STUDIA MATHEMATICA, T. LXXXIV. (1986)

Plurisubharmonic functions on quasi-Banach spaces

by

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Abstract. We study conditions under which a quasi-Banach space can be equipped with an equivalent plurisubharmonic quasi-norm. We show this is equivalent to validity of a weak form of the Maximum Modulus Principle for analytic functions valued in the space. We examine the relationship between the existence of a plurisubharmonic quasi-norm and the existence of "good" tensor products. We also prove that in a quasi-Banach algebra the spectral radius is plurisubharmonic, extending a theorem of Vesentini, and give some applications to the study of holomorphic functions on non-locally convex spaces.

1. Introduction. In [8] Etter observed that in the spaces L_p the natural quasi-norm

$$\|f\| = \{\int |f(t)|^p\,dt\}^{1/p}$$

is plurisubharmonic, i.e. for $f, g \in L_p$

$$\|f\| \leqslant (2\pi)^{-1}\int_0^{2\pi} \|f+e^{i\theta}g\|\,d\theta.$$

Recently several authors have considered this notion. Aleksandrov [1] calls a complex quasi-Banach space locally holomorphic if it has an equivalent plurisubharmonic quasi-norm. Peetre [17] calls such a space locally pseudo-convex and Davis, Garling and Tomczak-Jaegermann [5] call a space equipped with a plurisubharmonic quasi-norm PL-convex. See also Edgar [7].

We shall call a space with an equivalent plurisubharmonic quasi-norm A-convex (for analytically convex). Aleksandrov [1] notes that for $p < 1$, L_p/H_p is not A-convex.

* Research supported by NSF-grant DMS-8301099.

An A-convex space satisfies the Maximum Modulus Principle. Let Δ denote the open unit disc in $\boldsymbol{C}$ and let $\boldsymbol{T}$ be the unit circle so that $\bar{\Delta} = \Delta \cup \boldsymbol{T}$. Suppose X is A-convex and $f: \bar{\Delta} \to X$ is a continuous map which is analytic on Δ (cf. [14], [21] or Section 2 for the definition of analytic functions in a non-locally convex setting). Then we have

$$\|f(0)\| \leqslant C \max_{w \in \boldsymbol{T}} \|f(w)\|$$

where C is a constant independent of f. Peetre [17] asks essentially whether the converse is true, and one of our first results (Theorem 4.1) shows that this is the case, i.e. if X satisfies the Maximum Modulus Principle then X is A-convex. We refer to [14] for a general discussion.

We relate A-convex spaces to the so-called natural spaces introduced by the author in [12]. A quasi-Banach space is natural if and only if it embeds into an l_∞-product of L_p-spaces for some fixed p, $0 < p < 1$ (cf. Theorem 4.2 below). Natural spaces are A-convex and the converse is true for lattices (Theorem 4.4; compare the earlier results of Peetre [17]). We give an example to show that the converse is not true in general.

In Section 5, we first prove an annular version of the Maximum Modulus Principle which holds for every quasi-Banach space. Using this we study first quasi-Banach algebras and show that a theorem of Vesentini [23] can be generalized to this setting, so that the spectral radius is always a plurisubharmonic function. Thus quasi-Banach algebras have a special structure amongst general quasi-Banach spaces.

In Section 6, we study conditions on a p-normable space X so that whenever Y is a p-normable space there exists a p-normed space Z and a bilinear form $B: X \times Y \to Z$ so that

$$k\|x\|\,\|y\| \leqslant \|B(x, y)\| \leqslant \|x\|\,\|y\|, \qquad x, y \in X.$$

This is related to the existence of a p-normed tensor product. We show that if X satisfies these conditions for some fixed p, $0 < p < 1$, then X is A-convex; conversely, if X is natural then it will satisfy these conditions for small enough p.

In Section 7, we sketch the foundations of a theory of holomorphic functions on a complex quasi-Banach space. A sample result is that, essentially, every holomorphic function on an A-trivial space is entire. Here an A-trivial space is a space such as L_p/H_p $(0 < p < 1)$ which admits no continuous operators into an A-convex space.

I would like to thank Jean Bourgain for suggesting the example in Section 4 and Stephen Dilworth for many helpful comments.

2. Basic definitions. Throughout this paper all vector spaces are assumed to be complex. If X is a vector space then a map $x \to \|x\|$ $(X \to \boldsymbol{R}_+)$ is called

a *quasi-seminorm* if

(i) $\|\alpha x\| = |\alpha|\,\|x\|$, $\quad \alpha \in \boldsymbol{C},\ x \in X$,

(ii) $\|x_1 + x_2\| \leqslant C(\|x_1\| + \|x_2\|)$, $\quad x_1, x_2 \in X$,

where C is a constant independent of x_1, x_2.

We call $\|\cdot\|$ a *p-seminorm* where $0 < p \leqslant 1$ if in addition

(iii) $\|x_1 + x_2\|^p \leqslant \|x_1\|^p + \|x_2\|^p$, $\quad x_1, x_2 \in X$.

