STUDIA MATHEMATICA, T. LXXXIV. (1986)

On the orthogonal projections onto spaces of pluriharmonic functions and duality

by

EWA LIGOCKA (Warszawa)

Abstract. In the present paper the connections are established between the regularity of the Bergman projection B onto the space of square-integrable holomorphic functions and the regularity of the operators Q and S of orthogonal projection onto the space of square-integrable pluriharmonic functions and the complexified space of the real parts of holomorphic functions respectively. It turns out that the regularity of B in a Sobolev or Hölder norm is always equivalent to the regularity of S and in the case of pseudoconvex domains the regularity of B is equivalent to the regularity of Q. This result is applied to the study of duality between spaces of holomorphic and pluriharmonic functions.

1. Introduction and the statement of results. Let D be a domain in C^n. A function u on D is called *pluriharmonic* iff $\bar{\partial}\partial u = 0$. If $H^1(D, \boldsymbol{R}) = 0$ then the space of pluriharmonic functions is equal to the complexification of the space of the real parts of functions holomorphic on D. In this paper we shall deal with a bounded domain D and with the following spaces of pluriharmonic functions on D:

(1) The space $L^2 PH(D)$ of all square-integrable pluriharmonic functions on D with $L^2(D)$-norm.

(2) The space $L^2 H(D)$ of all square-integrable holomorphic functions on D with $L^2(D)$-norm.

(3) The real space $\operatorname{Re} L^2 H(D)$ of the real parts of functions from $L^2 H(D)$.

(4) The complexification $\operatorname{Re} L^2 H(D) \otimes C$ of $\operatorname{Re} L^2 H(D)$.

The spaces (1) and (2) are always closed subspaces of $L^2(D)$. If the boundary of D is very bad then the spaces (3) and (4) can be not closed in $L^2(D)$.

The necessary and sufficient condition for the closedness of the spaces (3) and (4) is the following estimate:

$$\|f\| \leqslant c\|\operatorname{Re} f\| \quad \text{for every } f \in L^2 H(D) \text{ s.t. } \int_D \operatorname{Im} f = 0.$$

If this estimate holds for D then we shall consider the following orthogonal projections from $L^2(D)$:

(1) The projector Q onto $L^2 PH(D)$.
(2) The Bergman projection B onto $L^2 H(D)$.
(3) The real projection S_r from $L^2_r(D)$ onto $\operatorname{Re} L^2 H(D)$.
(4) The projection S from $L^2(D)$ onto $\operatorname{Re} L^2 H(D) \otimes \boldsymbol{C}$.

The following theorem holds:

THEOREM 0. *Let D be a bounded domain with C^2-boundary. Then*

(1) *There exists $c > 0$ s.t. $\|f\| \leqslant c \|\operatorname{Re} f\|$ for all $f \in L^2 H(D)$ such that $\int_D \operatorname{Im} f = 0$.*

(2) *We have $L^2 PH(D) = (\operatorname{Re} L^2 H(D) \otimes \boldsymbol{C}) \oplus E$, where $\dim_{\boldsymbol{C}} E \leqslant \dim_{\boldsymbol{R}} H^1(D, \boldsymbol{R}) < \infty$, $E \perp \operatorname{Re} L^2 H(D) \otimes \boldsymbol{C}$.*

(3) *For every $f \in E$ there exists $u \in C^\infty(\bar{D})$ s.t.*

$$f = Q(u).$$

Theorem 0 seems to be known, but somehow we cannot find a reference to it. We shall include a short proof for the sake of completeness.

Part (1) of Theorem 0 can be obtained from Stout's estimate

$$\|f\|_{L^2(\partial D)} \leqslant c \, \|\operatorname{Re} f\|_{L^2(\partial D)} \quad \text{for } f \in H^2(\partial D)$$

(see [19]) but we shall give a straightforward proof of it; (2) and (3) are consequences of the de Rham Theorem and the topological fact that $\dim H^1(D, \boldsymbol{R}) < \infty$ if D is a bounded domain with boundary of class C^2. Part (1) of Theorem 1 has the following interesting consequence:

COROLLARY. *For every $f \in L^2 H(D)$ there exists $u \in \operatorname{Re} L^2 H(D)$ such that*

$$f = B(u) + i \int_D \operatorname{Im} f.$$

We shall study the connections between the regularity of B, Q and S in Sobolev and Hölder norms. We shall denote by Λ_α the space of functions whose kth derivatives satisfy the $\alpha - k$ Hölder condition, $k = [\alpha]$ (the integer part of α), $0 < \alpha - [\alpha] < 1$. W^s will denote the sth Sobolev space. We shall also denote by

$\operatorname{Harm}^s(D)$	the space of harmonic functions from $W^s(D)$,
$\Lambda_\alpha \operatorname{Harm}(D)$	the space of harmonic functions from $\Lambda_\alpha(D)$,
$H^s(D)$	the space of holomorphic functions from $W^s(D)$,
$\Lambda_\alpha H(D)$	the space of holomorphic functions from $\Lambda_\alpha(D)$,
$PH^s(D)$	the space of pluriharmonic functions from $W^s(D)$,
$\Lambda_\alpha PH(D)$	the space of pluriharmonic functions from $\Lambda_\alpha(D)$,
$\operatorname{Harm}^p(\partial D)$	the pth Hardy space of harmonic functions,
$H^p(\partial D)$	the pth Hardy space of holomorphic functions.

We have the following

THEOREM 1. *Let D be a bounded domain with C^∞-boundary. Then*

(1) *For every $s > 0$ the operators S_r and S map continuously W^s into W^s if and only if the Bergman projector B maps continuously W^s into W^s.*

If the projection Q maps continuously W^s into W^s then S_r, S and B are continuous from W^s into W^s.

(2) *For every $\alpha > 0$, S_r and S map continuously Λ_α into Λ_α iff B maps continuously Λ_α into Λ_α.*

If Q is continuous from Λ_α into Λ_α then so are S_r, S and B.

THEOREM 2. *Let D be a bounded pseudoconvex domain with C^∞-boundary. Then if one of the projectors B, Q, S, S_r is continuous from W^s into W^s then so are the others.*

If one of the projectors B, Q, S, S_r is continuous from Λ_α into Λ_α then the others are too.

Note that for pseudoconvex domains with C^2-boundary the space E from Theorem 0 is equal to $H^1(D, \boldsymbol{C}) = H^1(D, \boldsymbol{R}) \otimes \boldsymbol{C}$. It is an open problem whether B, Q, S are continuous from W^s into W^s for a smooth pseudoconvex domain D. The most important classes of domains for which our operators are continuous from W^s into W^s for every $s > 0$ are the smooth strictly pseudoconvex domains and pseudoconvex domains with real-analytic boundary. For a description of a larger class of "weakly regular" domains which include both previous classes see Catlin [9].

