a smoothness $W\subset Z^d_+$. Then for $1\leqslant p\leqslant\infty$ the space $L^p_W(S^d)$ (resp. $C_W(S^d)$) contains a complemented subspace isometrically isomorphic to $L^p_{(m)}(S)$ (resp. $C_{(m)}(S)$), where $m+1$ is the number of elements in V and (m) denotes the unique smoothness in Z_+ possessing $m+1$ elements.*

The proof of Lemma 6.2 is a nonessential modification of the proof of Lemma 5.3.

PROPOSITION 6.4. *Let $m = 1, 2, \ldots$ Then the space $C_{(m)}(S)$ (resp. $L^p_{(m)}(S)$ for $1 \leqslant p \leqslant \infty$) is isomorphic to $C(S)$ (resp. $L^p(S)$).*

Proof. Let E_m $(m = 0, 1, \ldots)$ denote either $L^p_{(m)}(\boldsymbol{T})$ for fixed p with $1 \leqslant p \leqslant \infty$, or $C_{(m)}(\boldsymbol{T})$. We shall show that the operator

$$Uf = f' + (2\pi)^{-1} \int_{-\pi}^{\pi} f(s)\, ds \cdot 1$$

is an isomorphism from E_m onto E_{m-1} for $m = 1, 2, \ldots$

Pick f and a sequence (f_n) in E_m so that $f_n \underset{E_m}{\to} f$. Then

$$\int f_n \to \int f \quad \text{and} \quad f_n' \underset{E_{m-1}}{\to} f',$$

hence $Uf_n \underset{E_{m-1}}{\to} Uf$ as $n \to \infty$. Conversely, if $Uf_n \underset{E_{m-1}}{\to} g$ for some g in E_{m-1} then $\int Uf_n \to \int g$. Note that $\int Uf_n = \int f_n$ for all n because for h with h' periodic $\int_{-\pi}^{\pi} h'(s)\, ds = 0$. Hence our assumption yields that for p which determines E_m,

$$\|f_n' - g\|_p \to 0 \quad \text{and} \quad \int f_n \to \int g$$

as $n \to \infty$. This implies that there is a periodic f such that $\|f_n - f\|_p \to 0$ and $f' = g$. Thus $f \in E_m$ and $f_n \underset{E_m}{\to} f$ as $n \to \infty$. Hence U is continuous and has closed graph. Moreover, U is an algebraic isomorphism on the trigonometric polynomials of degree $\leqslant k$ for every $k = 1, 2, \ldots$ Since the trigonometric polynomials are dense in E_m (except the case of $L^\infty_{(m)}(\boldsymbol{T})$ where they are weak-star dense), we infer that U is the desired isomorphism.

Now let F_m denote the closure of all algebraic polynomials restricted to the interval $[-\pi, \pi]$ either under the norm

$$\|f\|_{\infty,m} = \max_{0 \leqslant j \leqslant m} \|f^{(j)}\|_\infty,$$

or under the norm

$$\|f\|_{p,m} = \Big(\sum_{0 \leqslant j \leqslant m} \|f^{(j)}\|_p^p\Big)^{1/p}$$

for $1 \leqslant p < \infty$, or the closure in the weak-star topology of the closure of the algebraic polynomials in the norm $\|\cdot\|_{\infty,m}$. Clearly F_m is isomorphic to a subspace of finite codimension of the suitable space $E_m \times \boldsymbol{C}^{m+1}$. Hence F_m is also isomorphic to the corresponding $L^p(\boldsymbol{T})$ (resp. $C(\boldsymbol{T})$). Thus the same is true for the spaces $\sum F_m$, where $\sum F_m$ denotes the infinite l^p-sum (resp. c_0-sum) of copies of the same F_m (m and the p determining F_m are fixed). Using a simple version of the Whitney extension theorem for the interval (cf. [ST], Chap. VI for a general result) one can easily see that the space $\sum F_m$ can be decomposed into the direct sum $L^p_{(m)}(\boldsymbol{R}) \times l^p$ (resp. $C_{(m)}(\boldsymbol{R}) \times c_0$). Now the standard decomposition technique yields the desired conclusion also for the Sobolev space on $\boldsymbol{R}$.

Now we are ready to prove Theorems B and C.

Proof of Theorem B. Let $W \subset \boldsymbol{Z}^d_+$ be a smoothness. Since a complemented subspace of a space with HSFP (resp. NFP) has the same property, combining Theorem GR with Proposition 5.1 we infer that if W is not an interval then $L^1_W(S^d)$, $L^\infty_W(S^d)$, $C_W(S^d)$ are not isomorphic to complemented subspaces of L^1, L^∞, C spaces respectively. If W is an interval then combining Theorem 6.1 with Proposition 6.3 and using the standard decomposition method (cf. e.g. [MI] and [P]) we infer that the spaces $L^1_W(S^d)$, $L^\infty_W(S^d)$, $C_W(S^d)$ are isomorphic to $L^1(S^d)$, $L^\infty(S^d)$ and $C(S^d)$ respectively.

Proof of Theorem C. Combine Proposition 6.2 with Proposition 6.3 and use the standard decomposition method.

Remark 1. The analysis of the proof of Theorem C and Propositions 6.2 and 6.3 shows that the isomorphisms in question have been constructed in the same way for all p with $1 < p < \infty$ using operators which coincide on the common dense subset of all the scale L^p. Thus we have in fact established the following slightly stronger result.

THEOREM 6.2. *For an arbitrary smoothness $W \subset \boldsymbol{Z}^d_+$ there exists an operator*

$$T: \bigcap_{1 \leqslant p \leqslant \infty} L^p(S) \to \bigcap_{1 < p < \infty} L^p_W(S^d)$$

which is p-bounded for every p with $1 < p < \infty$ and which for each such p extends to an isomorphism from $L^p(S)$ onto $L^p_W(S^d)$.

Moreover, if W is an interval then the operator in question is also 1-bounded and ∞-bounded, and it extends to an isomorphism from $L^1(S)$ onto $L^1_W(S^d)$ and $L^\infty(S)$ onto $L^\infty_W(S^d)$ respectively; the latter isomorphism (in the case of $L^\infty(S)$) carries $C(S)$ onto $C_W(S^d)$.

Remark 2. B. S. Mityagin (private communication) observed that the desired isomorphism for $1 < p < \infty$ can be defined explicitly by

$$T(f) = \big((Q(W))^{-1/2}\hat{f}\big)^{\check{}}.$$

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2. The case of our Theorem B concerning C_W spaces has been obtained independently by N. G. Sidorenko (cf. the forthcoming paper [SI]).

3. Very recently S. V. Kisliakov and N. G. Sidorenko [K–S] have proved that if W is not an interval then the space C_W does not have any local unconditional structure.

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