If further $\|x\| = 0$ implies $x = 0$ then $\|\cdot\|$ is called a *quasi-norm* (if (i) and (ii) are satisfied) or a *p-norm* (if (i), (ii) and (iii) are satisfied). Since every quasi-norm is equivalent to a p-norm for some p, $0 < p \leqslant 1$, we shall always assume that quasi-norms are p-norms for some p and are, in particular, continuous.

If $\|\cdot\|$ is a quasi-norm on X defining a complete metrizable topology then X is called a *quasi-Banach space*.

An upper-semicontinuous function $\varphi: X \to [-\infty, \infty)$ is called *plurisubharmonic* if for every $x_1, x_2 \in X$

$$\varphi(x_1) \leqslant \int_{\boldsymbol{T}} \varphi(x_1 + w x_2)\, d\lambda(w)$$

where λ denotes the normalized Haar measure on $\boldsymbol{T}$, i.e. $d\lambda = (2\pi)^{-1} d\theta$.

If the quasi-norm $\|\cdot\|$ on a quasi-Banach space X is plurisubharmonic then X is called *PL-convex* by Davis, Garling and Tomczak-Jaegermann [5]. If X can be equivalently normed with a plurisubharmonic quasi-norm then we shall say that X is *A-convex* (the term locally pseudo-convex and locally holomorphic have been used by Peetre [17] and Aleksandrov [1]). We also say that X is *A-trivial* if there are no nonzero continuous plurisubharmonic quasi-seminorms on X. Thus L_p $(0 < p < 1)$ is A-convex (Etter [8]) and L_p/H_p is A-trivial (Aleksandrov [1]). Clearly if X is A-trivial and Y is A-convex then $\mathscr{L}(X, Y) = \{0\}$ as noted by Aleksandrov [1].

If Ω is an open subset of $\boldsymbol{C}$ then a map $f: \Omega \to X$ is called *analytic* if for every $z_0 \in \Omega$ there exists $\delta > 0$ so that f can be expanded in a power series for $|z - z_0| < \delta$, i.e.

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n, \qquad |z - z_0| < \delta.$$

Clearly $x_n = (1/n!) f^{(n)}(z_0)$ (see [14] and [21]). $A_0(X)$ will denote the space of all functions $f: \bar{\Delta} \to X$ which are continuous on $\bar{\Delta}$ and analytic on Δ where $\Delta = \{z: |z| < 1\}$.

X is said to satisfy the *Maximum Modulus Principle* (*MMP*) if there exists a constant $M > 0$ so that for all $f \in A_0(X)$ we have

$$(*) \qquad \|f(0)\| \leqslant M \max_{|z|=1} \|f(z)\|.$$

Clearly if this is the case we can show also

$$\|f(z_0)\| \leqslant M \max_{|z|=1} \|f(z)\|$$

for any z_0, $|z_0|<1$. We also noted that it suffices to establish $(*)$ for polynomials, since these are dense in $A_0(X)$ with the topology of uniform convergence on $\bar{\Delta}$. [Then if $f_r(z) = f(rz)$ for $0<r<1$ then f_r can be expanded uniformly in a power series and $\max_{|z|\leqslant 1} \|f_r(z)-f(z)\| \to 0$.] Spaces satisfying (MMP) are considered by Peetre [17].

The following lemma will be useful later on.

LEMMA 2.1. *Let X be a quasi-Banach space and let $F\colon X\to \mathbf{R}_+$, $G\colon X \to \mathbf{R}_+$ be two functions. Suppose that F is upper-semicontinuous, and satisfies $F(\alpha x) = |\alpha| F(x)$ for $\alpha\in \mathbf{C}$, $x\in X$. Then the following conditions on G are equivalent:*

(i) *For every polynomial $f\colon \mathbf{C}\to X$*

$$G(f(0)) \leqslant \max_{|z|=1} F(f(z)).$$

(ii) *For every $f\in A_0(X)$*

$$G(f(0)) \leqslant \max_{|z|=1} F(f(z)).$$

(iii) *For every $f\in A_0(X)$*

$$G(f(0)) \leqslant \exp\{\int_T \log F(f(w))\,d\lambda(w)\}.$$

Proof. (i) $\Rightarrow$ (ii). This is a simple density argument.

(ii) $\Rightarrow$ (iii). Suppose that $f\in A_0(X)$ and that $K = f(T)$. Let H be any continuous function on K so that $H(x) > F(x)$ for all $x\in K$. Note in particular $H(x)>0$ for $x\in K$ and hence $\log H$ is continuous on K. Thus $\log H(f(w))$ is continuous on T; if we define

$$u(z) = -\int_T \operatorname{Re}\left(\frac{w+z}{w-z}\right) \log H(f(w))\,d\lambda(w)$$

for $|z|<1$ and set $u(z) = -\log H(f(z))$ for $|z|=1$, then u is continuous on $\bar{\Delta}$.

Now define $h\in H^\infty$ by

$$h(z) = \exp\left\{-\int_T \frac{w+z}{w-z} \log H(f(w))\,d\lambda(w)\right\}.$$

Then $|h(z)| = \exp(u(z))$ for $z\in\Delta$.