Recently H. Boas [8] proved that the operator B is regular in Sobolev norms if D is a bounded circular pseudoconvex domain.

It should be mentioned here that for a complete circular bounded domain the results of this paper are trivial because in this case we have $Q = S$, $S = S_r \otimes \boldsymbol{C}$ and $S_r(u) = 2\operatorname{Re} B(u) - (\operatorname{vol} D)^{-1} \int u$ and thus the connections between our projectors are quite explicit.

Barrett [3] gave an example of a bounded domain with C^∞-boundary in $\boldsymbol{C}^2$ for which B is not continuous in any Sobolev or Hölder or L^p, $p > 2$, norm. Theorem 1 yields that Q and S are also not regular in these norms.

The Hölder estimates for B are known only for strictly pseudoconvex domains with the boundary of class C^{k+4}, $\alpha < k$ (see [14] and [2] for smooth domains).

The applications of Theorems 1 and 2 are in duality theory originated and developed by S. Bell ([4], [5]).

In [4] S. Bell constructed the following family of operators:

Let D be a smooth bounded domain in $\boldsymbol{R}^m$ and $u \in C^\infty(\bar{D})$. Let ϱ be a defining function for D, i.e. $\varrho \in C^\infty(\boldsymbol{R}^m)$, $D = \{x \in \boldsymbol{R}^m : \varrho(x) < 0\}$, grad $\varrho \neq 0$ on

∂D. Then

$$L^1 u = u - \Delta(\theta_0 \varrho^2), \qquad \theta_0 = \frac{1}{2}\frac{\varphi u}{|\nabla \varrho|^2},$$

$$\theta_t = \frac{\varphi}{(t+2)!}|\nabla\varrho|^{-2}\left(\frac{\partial}{\partial\eta}\right)^t L^t u, \qquad \frac{\partial}{\partial\eta} = \frac{\sum_{i=1}^{m} \frac{\partial\varrho}{\partial x_i}\cdot\frac{\partial}{\partial x_i}}{|\nabla\varrho|^2},$$

$$L^s u = u - \Delta\Big(\sum_{k=0}^{s-1} \theta_k \varrho^{k+2}\Big).$$

φ denotes here an arbitrarily chosen C^∞-function equal to 1 in a neighborhood of ∂D and equal to zero in a neighborhood of the set $\{\nabla\varrho = 0\}$. The fundamental property of the operators L^s is that for every u, $L^s u$ vanishes on ∂D up to order $s-1$ and $u - L^s u$ is orthogonal to the space of square-integrable harmonic functions on D. Denote this space by $L^2\,\mathrm{Harm}(D)$ and by P the orthogonal projection of $L^2(D)$ onto $L^2\,\mathrm{Harm}(D)$. If $v, u \in L^2\,\mathrm{Harm} \cap C^\infty(\bar{D})$ then

$$\langle v, u\rangle = \langle v, L^s u\rangle = \langle L^s v, u\rangle \quad \text{for every } s > 0.$$

S. Bell defined then the sesquilinear pairings

$$\langle v, u\rangle_s = \langle v, L^s u\rangle, \qquad s = 1, 2, \ldots$$

which extend the scalar product from $L^2\,\mathrm{Harm}(D)$ and can be used in the study of duality relations between various spaces of harmonic functions (see [4], [7]). The integer s is chosen according to the needs of the study.

In [15] it was proved that L^k maps continuously $\mathrm{Harm}^s(D)$ into $W^s(D)$ for all integers s and k, $k > 0$. This implies in particular that $\langle v, \rangle_s = \langle v, \rangle$ for every v from $L^2\,\mathrm{Harm}(D)$.

In [16] it was proved that L^s maps continuously $\Lambda_\alpha\,\mathrm{Harm}(D)$ into $\Lambda_\alpha(D)$ and that if $s = [\alpha]+1$, $0 < \alpha - [\alpha] < 1$, then $L^s u = |\varrho|^\alpha \varphi$ for $u \in \Lambda_\alpha\,\mathrm{Harm}(D)$ where φ is bounded on D. Thus the fact that P is bounded from $\Lambda_\alpha \to \Lambda_\alpha$ and $W^s \to W^s$ for α, $s > 0$ implies that

(1) $\mathrm{Harm}^{-s}(D)$ and $\mathrm{Harm}^s(D)$ are mutually dual via the pairing $\langle\,,\,\rangle_s$. The space $\mathrm{Harm}^{-s}(D)$ is equal to the space $L^2\,\mathrm{Harm}(D, |\varrho|^{2s})$.

(2) The space $\Lambda_\alpha\,\mathrm{Harm}(D)$ represents the dual space of the space $\mathring{L}^1\,\mathrm{Harm}(D), |\varrho|^\alpha)$ via the pairing $\langle\,,\,\rangle_s$, $s = [\alpha]+1$; $\mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^\alpha)$ is the closure of $L^2\,\mathrm{Harm}(D)$ in $L^1(D, |\varrho|^\alpha)$. We do not know whether the space $\mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^\alpha)$ is equal to the space $L^1\,\mathrm{Harm}(D, |\varrho|^\alpha)$.

We can now return to our operators B, S and Q. First we define the spaces

$\mathring{L}^2 H(D, |\varrho|^{2s})$ = the closure of $L^2 H(D)$ in $L^2(D, |\varrho|^{2s})$,

$\mathring{L}^2 R(D, |\varrho|^{2s})$ = the closure of $\mathrm{Re}\, L^2 H(D) \otimes C$ in $L^2(D, |\varrho|^{2s})$,
$\mathring{L}^2 PH(D, |\varrho|^{2s})$ = the closure of $L^2 PH(D)$ in $L^2(D, |\varrho|^{2s})$,
$\mathring{L}^1 H(D, |\varrho|^\alpha)$ = the closure of $L^2 H(D)$ in $L^1(D, |\varrho|^\alpha)$,
$\mathring{L}^1 R(D, |\varrho|^\alpha)$ = the closure of $\mathrm{Re}\, L^2 H(D) \otimes C$ in $L^1(D, |\varrho|^\alpha)$,
$\mathring{L}^1 PH(D, |\varrho|^\alpha)$ = the closure of $L^2 PH(D)$ in $L^1(D, |\varrho|^\alpha)$.

There is an open problem: is the space $\mathring{L}^2 H(D, |\varrho|^{2s})$ equal to the whole space $L^2 H(D, |\varrho|^{2s})$? S. Bell proved that $\mathring{L}^2 H(D, |\varrho|^{2s}) = L^2 H(D, |\varrho|^{2s})$ if D is pseudoconvex. The similar problems for our other spaces are also unsolved yet.