For $r<1$ set $h_r(z) = h(rz)$. Then $h(0)^{-1} h_r(z) f(z) \in A_0(X)$ and hence

$$G(f(0)) \leqslant \max_{|z|=1} |h(0)|^{-1} |h_r(z)|\, F(f(z)).$$

Letting $r\to 1$ we obtain

$$G(f(0)) \leqslant |h(0)|^{-1} = \int_T \log H(f(w))\,d\lambda(w).$$

As $H > F$ is arbitrary and F is u.s.c., the lemma follows.

LEMMA 2.2. *Let X be a quasi-Banach space and suppose $F\colon X\to\mathbf{R}_+$ is upper-semicontinuous and satisfies $F(\alpha x) = |\alpha| F(x)$ for $\alpha\in\mathbf{C}$, $x\in X$. Then the following conditions on F are equivalent:*

(i) *F is plurisubharmonic.*

(ii) *For every $f\in A_0(X)$*

$$F(f(0)) \leqslant \max_{|z|=1} F(f(z)).$$

(iii) *Log F is plurisubharmonic.*

This is immediate from Lemma 2.1.

3. Plurisubharmonic functions on a quasi-Banach space.

LEMMA 3.1. *Let X be a quasi-Banach space and suppose $F\colon X\to \mathbf{C}$ is a continuous function. Suppose $f, g\colon \mathbf{C}\to X$ are polynomials. Then*

$$\lim_{n\to\infty} \int_T F(f(w)+w^n g(w))\,d\lambda(w) = \int_T\int_T F(f(w)+zg(w))\,d\lambda(z)\,d\lambda(w)$$

Proof. We suppose

$$f(z) = \sum_{k=0}^{N} x_k z^k, \qquad g(z) = \sum_{k=0}^{N} y_k z^k,$$

and let X_0 be a finite-dimensional subspace of X containing $f(T)\cup g(T)$. Let $K = f(T)+g(T)$. Then we define linear functionals $\mu_n\colon C(K)\to\mathbf{C}$ and $\mu\colon C(K)\to \mathbf{C}$ by

$$\mu_n(F) = \int_T F(f(w)+w^n g(w))\,d\lambda(w),$$

$$\mu(F) = \int_T\int_T F(f(w)+zg(w))\,d\lambda(z)\,d\lambda(w).$$

Note that $\mu_n(1) = \mu(1) = 1$ and $\|\mu_n\| = \|\mu\| = 1$. In order to show that $\mu_n\to\mu$ weak* as required, we need only check on a dense subset of F's. We therefore consider F of the form

$$F(x) = \prod_{j=1}^{l} \varphi_j(x) \prod_{j=l+1}^{m} \overline{\varphi_j(x)}$$

where $\varphi_1, \ldots, \varphi_m \in X_0^*$. By the Stone–Weierstrass theorem the linear span of such F's, and 1, is dense in $C(K)$. Let

$$\Phi(\xi_1, \ldots, \xi_m) = \prod_{j=1}^{l} \varphi_j(\xi_j) \prod_{j=l+1}^{m} \overline{\varphi_j(\xi_j)}$$

for $\xi_1, \ldots, \xi_m \in X_0$. Φ is then linear in the first l coordinates and conjugate-linear in the remaining $m-l$. Note $F(x) = \Phi(x, \ldots, x)$.

For $A \subset \{1, 2, \ldots, l\}$ and $B \subset \{l+1, l+2, \ldots, m\}$ set $h_j(w) = f(w)$ if $j \notin A \cup B$ and $h_j(w) = g(w)$ if $j \in A \cup B$. Then

$$F\big(f(w)+w^n g(w)\big) = \sum_{A,B} w^{n(|A|-|B|)} \Phi\big(h_1(w), \ldots, h_m(w)\big).$$

By the Riemann–Lebesgue lemma,

$$\lim_{n\to\infty} \int_T F\big(f(w)+w^n g(w)\big)\, d\lambda(w) = \sum_{|A|=|B|} \int_T \Phi(h_1, \ldots, h_m)\, d\lambda(w).$$

Similarly

$$F\big(f(w)+zg(w)\big) = \sum_{A,B} z^{(|A|-|B|)} \Phi\big(h_1(w), \ldots, h_m(w)\big)$$

and so

$$\int_T F\big(f(w)+zg(w)\big)\, d\lambda(z) = \sum_{|A|=|B|} \Phi\big(h_1(w), \ldots, h_m(w)\big)$$

and the lemma follows.

LEMMA 3.2. *Let X be a p-normed quasi-Banach space. Let $\varphi\colon T \to X$ be a bounded Borel function. Suppose $\varepsilon > 0$ and $0 < r < 1$. Then there exists a polynomial $g\colon C \to X$ so that*

(a) $g(0) = 0$.

(b) $\|g(z)\| < \varepsilon$ *for* $|z| \leqslant r$.