It will turn out in the sequel that

$$\mathring{L}^1 R(D, |\varrho|^\alpha) = \mathrm{Re}\, \mathring{L}^1 H(D, |\varrho|^\alpha) \otimes C,$$

$$\mathring{L}^2 R(D, |\varrho|^{2s}) = \mathrm{Re}\, \mathring{L}^2 H(D, |\varrho|^{2s}) \otimes C.$$

Now we can state the following:

THEOREM 3. *Let D be a bounded domain with C^∞-boundary and ϱ a defining function for D. Let us consider the following conditions:*

I(a) *The projectors B and S are continuous from W^s to W^s.*

I(b) *The projection B extends to a continuous mapping from $L^2\,\mathrm{Harm}(D, |\varrho|^{2s})$ to $\mathring{L}^2 H(D, |\varrho|^{2s})$.*

I(c) *The projection S extends to a continuous mapping from $L^2\,\mathrm{Harm}(D, |\varrho|^{2s})$ to $\mathring{L}^2 R(D, |\varrho|^{2s})$.*

I(d) *The spaces $\mathring{L}^2 H(D, |\varrho|^{2s})$ and $H^s(D)$ are mutually dual via the pairing $\langle\,,\,\rangle_s$.*

I(e) *The spaces $\mathring{L}^2 R(D, |\varrho|^{2s})$ and $\mathrm{Re}\, H^s(D) \otimes C$ are mutually dual via the pairing $\langle\,,\,\rangle_s$.*

I(f) *The $L^2(D, |\varrho|^{2s})$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v \in H^s\\ \|v\|_s \leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u \in L^2 H(D).$$

I(g) *The $W^s(D)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v \in L^2 H(D)\\ \|v\|_{L^2(D, |\varrho|^{2s})} \leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u \in L^2 H(D).$$

I(h) *The $L^2(D, |\varrho|^{2s})$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v \in \mathrm{Re}\, H^s \otimes C\\ \|v\|_s \leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u \in \mathrm{Re}\, L^2 H(D) \otimes C.$$

I(i) *The $W^s(D)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v \in \mathrm{Re}\, L^2 H(D) \otimes C\\ \|v\|_{L^2(D, |\varrho|^{2s})} \leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u \in \mathrm{Re}\, L^2 H(D) \otimes C.$$

I(j) *The projection Q is continuous from W^s into W^s.*

I(k) *The projection Q extends to a continuous mapping from $L^2\,\mathrm{Harm}(D, |\varrho|^{2s})$ to $\mathring{L}^2\,PH(D, |\varrho|^{2s})$.*

I(l) *The spaces $\mathring{L}^2\,PH(D, |\varrho|^{2s})$ and $PH^s(D)$ are mutually dual via the pairing $\langle\, , \rangle_s$.*

I(m) *The $L^2(D, |\varrho|^{2s})$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in PH^s(D)\\ \|v\|_s\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in L^2\,PH(D).$$

I(n) *The $W^s(D)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in L^2 PH(D)\\ \|v\|_{L^2(D,|\varrho|^{2s})}\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in L^2\,PH(D).$$

II(a) *The projectors B and S are continuous from Λ_α to Λ_α.*

II(b) *The projection B extends to a continuous mapping from $\mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^\alpha)$ to $\mathring{L}^1\,H(D, |\varrho|^\alpha)$.*

II(c) *The projection S extends to a continuous mapping from $\mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^\alpha)$ to $\mathring{L}^1\,R(D, |\varrho|^\alpha)$.*

II(d) *The space $\Lambda_\alpha H(D)$ represents the dual of $\mathring{L}^1\,H(D, |\varrho|^\alpha)$ via the pairing $\langle\, , \rangle_s$, $s = [\alpha]+1$.*

II(e) *The space $\mathrm{Re}\,\Lambda_\alpha H(D)\otimes \boldsymbol{C}$ represents the dual of $\mathring{L}^1\,R(D, |\varrho|^\alpha)$ via the pairing $\langle\, , \rangle_s$, $s = [\alpha]+1$.*

II(f) *The $L^1(D, |\varrho|^\alpha)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in \Lambda_\alpha H\\ \|v\|_\alpha\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in L^2\,H(D).$$

II(g) *The $\Lambda_\alpha(D)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in L^2 H(D)\\ \|v\|_{L^1(D,|\varrho|^\alpha)}\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in L^2\,H(D).$$

II(h) *The $L^1(D, |\varrho|^\alpha)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in \mathrm{Re}\,\Lambda_\alpha H\otimes \boldsymbol{C}\\ \|v\|_\alpha\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in \mathrm{Re}\,L^2\,H(D)\otimes \boldsymbol{C}.$$

II(i) *The $\Lambda_\alpha(D)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in \mathrm{Re}\,L^2 H(D)\otimes \boldsymbol{C}\\ \|v\|_{L^1(D,|\varrho|^\alpha)}\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in \mathrm{Re}\,L^2\,H(D)\otimes \boldsymbol{C}.$$

II(j) *The projection Q is continuous from Λ_α to Λ_α.*

II(k) *The projection Q extends to a continuous mapping from $\mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^\alpha)$ to $\mathring{L}^1\,PH(D, |\varrho|^\alpha)$.*

II(l) *The space $\Lambda_\alpha PH$ represents the dual of $\mathring{L}^1\,PH(D, |\varrho|^\alpha)$ via the pairing $\langle\, , \rangle_s$, $s = [\alpha]+1$.*

II(m) *The $L^1(D, |\varrho|^\alpha)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in \Lambda_\alpha PH(D)\\ \|v\|_\alpha\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in L^2\,PH(D).$$

II(n) *The $\Lambda_\alpha(D)$ norm is equivalent to the norm*

$$\|u\| = \sup_{\substack{v\in L^2 PH(D)\\ \|v\|_{L^1(D,|\varrho|^\alpha)}\leqslant 1}} |\langle u, v\rangle| \quad \textit{for } u\in L^2\,PH(D).$$

Then

(1) *Conditions* I(a)–I(i) *are equivalent.*

(2) *Conditions* I(j)–I(n) *are equivalent.*

(3) *Conditions* I(j)–I(n) *imply conditions* I(a)–I(i).

(4) *If the domain D is pseudoconvex then all conditions* I(a)–I(n) *are equivalent.*

(5) *Conditions* II(a)–II(i) *are equivalent.*

(6) *Conditions* II(j)–II(n) *are equivalent.*

(7) *Conditions* II(j)–II(n) *imply conditions* II(a)–II(i).

(8) *If the domain D is pseudoconvex then all conditions* II(a)–II(n) *are equivalent.*

This ends at last the statement of Theorem 3. It should be mentioned that clearly all conditions in Theorem 3 which concern the projection S have their equivalent real counterparts for the operator S_r, e.g. I(c) is equivalent to the fact that S_r is continuous from $L^2_r\,\mathrm{Harm}(D, |\varrho|^{2s})$ to $\mathrm{Re}\,\mathring{L}^2\,H(D, |\varrho|^{2s})$, II(c) is equivalent to the fact that S_r is continuous from $\mathring{L}^2_r\,\mathrm{Harm}(D, |\varrho|^\alpha)$ to $\mathrm{Re}\,\mathring{L}^1\,H(D, |\varrho|^\alpha)$ and the duality relations in I(e) and II(e) are equivalent to the duality relations between the corresponding real spaces (r always stands for real).

Some parts of our Theorem 3 are already known. The equivalence of conditions I(a), I(d), I(f) and I(g) for B on smooth strictly pseudoconvex domains was proved by S. Bell in [5]. The equivalence of the regularity of B and the duality theorems was studied also in [7]. In [15] it was proved that I(a), I(b), I(d), I(f) and I(g) for B are equivalent.

The fact that the continuity of B in Hölder norms yields the duality relation II(d) seems to be known at least for $0<\alpha<1$ and D equal to the unit ball in $\boldsymbol{C}^n$. There is also the paper of S. Bell [6] on a Sobolev inequality for pluriharmonic functions. The interesting consequence of Theorem 3 is that our operators B, S, S_r and Q behave better on harmonic functions than on arbitrary functions. This means that the restriction of B, S, S_r and Q to the space of harmonic functions is bounded in $L^1(D, |\varrho|^\alpha)$ norm or $L^2(D, |\varrho|^{2s})$

norm although they cannot be extended to continuous mappings on the whole space $L^1(D, |\varrho|^\alpha)$ and $L^2(D, |\varrho|^{2s})$.

We want now to signalize other examples of such behavior of B on harmonic functions. If D is a bounded strictly pseudoconvex domain with C^5-boundary then

(1) B is bounded from $L^1\,\mathrm{Harm}(D)$ into $L^1 H(D)$.

(2) Let $\mathrm{B\,Harm}$ denote the class of harmonic Bloch functions, i.e. functions u for which

$$\|u\|_B = \sup_{z\in D}|\varrho u(z)| + \sup_{\substack{z\in D\\ |\alpha|=1}} |\varrho D^\alpha u(z)| < \infty.$$

Then B maps $L^\infty(D)$ onto $\mathrm{B}H(D)$, $\mathrm{B\,Harm}(D)$ onto $\mathrm{B}H(D)$ and thus the space $\mathrm{B}H(D)$ of holomorphic Bloch functions is the dual of $L^1 H(D)$ via the pairing $\langle\, , \rangle_1$.

(3) The operator B maps the Hardy spaces $\mathrm{Harm}^p(\partial D)$ into $H^p(\partial D)$, $2 \leqslant p < \infty$.

(4) The operator B maps continuously the space Harm^∞ of bounded harmonic functions onto the dual space of $H^1(\partial D)$.

Statements (1)–(4) hold also for S and Q. We shall not prove (1)–(4) in the present paper. We shall give a detailed account of this subject in the subsequent paper "The Bergman projection on harmonic functions" (in preparation).

2. Proofs.

1. Proof of Theorem 0. A pluriharmonic real function u belongs to $\mathrm{Re}L^2 H(D)$ iff the differential form ∂u is d-exact. Thus, the first part of Theorem 1 will be proved if we show that the space of d-exact differential 1-forms with harmonic coefficients is closed in $W^{-1}_{(1)}(D)$ ($W^{-1}_{(1)}(D)$ is the space of differential 1-forms with coefficients from $W^{-1}(D)$). Since du maps continuously $L^2\,\mathrm{Harm}(D)$ to $W^{-1}_{(1)}(D)$, it can be proved that there exists an operator T from the space of harmonic d-exact forms from $W^{-1}_{(1)}(D)$ into $L^2\,\mathrm{Harm}(D)$ such that $\|Tw\| \leqslant C\|w\|_{-1}$ and $dTw = w$.

This permits to prove the above fact for forms with real coefficients and the real space $L^2_r\,\mathrm{Harm}(D)$ and then take the complexification.

Let $w \in W^{-1}_{(1)}(D)$, $w = du$ and the coefficients of w are harmonic. Thus u is also harmonic, $w_i = \partial u/\partial x_i$. As was proved in [15] (following [7]), $\varrho \dfrac{\partial u}{\partial x_i} \in L^2(D)$ and

$$\left\|\varrho \frac{\partial u}{\partial x_i}\right\|_{L^2} \leqslant \left\|\frac{\partial u}{\partial x_i}\right\|_{-1}$$

(the C^2-regularity suffices here because we deal with $s = -1$ only). The function ϱ is as usual a C^2 defining function of D.

Let $D_1 \Subset D$ be a domain with C^∞-boundary s.t.

$$\sum_i \left(\frac{\partial \varrho}{\partial x_i}\right)^2 + \varrho\Delta\varrho > C > 0 \quad \text{on } D\setminus D_1.$$

Let z_0 be a point from D_1. We can assume now that $u(z_0) = 0$. Then we have

$$\sum_i \left\|\varrho\frac{\partial u}{\partial x_i}\right\|_2^2 = \int_D \sum_i \varrho^2\left(\frac{\partial u}{\partial x_i}\right)^2 = 2\int_D \left(\sum_i\left(\frac{\partial \varrho}{\partial x_i}\right)^2 + \varrho\Delta\varrho\right)u^2.$$

Thus

$$c\int_{D\setminus D_1}|u|^2 \leqslant \|du\|_{-1} + c_1\int_{D_1}|u|^2.$$

Since u is harmonic and $\|u\|_{L^2(D_1)} \leqslant c_0\|du\|_{W^s(D_1)}$ for large s with $\|du\|_{W^s(D_1)} \leqslant c_1\|du\|_{-1}$, we have

$$c_2\|u\|_{L^2(D)} \leqslant (1 + c_0 c_1)\|du\|_{-1}.$$

Thus the correspondence $du \leftrightarrow u$, $u(z_0) = 0$ defines the needed operator T.

The subspace E of functions from $L^2 PH(D)$ orthogonal to $\mathrm{Re}\,L^2 H(D)$ is the complexification of the space E_r of real plurisubharmonic functions orthogonal to $\mathrm{Re}\,L^2 H(D)$ with respect to the real scalar product.

Functions $F_1, \ldots, F_k$ from E_r are linearly independent if and only if the cohomology classes of the differential forms $w_i = \mathrm{Im}\,\partial f_i$ are linearly independent in $H^1(D, \boldsymbol{R})$. Since D has C^2-boundary, we have $\dim H^1(D, \boldsymbol{R}) < \infty$, $\dim_{\boldsymbol{R}} E_r \leqslant \dim H^1(D, \boldsymbol{R})$ and $\dim_{\boldsymbol{C}} E = \dim_{\boldsymbol{R}} E_r$.