(c) *There exists a Borel map $v\colon T \to T$ so that*

$$\int_T \|g(w) - v(w)\varphi(w)\|^p\, d\lambda(w) < \varepsilon^p.$$

Proof. It clearly suffices to consider the case when φ is simple, i.e.

$$\varphi(w) = \sum_{j=1}^{N} x_j 1_{E_j}$$

where $x_j \in X$ $(1 \leqslant j \leqslant N)$ and $E_1, \ldots, E_N$ are disjoint Borel sets in T so that $E_1 \cup \ldots \cup E_N = T$.

For each j, there exists a polynomial $u_j\colon C \to C$ so that $u_j(0) = 0$ and

$$\int_T \big| |u_j(w)| - 1_{E_j}(w)\big|^p\, d\lambda(w) < N^{-1} \varepsilon^p \|\varphi\|_\infty^{-p}.$$

We let

$$g(z) = z^m \sum_{j=1}^{N} u_j(z) x_j$$

where m is chosen so large that $\|g(z)\| < \varepsilon$ for $|z| \leqslant r$.

Let $v\colon T \to T$ be a Borel map satisfying

$$v(w)\overline{u_j(w)} = |u_j(w)|, \quad w \in E_j.$$

If $w \in E_k$ then

$$\|g(w) - v(w)x_k\|^p \leqslant |v(w) - u_k(w)|^p \|x_k\|^p + \sum_{j \neq k} |u_j(w)|^p \|x_j\|^p$$
$$\leqslant \|\varphi\|_\infty^p \sum_{j=1}^{N} \big| |u_j(w)| - 1_{E_j}(w)\big|^p.$$

Hence

$$\int_T \|g(w) - v(w)\varphi(w)\|^p\, d\lambda(w) < \varepsilon^p.$$

Clearly g satisfies the lemma.

We now suppose X is p-normed and set $F_0(x) = \|x\|^p$. For each $n \in N$ define

$$F_n(x) = \inf_{y \in X} \int_T F_{n-1}(x+wy)\, d\lambda(w), \qquad G_n(x) = \inf_{\varphi} \int_T F_{n-1}(\varphi(w))\, d\lambda(w)$$

where the infimum is taken over all polynomials φ such that $\varphi(0) = x$.

Note that if $x_1, x_2 \in X$ then

$$|F_0(x_1) - F_0(x_2)| \leqslant \|x_1 - x_2\|^p.$$

LEMMA 3.3. *For all $n \in N$ and x_1, x_2*

$$|F_n(x_1) - F_n(x_2)| \leqslant \|x_1 - x_2\|^p, \qquad |G_n(x_1) - G_n(x_2)| \leqslant \|x_1 - x_2\|^p.$$

This is an easy induction proof which we omit. In particular each F_n, G_n is continuous. Note also that F_n is monotone decreasing and so $\lim_{n\to\infty} F_n = F_\infty$ exists and further

$$|F_\infty(x_1) - F_\infty(x_2)| \leqslant \|x_1 - x_2\|^p$$

for $x_1, x_2 \in X$.

LEMMA 3.4 (Edgar [7]). *F_∞ is plurisubharmonic.*

We remark that this follows from the fact that

$$F_\infty(x) \leqslant F_{n+1}(x) \leqslant \int_T F_n(x+wy)\, d\lambda(w)$$

for all $n \in N$, $x, y \in X$.

LEMMA 3.5. *For every $n \in N$, $G_n = G_{n+1}$.*

Proof. Suppose $x \in X$ and $\varepsilon > 0$. Then there is a polynomial $f\colon C \to X$ so that

$$\int_T F_n(f(w))\, d\lambda(w) < G_{n+1}(x) + \varepsilon.$$

Now we can partition T into N disjoint Borel sets $E_1, \ldots, E_N$ and find $w_j \in E_j$ so that

$$\|f(w) - f(w_j)\|^p < \varepsilon, \qquad w \in E_j.$$

Pick $y_j \in X$ $(1 \leqslant j \leqslant N)$ so that

$$\int_T F_{n-1}(f(w_j) + wy_j)\, d\lambda(w) < F_n(f(w_j)) + \varepsilon.$$

Thus

$$\int_{E_j}\int_T F_{n-1}(f(w) + zy_j)\, d\lambda(z)\, d\lambda(w) < \big(F_n(f(w_j)) + 2\varepsilon\big)\lambda(E_j)$$
$$< \int_{E_j} F_n(f(w))\, d\lambda(w) + 3\varepsilon\lambda(E_j).$$

Thus if we let $\varphi\colon T \to X$ be defined by $\varphi(w) = y_j$, $w \in E_j$, we have

$$\int_T\int_T F_{n-1}(f(w) + z\varphi(w))\, d\lambda(z)\, d\lambda(w) < \int_T F_n(f(w))\, d\lambda(w) + 3\varepsilon.$$

Now use Lemma 3.2 to find a polynomial $g\colon C \to X$ so that $g(0) = 0$ and

$$\int_T \|g(w) - v(w)\varphi(w)\|^p\, d\lambda(w) < \varepsilon$$

where $v\colon T \to T$ is a Borel function.

Let $f_m(z) = f(z) + z^m g(z)$. Then

$$\lim_{m\to\infty} \int_T F_{n-1}(f_m(w))\, d\lambda(w) = \int_T\int_T F_{n-1}(f(w) + zg(w))\, d\lambda(z)\, d\lambda(w)$$

(by Lemma 3.1). Now

$$\int_T\int_T F_{n-1}(f(w) + zg(w))\, d\lambda(z)\, d\lambda(w)$$
$$< \varepsilon + \int_T\int_T F_{n-1}(f(w) + zv(w)\varphi(w))\, d\lambda(z)\, d\lambda(w)$$
$$= \varepsilon + \int_T\int_T F_{n-1}(f(w) + z\varphi(w))\, d\lambda(z)\, d\lambda(w)$$
$$< \int_T F_n(f(w))\, d\lambda(w) + 4\varepsilon.$$

Hence for large enough m

$$\int_T F_{n-1}(f_m(w))\, d\lambda(w) < \int_T F_n(f(w))\, d\lambda(w) + 4\varepsilon < G_{n+1}(x) + 5\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $G_n(x) \leqslant G_{n+1}(x)$. As $G_n \geqslant G_{n+1}$ trivially the lemma is proved.

LEMMA 3.6. $G_1(x) = F_\infty(x)$ *for* $x \in X$.

Proof. Clearly $G_n \leqslant F_n$ so that by Lemma 3.5, $G_1 \leqslant F_\infty$. Conversely, if φ is a polynomial then $F_\infty \circ \varphi$ is subharmonic and so

$$F_\infty(\varphi(0)) \leqslant \int_T F_\infty(\varphi(w))\, d\lambda(w) \leqslant \int_T \|\varphi(w)\|^p\, d\lambda(w).$$

Hence $F_\infty \leqslant G_1$.

THEOREM 3.7. *Let X be a p-normed quasi-Banach space. For $x \in X$ define*

$$\|x\|_A = \inf \max_{|w|=1} \|\varphi(w)\|$$

where the infimum is taken over all $\varphi \in A_0(X)$ so that $\varphi(0) = x$. Then

(i) $\|\cdot\|_A$ *is a p-seminorm on X.*

(ii) $\|\varphi(0)\|_A \leqslant \exp \int_T \log\|\varphi(w)\|\, d\lambda(w)$ *for all polynomials $\varphi\colon C \to X$.*

(iii) $\|\cdot\|_A^p$ *is plurisubharmonic.*

(iv) $\log\|\cdot\|_A$ *is plurisubharmonic.*

Proof. (i) is trivial. (ii) follows from Lemma 2.1.

(iii) From (ii) it follows that

$$\|\varphi(0)\|_A^p \leqslant \int_T \|\varphi(w)\|^p\, d\lambda(w)$$

and hence $\|x\|_A^p \leqslant G_1(x)$, but $G_1(x) \leqslant \|x\|_A^p$ by definition so that $\|\cdot\|_A$ is plurisubharmonic.

(iv) This follows from Lemma 2.2.

We remark that $\|\cdot\|_A$ is a PL-convex quasi-seminorm in the sense of [4].

We note also that $\|\cdot\|_A$ is the largest plurisubharmonic function dominated by $\|\cdot\|$. X is A-trivial if and only if $\|x\|_A = 0$ for every $x \in X$. X is A-convex if and only if $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent.

For a general space X, we may form a space X_A by quotienting by $N = \{x\colon \|x\|_A = 0\}$ and then completing X/N with respect to $\|\cdot\|_A$. X_A is A-convex and it is easy to see that every operator $T\colon X \to Y$ where Y is A-convex factors through X_A. In particular, X is A-trivial if and only if every operator from X into an A-convex space is zero.

4. Natural spaces and A-convex spaces. Our first result is a very simple application of Theorem 3.7, which partially answers a question of Peetre [17].

THEOREM 4.1. *In order that a quasi-Banach space X is A-convex it is necessary and sufficient that X satisfies the Maximum Modulus Principle (MMP).*

Proof. X satisfies (MMP) if and only if $\|\cdot\|_A$ is equivalent to $\|\cdot\|$.

Next we turn to quasi-Banach lattices motivated by the results of Peetre [17]. Here a complex quasi-Banach lattice X is simply the complexification of a real lattice which we denote ReX. If $x_1, \ldots, x_n \geqslant 0$ then it is possible to define unambiguously $(x_1 \ldots x_n)^{1/n}$ as an element of X using the Krivine calculus (see [16]).

We recall that a quasi-Banach lattice is *L-convex* [12] if there exists $\delta > 0$ so that if $u \in X_+$, $\|u\| = 1$ and $0 \leqslant x_i \leqslant u$ are such that $(1/n)(x_1 + \ldots + x_n) \geqslant (1-\delta) u$ then

$$\max_{1 \leqslant i \leqslant n} \|x_i\| \geqslant \delta.$$

A quasi-Banach space is called *natural* [13] if it is isomorphic to a subspace of an L-convex quasi-Banach lattice. In [12] it is shown, essentially, that a natural quasi-Banach lattice is automatically L-convex so that this definition is consistent. The following theorem [13] characterizes natural spaces and can be thought of as supplying an alternative definition.