Now let $u_1, \ldots, u_k$ form a basis of E. There exist $\gamma_1, \ldots, \gamma_k$, closed smooth curves in $\boldsymbol{C}^n$, $\gamma_i = \gamma_i(t)$, $\gamma_i(0) = \gamma_i(1)$, such that $\int_{\gamma_i} \partial u_j = 1$ for $i = j$ and 0 for $i \neq j$. This is an immediate consequence of the de Rham theorem. We have

$$\begin{aligned}\int_{\gamma_i}\partial u &= \sum_{k=1}^n \int_0^1 \frac{\partial u}{\partial z_k}(\gamma_i(t))\,d\gamma_i(t)\\ &= \sum_{k=1}^n \int_0^1\int_D \frac{\partial u}{\partial z_k}(z)\,\theta(z-\gamma_i(t))\,dz\wedge d\bar z\,d\gamma_i(t)\\ &= \int_D u(z)\int_0^1\sum_{k=1}^n \frac{\partial}{\partial z_k}\theta(z-\gamma_i(t))\,d\gamma_i(t)\,dz\wedge d\bar z\end{aligned}$$

for every $u \in PH(D)$. θ denotes here a nonnegative radially symmetric function from $C_0^\infty(C^n)$ such that $\int_{C^n} \theta = 1$ and $\{\gamma_i + \operatorname{supp}\theta\} \subset D$. Thus $u_i = Q(\varphi_i)$ where

$$\varphi_i(z) = \int_0^1 \sum_{k=1}^n \frac{\partial}{\partial z_k} \theta(z - \gamma_i(t))\, d\gamma_i(t) \in C^\infty(\bar{D}).$$

This ends the proof of Theorem 0.

2. Proof of the Corollary. Let $G(z, t)$ denote the reproducing kernel of the space $\operatorname{Re} L^2 H(D)$. The function $G(z, t)$ is also the reproducing kernel of the space $\operatorname{Re} L^2 H(D) \otimes C$. Thus $B_z(G(z, t)) = K(z, t)$, where $K(z, t)$ is the Bergman function of the domain D. The set of linear combinations $\sum_i c_i G(z, t_i)$, c_i real numbers, is dense in $\operatorname{Re} L^2 H(D)$. The closure of the set of linear combinations $\sum_i c_i K(z, t_i)$ is the real subspace H_0 of $L^2 H(D)$ consisting of all functions f from $L^2 H(D)$ for which $\int_D \operatorname{Im} f = 0$. We shall prove that B is an isomorphism from $\operatorname{Re} L^2 H(D)$ onto H_0. To prove this we shall show that

$$\|B(u)\|^2 \geqslant C\|u\|^2 \quad \text{for } u = \sum_i c_i G(z, t_i).$$

We have

$$\begin{aligned}\|B(u)\|^2 &= \Big\|\sum_i c_i K(z, t_i)\Big\|^2 = \sup_{\substack{f \in H_0\\ \|f\| \leqslant 1}} \Big|\sum_i c_i f(t_i)\Big|^2 \\ &= \sup_{\substack{f \in H_0\\ \|f\| \leqslant 1}} \Big(\big(\sum_i c_i \operatorname{Re} f(t_i)\big)^2 + \big(\sum_i c_i \operatorname{Im} f(t_i)\big)^2\Big) \\ &\geqslant \sup_{\substack{f \in H_0\\ \|f\| \leqslant 1}} \big(\sum_i c_i \operatorname{Re} f(t_i)\big)^2 \geqslant c \sup_{\substack{u \in \operatorname{Re} L^2 H(D)\\ \|u\| \leqslant 1}} \big(\sum_i c_i u(t_i)\big) \\ &= c\Big\|\sum_i c_i G(z, t_i)\Big\|^2 = \|u\|^2.\end{aligned}$$

The last inequality follows from part (1) of Theorem 0.

3. Proof of Theorem 1. We begin with the following

Lemma. *Let $f \in L^2 H(D)$. Then for every $s \geqslant 1$*

$$B(\bar{f}) = B(\bar{f} L^s(1)).$$

Proof. If f is bounded then clearly for every $g \in L^2 H(D)$,

$$\langle \bar{f} - \bar{f} L^s(1), g\rangle = \langle 1 - L^s(1), fg\rangle = 0.$$

Consider the domains $D_\varepsilon = \{z \in D: \varrho(z) < -\varepsilon\}$, ϱ a defining function for D. Let L_ε^s denote the sth Bell operator constructed for the domain D_ε and $\varrho_\varepsilon = \varrho + \varepsilon$. It follows from the construction of L_ε^s that $L_\varepsilon^s \to L^s$ if $\varepsilon \to 0$. Then for every $g \in L^2 H(D)$

$$\langle \bar{f} - \bar{f} L^s(1), g\rangle = \lim_{\varepsilon \to 0} \langle \bar{f} - \bar{f} L_\varepsilon^s(1), g\rangle_{D_\varepsilon} = 0.$$

Thus $B(\bar{f}) = B(\bar{f} L^s(1))$.

From the results of [13] it follows that the mapping $f \to \bar{f} L^s(1)$ imbeds continuously $L^2 H(D)$ into W^s. Suppose B is continuous from W^s into W^s. Let u be a real function from W^s. Then

$$\begin{aligned}2B(u) &= B(S(u)) = B(S_r(u)) = B(S_r^0(u)) = B\big(S_r^0(u) + \overline{S_r^0(u)}\big) \\ &= S_r^0(u) + B\big(\overline{S_r^0(u)}\big) = S_r^0(u) + B\big(\overline{S_r^0(u)}\, L^s(1)\big).\end{aligned}$$

$S_r^0(u)$ denotes here the holomorphic function such that $\operatorname{Re} S_r^0(u) = S_r(u)$ and $\int_D \operatorname{Im} S_r^0(u) = 0$.

Thus $S_r u = 2\operatorname{Re} B(u) - \operatorname{Re} B\big(\overline{S_r^0(u)}\, L^s(1)\big)$ maps continuously $W_r^s(D)$ into $W^s(D)$. Since $S = S_r \otimes C$, it follows that S is also continuous in the sth Sobolev norm.