THEOREM 4.2. *Let X be a quasi-Banach space. Then X is natural if and only if there is a constant $M > 0$ and $q > 0$ so that if $x \in X$ is nonzero, there exists a probability space (Ω, Σ, P) and an operator $T\colon X \to L_q(\Omega, \Sigma, P)$ with $\|T\| \leqslant M\|x\|^{-1}$ and $Tx = 1_\Omega$. If X is p-normable, any $q < p$ suffices.*

Before our main theorem we prove a simple lemma.

LEMMA 4.3. *Let (Ω, P) be a probability space and suppose $f\colon \Omega \to \boldsymbol{R}$ is a random variable, $\theta \leqslant f \leqslant 1$ a.e. Then*

$$\mathscr{E}(\log f) \geqslant \frac{1-\mathscr{E}(f)}{1-\theta} \log \theta.$$

Proof. Let $W \subset L_1(\Omega)$ be the set of g so that $\theta \leqslant g \leqslant 1$ and $\mathscr{E}(g) = \mathscr{E}(f)$. Note that for any $x, a \in \boldsymbol{R}_+$ we have

$$\log x \leqslant \log a + \frac{1}{a}(x-a).$$

For all $g \in W$,

$$\mathscr{E}(\log g) \leqslant \mathscr{E}(\log f) + \mathscr{E}(f^{-1}(g-f))$$

and so there is an extreme point g so that

$$\mathscr{E}(\log g) \leqslant \mathscr{E}(\log f).$$

At an extreme point, $g = \theta$ on a set of measure $\dfrac{1-\mathscr{E}(f)}{1-\theta}$ and $g = 1$ on a set of measure $\dfrac{\mathscr{E}(f)-\theta}{1-\theta}$. Hence

$$\mathscr{E}(\log f) \geqslant \frac{1-\mathscr{E}(f)}{1-\theta} \log \theta.$$

THEOREM 4.4. *Let X be a complex quasi-Banach lattice. Then the following conditions are equivalent:*

(i) *X is A-convex.*

(ii) *There exists a constant C so that if $x_1, \ldots, x_n \geqslant 0$ in X then*

$$\|(x_1 \ldots x_n)^{1/n}\| \leqslant C \max_{1 \leqslant i \leqslant n} \|x_i\|$$

(iii) *X is L-convex.*

Proof. We begin with a well-known remark. Suppose $u \in X$ and $u \geqslant 0$. Then the order interval $[-u, u]$ in ReX generates in a natural way an abstract M-space in the sense of Kakutani and so we can produce a compact Hausdorff space $\Omega = \Omega_u$ and a lattice embedding $\varrho\colon C_{\boldsymbol{R}}(\Omega) \to \mathrm{Re}X$ so that $\varrho 1 = u$. ϱ extends naturally to a map $\varrho\colon C(\Omega) \to X$. We refer to ϱ as the *Kakutani map* associated to u.

(i) $\Rightarrow$ (ii). We suppose X satisfies (MMP) with constant M. Suppose $x_1, \ldots, x_n \geqslant 0$ and $u = x_1 \vee \ldots \vee x_n$. Let $\varrho\colon C(\Omega) \to X$ be the Kakutani map associated to u. Suppose $\delta > 0$.

Let $\varphi\colon \boldsymbol{T} \to C(\Omega)$ be a C^∞-map so that $\delta 1_\Omega \leqslant \varphi(w) \leqslant 1_\Omega$ for all $w \in \boldsymbol{T}$. We write $\varphi(w, s)$ for $\varphi(w)(s)$ when $w \in \boldsymbol{T}$, $s \in \Omega$. For $z \in \Delta$, $s \in \Omega$ set

$$f(z, s) = \exp\left\{\int_{\boldsymbol{T}} \frac{w+z}{w-z} \log \varphi(w, s)\, d\lambda(w)\right\}.$$

Then $z \to f(z)$ is analytic into $C(\Omega)$ and extends continuously to $\bar{\Delta}$. On $\boldsymbol{T}$, $|f(z, s)| = \varphi(z, s)$. Now

$$f(0, s) = \exp\{\int_{\boldsymbol{T}} \log \varphi(w, s)\, d\lambda(w)\} = \varphi_0(s)$$

say. By the Maximum Modulus Principle in X,

$$\|\varrho(\varphi_0)\| \leqslant M \max_{|w|=1} \|\varrho(\varphi(w))\|.$$

Now pick $\xi_1, \ldots, \xi_n \in C(\Omega)$ so that $\varrho(\xi_i) = x_i \vee \delta u$. Clearly $\delta 1_\Omega \leqslant \xi_i \leqslant 1_\Omega$. We may pick C^∞-functions $v_1, \ldots, v_n\colon \boldsymbol{T} \to [0, 1]$ so that

(a) $\sum_{j=1}^{n} v_j(w) = 1, \quad w \in \boldsymbol{T}$.

(b) For each $w \in T$ at most two of $v_j(w)$ are nonzero.

(c) $\lambda\{v_j = 1\} \geqslant \frac{1}{n}(1-\delta)$.

Set $\varphi(w) = \sum_{i=1}^{n} v_i(w)\xi_i$. Then

$$\varphi_0(s) \geqslant \exp\left(\frac{1-\delta}{n}\sum_{i=1}^{n}\log\xi_i(s)+\delta\log\delta\right)$$
$$= \delta^\delta(\xi_1(s)\ldots\xi_n(s))^{(1-\delta)/n}.$$

Hence $\varrho(\varphi_0) \geqslant u^\delta((x_1 \vee \delta u)\ldots(x_n \vee \delta u))^{(1-\delta)/n}\delta^\delta$.

Note if $|w| = 1$

$$\|\varrho(\varphi(w))\| \leqslant 2^{1/p-1}\max_j\|x_j \vee \delta u\|$$

and thus

$$\delta^\delta\|u^\delta((x_1 \vee \delta u)\ldots(x_n \vee \delta u))^{(1-\delta)/n}\| \leqslant 2^{1/p-1}M\max_{j\leqslant n}\|x_j \vee \delta u\|.$$

Letting $\delta \to 0$ we obtain the result with $C = 2^{1/p-1}M$.

(ii) $\Rightarrow$ (iii). Choose $\delta > 0$ so that $\delta^2 < 1/2$ and

$$\delta^{2p}+\delta^p \leqslant C^{-p}\delta^{4p\delta}.$$

Suppose $x_j \in X$, $0 \leqslant x_j \leqslant u$, are such that

$$\frac{1}{n}(x_1+\ldots+x_n) \geqslant (1-\delta)u.$$

We again use a Kakutani map associated to u, say $\varrho\colon C(\Omega) \to X$. Suppose $x_j = \varrho(f_j)$. Then by Lemma 4.3 for each $s \in \Omega$

$$\frac{1}{n}\sum_{j=1}^{n}\log(\max\{f_j(s), \delta^2\}) \geqslant \frac{2\log\delta}{1-\delta^2}\left(1-\frac{1}{n}\sum_{j=1}^{n}f_j(s)\right) \geqslant \frac{2\delta\log\delta}{1-\delta^2}$$

(it may be useful to note that $\log\delta < 0$). Hence

$$\Big(\prod_{j=1}^{n}(f_j \vee \delta^2 1_\Omega)\Big)^{1/n} \geqslant \delta^{2\delta/(1-\delta^2)} \geqslant \delta^{4\delta}$$

provided $\delta^2 < 1/2$. Thus

$$\Big(\prod_{j=1}^{n}(x_j \vee \delta^2 u)\Big)^{1/n} \geqslant \delta^{4\delta}u$$

and so by (ii)

$$\delta^{4\delta} \leqslant C\max_{j\leqslant n}\|x_j \vee \delta^2 u\|.$$

Thus there exists $k \leqslant n$ so that $C^p(\|x_k\|^p+\delta^{2p}) \geqslant \delta^{4p\delta}$, i.e.

$$\|x_k\|^p \geqslant C^{-p}\delta^{4p\delta}-\delta^{2p} \geqslant \delta^p.$$

Thus X is L-convex.

(iii) $\Rightarrow$ (i). If X is L-convex then there exists $q > 0$ and a constant $M > 0$ so that if $x \in X$, with $\|x\| = 1$, there exists a probability space (Ω, Σ, P) and an operator $T_x\colon X \to L_q(\Omega, \Sigma, P)$ so that $T_x x = 1_\Omega$ and $\|T_x\| \leqslant M$.

Note that $y \to M^{-1}\|T_x y\|$ is a q-seminorm on X dominated by $\|\cdot\|$ which is plurisubharmonic. Hence

$$M^{-1}\|T_x y\| \leqslant \|y\|_A, \quad y \in X.$$

In particular, $M^{-1} \leqslant \|x\|_A$ if $\|x\| = 1$. Thus $\|\cdot\|_A$ is equivalent to $\|\cdot\|$.

Corollary 4.5. *Every natural quasi-Banach space is A-convex.*

We conclude this section by giving an example of an A-convex quasi-Banach space which fails to be natural. The example is the Schatten class S_p where $0 < p < 1$; that this might provide an example was suggested to the author by Jean Bourgain.

We shall need first a preparatory theorem which is probably of independent interest. Let us recall that a quasi-Banach space X is *of type* p $(0 < p \leqslant 2)$ if there is a constant C so that

$$[\mathscr{E}(\|\sum_{i=1}^{n}\varepsilon_i x_i\|)]^{1/p} \leqslant C(\sum_{i=1}^{n}\|x_i\|^p)^{1/p}$$

where $\varepsilon_1, \ldots, \varepsilon_n$ is any sequence of independent Bernoulli random variables with $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$. For $0 < p < 1$, X is of type p if (and only if) X is p-normable; if $p > 1$ and X is of type p then X is a Banach space (cf. [15], p. 99 and p. 107).