Now let S be continuous in the sth Sobolev norm. Let $\varphi = u + iv \in W^s$. Then we have

$$B\left(\varphi - \frac{\overline{S_r^0(v) + iS_r^0(u)}}{2} L^s(1)\right) = B\left(\varphi - \frac{\overline{S_r^0(v) + iS_r^0(u)}}{2}\right) = S_r^0(u) + iS_r^0(v).$$

It is well known that $\|\operatorname{Re} f\|_s$ and $\|f\|_s$ are equivalent on the space of holomorphic functions s.t. $\int \operatorname{Im} f = 0$, $s \geqslant 1$. The above fact and the continuity of S yields that the mapping $\varphi \to S_r^0(u) + iS_r^0(v)$, $\varphi = u + iv$, is continuous with respect to $\|\ \|_s$, and thus the mapping

$$F(\varphi) = \varphi - \frac{\overline{S_r^0(v) + iS_r^0(u)}}{2} L^s(1)$$

is a Fredholm map on W^s (this follows from Theorem 1 [15]) and therefore has continuous inverse. We have already proved that BF maps continuously W^s into W^s. Thus B must also be continuous.

If the operator Q maps continuously W^s into W^s then all elements of E must belong to W^s as the images of smooth functions. Thus the projection S is also regular in the sth Sobolev space.

The proof of the equivalence (2) is the same as (1). It suffices only to find, for every α, an s so large that the mapping $\bar{f} L^s(1)$ maps $L^2 H(D)$ compactly into Λ_α (Sobolev imbedding theorem) and the rest of the proof will be the same.

4. Proof of Theorem 2. We must prove that each element of the space E from Theorem 0 belongs to $W^s(D)$ if D is pseudoconvex and if the projector S maps continuously $W^s D)$ into $W^s(D)$.

Let $u \in E$. Consider the cohomology class $[\bar\partial u] \in H^1(D, C)$. It is a well-known fact from differential topology that all cohomology classes of D have representatives from $C^\infty_{(1)}(\bar D)$ (the space of differential 1-forms with coefficients from $C^\infty(\bar D)$).

This means that there exists $\varphi \in C^\infty(\bar D)$ such that $\partial u + d\varphi = w \in C^\infty_{(1)}(\bar D)$. We can write $w = w_1 + w_2$ where w_1 is a $\langle 1, 0\rangle$-form and w_2 is a $\langle 0, 1\rangle$-form. Then $\partial u + \partial\varphi = w_1$ and $\bar\partial\varphi = w_2$. By the J.J. Kohn estimates there exists $f \in C^\infty(\bar D)$ such that $\bar\partial f = w_2$. Then $\varphi = f + h_1$ with h_1 holomorphic on D and $h_1 \in L^2(D)$ because $d\varphi \in W^{-1}(D)$ and thus $\varphi \in L^2(D)$. Similarly we can find $g \in C^\infty(\bar D)$ such that $\partial g = w_1$. Thus

$$\varphi = f + g + h_1 + \bar h_2,$$

where

$$f, g \in C^\infty(\bar D) \quad \text{and} \quad h_1, h_2 \in L^2 H(D).$$

Since $S(\varphi) = 0$ we have $h_1 + \bar h_2 = -S(g+f) \in W^s(D)$. Thus φ is also in $W^s(D)$.

In exactly the same way we can prove that all elements of E belong to $\Lambda_\alpha(D)$ if S is a continuous map from $\Lambda_\alpha(D)$ to $\Lambda_\alpha(D)$.

5. Proof of Theorem 3. Theorem 3 will be proved if we show that

(i) Conditions I(b), I(d), I(f), I(g) are equivalent to the continuity of $B\colon W^s(D) \to W^s(D)$.

(ii) Conditions I(c), I(e), I(h), I(i) are equivalent to the continuity of $S\colon W^s(D) \to W^s(D)$.

(iii) Conditions I(k), I(l), I(m), I(n) are equivalent to the continuity of $Q\colon W^s(D) \to W^s(D)$ (to condition I(j)).

The relations between the regularity of B, Q and S are already established by Theorems 1 and 2. Thus we shall get a proof of statements (1)–(4) of Theorem 3.

(i) was proved in [15]. The proofs of (ii) and (iii) are exactly the same, and we shall not repeat them here.

In the case of Hölder norm we must prove that

(j) Conditions II(b), II(d), II(f), II(g) are equivalent to the continuity of $B\colon \Lambda_\alpha(D) \to \Lambda_\alpha(D)$.

(jj) Conditions II(c), II(e), II(h), II(i) are equivalent to the continuity of $S\colon \Lambda_\alpha(D) \to \Lambda_\alpha(D)$.

(jjj) Conditions II(k), II(l), II(m), II(n) are equivalent to the continuity of $Q\colon \Lambda_\alpha(D) \to \Lambda_\alpha(D)$ (to condition II(j)).

Thus we shall again use Theorems 1 and 2 to get a proof of statements (5)–(8) of Theorem 3.

The proofs of (j), (jj), (jjj) are also the same and very similar to the proofs of (i), (ii), (iii), with one significant difference. The spaces $\Lambda_\alpha H$ are not reflexive and thus we cannot have the mutual duality via $\langle\, ,\, \rangle_s$. Instead we must use the fact that a Banach space F imbeds isometrically into its second adjoint space F^{**}. We shall now prove (j). The proof of (jj) and (jjj) will be the same.

In the proof of (j) we shall constantly use the following facts from [16]:

(a) The space $\Lambda_\alpha \operatorname{Harm}(D)$ is the dual of $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$ via the pairing $\langle\, ,\, \rangle_s$, $s = [\alpha]+1$.

(b) The projector $P\colon L^2(D) \to L^2 \operatorname{Harm}(D)$ is regular on $\Lambda_\alpha(D)$ for every $\alpha > 0$.

(c) If u is harmonic then

$$\|u\|_{\mathring{L}^1(D,|\varrho|^\alpha)} \approx \sup_{\substack{v \in \Lambda_\alpha \operatorname{Harm}(D)\\ \|v\|_\alpha \leqslant 1}} |\langle u, v\rangle|, \qquad \|u\|_\alpha \approx \sup_{\substack{v \in L^2 \operatorname{Harm}(D)\\ \|v\|_{L^1(D,|\varrho|^\alpha)} \leqslant 1}} |\langle u, v\rangle|.$$

1. *The continuity of B implies* II(b). We have for $u \in L^2 \operatorname{Harm}(D)$

$$\|Bu\|_{L^1(D,|\varrho|^\alpha)} \approx \sup_{\substack{v \in \Lambda_\alpha \operatorname{Harm}(D)\\ \|v\|_\alpha \leqslant 1}} |\langle Bu, v\rangle| = \sup_{\substack{v \in \Lambda_\alpha \operatorname{Harm}(D)\\ \|v\|_\alpha \leqslant 1}} |\langle u, Bv\rangle| \leqslant c\|u\|_{L^1(D,|\varrho|^\alpha)} \|Bv\|_\alpha$$
$$\leqslant c_1 \|u\|_{L^1(D,|\varrho|^\alpha)} \|v\|_\alpha \leqslant C_1 \|u\|_{L^1(D,|\varrho|^\alpha)}.$$

Thus B extends to a continuous mapping from $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$ into itself.

2. II(b) *implies the continuity of B in Λ_α.*

$$\|Bv\|_\alpha \approx \sup_{\substack{u \in L^2 \operatorname{Harm}(D)\\ \|u\|_{L^1(D,|\varrho|^\alpha)} \leqslant 1}} |\langle u, Bv\rangle| = \sup_{\substack{u \in L^2 \operatorname{Harm}(D)\\ \|u\|_{L^1(D,|\varrho|^\alpha)} \leqslant 1}} |\langle Bu, v\rangle|$$
$$\leqslant c\|Bu\|_{L^1(D,|\varrho|^\alpha)} \|v\|_\alpha \leqslant c_1 \|v\|_\alpha.$$

3. *The continuity of B implies* II(d). Let φ be a continuous functional on $\mathring{L}^1 H(D, |\varrho|^\alpha)$. By the Hahn–Banach theorem it can be extended to a continuous functional on $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$ and therefore can be represented by a harmonic function $h_\varphi \in \Lambda_\alpha \operatorname{Harm}(D)$. For every $u \in L^2 H(D)$ we have $\varphi(u) = \langle u, h_\varphi\rangle = \langle u, Ph_\varphi\rangle$ and

$$|\langle u, Ph_\varphi\rangle| \leqslant c\|u\|_{L^1(D,|\varrho|^\alpha)} \|Ph_\varphi\|_\alpha.$$

Since $\mathring{L}^1 H(D, |\varrho|^\alpha)$ is the closure of $L^2 H(D)$ in $L^1(D, |\varrho|^\alpha)$, Ph_φ represents φ via $\langle\, ,\, \rangle_s$ ($\langle\, ,\, \rangle = \langle\, ,\, \rangle_s$ on $L^2 H(D) \times L^2 H(D)$).

4. II(d) $\Rightarrow$ II(f) *and* II(g). II(g) follows from the very definition of the dual space and II(f) follows from the fact that a Banach space F imbeds isometrically into its second dual F^{**}.

5. II(g) *implies the continuity of* B. Let $v \in \Lambda_\alpha \operatorname{Harm}(D)$. We have

$$\|Bv\|_\alpha \approx \sup_{\substack{u \in L^2 H(D) \\ \|u\|_{L^1(D,|\varrho|^\alpha)} \leqslant 1}} |\langle u, Bv\rangle| = \sup_{\substack{u \in L^2 H(D) \\ \|u\|_{L^1(D,|\varrho|^\alpha)} \leqslant 1}} |\langle u, v\rangle|$$

$$\leqslant \sup_{\substack{u \in L^2 \operatorname{Harm}(D) \\ \|u\|_{L^2(D,|\varrho|^\alpha)} \leqslant 1}} |\langle u, v\rangle| \approx \|v\|_\alpha .$$

6. II(f) *implies* II(b). Let $u \in L^2 \operatorname{Harm}(D)$. We have

$$\|Bu\|_{L^1(D,|\varrho|^\alpha)} = \sup_{\substack{v \in \Lambda_\alpha H(D) \\ \|v\|_\alpha \leqslant 1}} |\langle Bu, v\rangle| = \sup_{\substack{v \in \Lambda_\alpha H(D) \\ \|v\|_\alpha \leqslant 1}} |\langle u, v\rangle|$$

$$\leqslant \sup_{\substack{v \in \Lambda_\alpha \operatorname{Harm}(D) \\ \|v\|_\alpha \leqslant 1}} |\langle u, v\rangle| \leqslant c\|u\|_{L^1(D,|\varrho|^\alpha)} .$$

3. Remarks.

1. The main obstacle to establish the continuity of the operator Q is the fact that we do not know whether the elements of the space $E = L^2 PH(D) \ominus \operatorname{Re} L^2 H(D) \otimes C$ are smooth. If B and S are continuous in the sth Sobolev (or αth Hölder) norm then Q is also regular iff the space $W^s \cap L^2 PH(D)$ ($\Lambda_\alpha \cap L^2 PH(D)$ respectively) is dense in $L^2 PH(D)$.

2. The construction of Bell's operator L^s can be improved by taking, instead of an arbitrary defining function ϱ, the function ϱ_0 defined as follows: ϱ_0 is the unique solution of the Dirichlet problem $\Delta^2 \varrho_0 = 0$, $\varrho_0 = 0$ on D and $(\partial/\partial\eta)\varrho_0 = 1$ on ∂D. If the boundary of ∂D is of class $C^{4+\alpha}$ then ϱ_0 is also of class $C^{4+\alpha}$ (see [1]). Such a choice of ϱ_0 permits us to minimize the loss of smoothness in the construction of L^s since if $\Delta^2 u = 0$ and $u \in \Lambda_\beta$ then $\varrho_0^k u \in \Lambda_{\beta+k}$.

3. In the present paper we restricted our attention to the domains whose boundary is of class C^∞. However, there exists an important class of bounded strictly pseudoconvex domains with C^{k+4}-smooth boundary for which the Hölder regularity of the Bergman projection B is already established for $0 < \alpha < k$ ([14]). It is a natural question whether the operators S and Q are also regular in Λ_α norms $(0 < \alpha < k)$. The answer is affirmative although the proof is different from those given above since we can use only $L^1(1)$ in the proof that $B(\overline{S_r^0} L^1(1))$ is compact. To overcome this difficulty we must use the following facts:

1) If D is a strictly pseudoconvex domain with C^5-boundary then the projector B maps continuously $L^p(D)$ into $L^p(D)$ for every $p > 1$.

This fact is a consequence of the representation of B given in [14]. The nedeed L^p-estimates can be done in the same manner as in [18]. In fact we shall only need that B maps $L^p(D)$ into $L^{p-\varepsilon}(D)$ for $\varepsilon > 0$. These estimates are far more elementary and follow directly from the results of [14] and the Riesz–Thorin theorem.

2) If u is a harmonic function from $L^p(D)$ then $\varrho u \in \mathring{W}_p^1(D)$.

To prove this fact it suffices to observe that $\Delta\varrho u \in W_p^{-1}(D)$ since it contains only the first derivatives of u.

It follows from [1] and especially from [17], Th. 5.4, that the operator G solving the Dirichlet problem $\Delta g = f$, $f = 0$ on ∂D is an isomorphism between $W_p^{-1}(D)$ and $\mathring{W}_p^1(D)$. For $u \in W_p^1(D)$ we have $G(\Delta\varrho u) = \varrho u$. The functions from $\operatorname{Harm}_p^1(D)$ are dense in $L^p \operatorname{Harm}(D)$ since the projector P maps $L^p(D)$ onto $L^p \operatorname{Harm}(D)$ and $W_p^1(D)$ onto $\operatorname{Harm}_p^1(D)$. This last fact follows again from [1] and [17]. Thus $\varrho u \in \mathring{W}_p^1(D)$ for every $u \in L^p \operatorname{Harm}(D)$.

3) The third fact is the Sobolev imbedding theorem: $\mathring{W}_m^1(D)$ is compactly imbedded in $L^q(D)$, $q < 2nm/(2n-m)$ for $m \leqslant 2n$ and in $\Lambda_\alpha(D)$, $\alpha < 1 - 2n/m$, $m > 2n$.

These three facts permit us to prove that

$$B(\overline{S_r^0 u} L^1(1)) \in W_2^1 \quad \text{since} \quad L^1(1) = 1 - \tfrac{1}{2}\Delta(\varrho_0^2) = -\tfrac{1}{2}\varrho_0 \Delta\varrho_0$$

and thus

$$\overline{S_r^0 u} L^1(1) \in \mathring{W}_2^1(D) \Subset L^q, \qquad q = \frac{4n}{2n-2} - \varepsilon,$$

so that S_r maps L^q into L^q (as in the proof of Theorem 1). This implies that

$$\overline{S_r^0 u} L^1(1) \in \mathring{W}_q^1 \Subset L^{q_1}, \qquad q_1 = 2nq/(2n-q) - \varepsilon$$

and thus S_r maps L^{q_1} into L^{q_1}. After a finite number of such steps we shall prove that S_r maps Λ_α into Λ_α for some $\alpha > 0$, and continuing this process we find that S maps Λ_α into Λ_α for every α.

The proof of Theorem 2 is the same as in the smooth case. It suffices to use the well-known Hölder estimates for the solutions of the $\bar\partial$-problem for strictly pseudoconvex domains (see [10] for the best estimates). Thus we get the following

THEOREM. *Let D be a strictly pseudoconvex domain with boundary of class C^{k+4}. Then the projection Q maps continuously $\Lambda_\alpha(D)$ onto $\Lambda_\alpha PH(D)$ and the projection S maps continuously $\Lambda_\alpha(D)$ onto* $\operatorname{Re}\Lambda_\alpha H(D) \otimes C$ *for $\alpha < k$. If the boundary of D is of class C^5 (or more) then S and Q map continuously L^p into L^p for $p > 1$* (for $p < 2$ it suffices to use the fact that S and Q are selfadjoint).

If we use the construction of L^s from the preceding remark and carefully count the derivatives then it turns out that if ∂D is of class C^{k+4} and $u \in \Lambda_\alpha \operatorname{Harm}(D)$, $\alpha < k$, then $L^k(u)$, $k = [\alpha]+1$, has the form $|\varrho|^\alpha m$ where m is bounded, $\|m\|_\infty \leqslant c\|u\|_\alpha$. Since the estimates for the Dirichlet problem used in [16] are valid in our case (see [1]), the duality theorem holds for $\Lambda_\alpha \operatorname{Harm}(D)$ and thus we get the following

PROPOSITION. *All conditions* II(a)–II(m) *from Theorem* 3 *are satisfied if* D *is a bounded strictly pseudoconvex domain with the boundary of class* C^{k+4} *and* $0 \leqslant \alpha \leqslant k$. (*We can take* $\alpha < k$ *since if* $\Delta^2 u = 0$ *and* $u \in \Lambda_\beta$ *then* $u\varrho_0^k \in \Lambda_{\beta+k}$, $k = 1, 2, \ldots$)

We would like to inform the reader that the representation of the projector B can also be found in the works of I. Lieb and M. Range ([11], [12], [13]) who made some improvements in the formulae from [14] to make them work in the case of domains in hermitian manifolds. They got the integral representation for the Kohn solution of the $\bar\partial$-problem on $\langle 0, q\rangle$-forms and the Hölder and L^p estimates for this solution.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. 12 (1959), 623–727.

[2] P. Ahern and R. Schneider, *Holomorphic Lipschitz functions in pseudoconvex domains*, Amer. J. Math. 101 (1979), 543–562.

[3] D.E. Barrett, *Irregularity of the Bergman projection on a smooth bounded domain in* C^2, Ann. of Math. 119 (1984), 431–436.

[4] S. Bell, *A duality theorem for harmonic functions*, Michigan Math. J. 29 (1982), 123–128.

[5] —, *A representation theorem in strictly pseudoconvex domains*, Illinois J. Math. 26 (1982), 19–26.

[6] —, *A Sobolev inequality for pluriharmonic functions*, Proc. Amer. Math. Soc. 85 (1982), 350–352.

[7] S. Bell and H. Boas, *Regularity of the Bergman projection and duality of holomorphic function spaces*, Math. Ann. 267 (1984), 473–478.

[8] H.P. Boas, *Holomorphic reproducing kernels in Reinhardt domains*, Pacific J. Math. 112 (1984), 273–292.

[9] D. Catlin, *Global regularity of the* $\bar\partial$*-Neumann problem*, in: Complex Analysis of Several Variables, Proc. Sympos. Pure Math. 41 (1984), 39–49.

[10] S. Krantz, *Optimal Lipschitz and* L^p *regularity for the equation* $\bar\partial u = f$ *on strongly pseudo-convex domains*, Math. Ann. 219 (1976), 233–260.

[11] I. Lieb and R.M. Range, *On integral representations and a priori Lipschitz estimates for the canonical solution of the* $\bar\partial$*-equation*, Math. Ann. 265 (1983), 221–251.

[12] —, —, *Integral representations on Hermitian manifolds: the* $\bar\partial$*-Neumann solution of the Cauchy–Riemann equations*, Bull. Amer. Math. Soc. 11 (1984), 355–358.

[13] —, —, *Integral representations and estimates in the theory of the* $\bar\partial$*-Neumann problem*, preprint.

[14] E. Ligocka, *The Hölder continuity of the Bergman projection and proper holomorphic mappings*, Studia Math. 80 (1984), 89–107.

[15] —, *The Sobolev spaces of harmonic functions*, this volume, 79–87.

[16] —, *The Hölder duality for harmonic functions*, this volume, 269–277.

[17] V.G. Maz'ya and T.O. Shaposhnikova, *On the regularity of the boundary in* L_p*-theory of elliptic boundary problems I, II*, Trudy Sem. S. L. Soboleva 1980, no. 2, 39–56, 1981, no. 1, 57–102 (in Russian); English version: preprint P-28/80, Akad. Wiss. DDR, Berlin 1980.

[18] D.H. Phong and E.M. Stein, *Estimates for the Bergman and Szegö projections on strongly pseudo-convex domains*, Duke Math. J. 44 (1977), 695–704.

[19] E.L. Stout, H^p*-functions on strictly pseudoconvex domains*, Amer. J. Math. 98 (1976), 821–852.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa, Poland

Received April 18, 1985 (2050)