Theorem 4.6. *Suppose X and Y are quasi-Banach spaces; suppose X is of type p and Y is of type q where $0 < p, q \leqslant 2$. Suppose $0 < r < 1$. Then there is a constant $C = C(p, q, r, X, Y)$ so that if (Ω, Σ, μ) is a probability measure space and $B\colon X \times Y \to L_r(\Omega, \Sigma, \mu)$ is a bilinear form then for $x_1, \ldots, x_n \in X$, $y_1, \ldots, y_n \in Y$*

$$\|\sum_{j=1}^{n}B(x_j, y_j)\|_r \leqslant C\|B\|(\sum_{j=1}^{n}\|x_j\|^s\|y_j\|^s)^{1/s}$$

where $1/s = 1/p+1/q$.

Proof. (In the ensuing argument, C will represent a constant depending on X, Y and r, which may vary from line to line.) Let $\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_n$ be any sequence of $2n$ independent Bernoulli random variables defined on some probability space (Ω', Σ', P'). Then for $\omega' \in \Omega'$

$$\|B(\sum_{j=1}^{n}\varepsilon_j(\omega')x_j, \sum_{j=1}^{n}\eta_j(\omega')y_j)\|_r \leqslant \|B\|\,\|\sum_{j=1}^{n}\varepsilon_j(\omega')x_j\|\,\|\sum_{j=1}^{n}\eta_j(\omega')y_j\|.$$

Let $f_{jk} = B(x_j, y_k)$. Then

$$\mathscr{E}(\|\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j\eta_k f_{jk}\|_r^r) \leqslant \|B\|^r \mathscr{E}(\|\sum_{j=1}^{n}\varepsilon_j x_j\|^r \|\sum_{j=1}^{n}\eta_j y_j\|^r)$$
$$= \|B\|^r \mathscr{E}(\|\sum_{j=1}^{n}\varepsilon_j x_j\|^r)\mathscr{E}(\|\sum_{j=1}^{n}\eta_j y_j\|^r)$$
$$\leqslant C\|B\|^r(\sum_{j=1}^{n}\|x_j\|^p)^{r/p}(\sum_{j=1}^{n}\|y_j\|^q)^{r/q}.$$

Similarly

$$\mathscr{E}(\|\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j\varepsilon_k f_{jk}\|_r^r) \leqslant \|B\|^r \mathscr{E}(\|\sum_{j=1}^{n}\varepsilon_j x_j\|^r \|\sum_{j=1}^{n}\varepsilon_j y_j\|^r)$$
$$\leqslant \|B\|^r \mathscr{E}(\|\sum_{j=1}^{n}\varepsilon_j x_j\|^{2r})^{1/2}\mathscr{E}(\|\sum_{j=1}^{n}\varepsilon_j y_j\|^{2r})^{1/2}$$
$$\leqslant C\|B\|^r(\sum_{j=1}^{n}\|x_j\|^p)^{r/p}(\sum_{j=1}^{n}\|y_j\|^q)^{r/q}.$$

Now

$$\|\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j\eta_k f_{jk}\|_r^r = \int_\Omega|\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j(\omega')\eta_k(\omega')f_{jk}(\omega)|^r\,d\mu(\omega).$$

By a generalization of Khintchine's inequality due to Bonami [3] (cf. also Pisier [19]) we have

$$\mathscr{E}(|\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j(\omega')\eta_k(\omega')f_{jk}(\omega)|^r) \leqslant C(\sum_{j,k}|f_{jk}(\omega)|^2)^{r/2}$$

and hence we conclude

$$\|(\sum_{j=1}^{n}\sum_{k=1}^{n}|f_{jk}|^2)^{1/2}\|_r \leqslant C\|B\|(\sum_{j=1}^{n}\|x_j\|^p)^{1/p}(\sum_{j=1}^{n}\|y_j\|^q)^{1/q}.$$

In particular

$$\|(\sum_{j\neq k}|f_{jk}|^2)^{1/2}\|_r \leqslant C\|B\|(\sum_{j=1}^{n}\|x_j\|^p)^{1/p}(\sum_{j=1}^{n}\|y_j\|^q)^{1/q}$$

and hence

$$\|(\sum_{j>k}|f_{jk}+f_{kj}|^2)^{1/2}\|_r \leqslant C\|B\|(\sum\|x_j\|^p)^{1/p}(\sum\|y_j\|^q)^{1/q}.$$

Now

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j\varepsilon_k f_{jk} = \sum_{k=1}^{n} f_{kk} + \sum_{j>k}\varepsilon_j\varepsilon_k(f_{jk}+f_{kj}).$$

Again by use of Bonami's theorem

$$(\mathscr{E}(\|\sum_{j>k}\varepsilon_j\varepsilon_k(f_{jk}+f_{kj})\|_r^r))^{1/r} \leqslant C\|(\sum_{j>k}|f_{jk}+f_{kj}|^2)^{1/2}\|_r.$$

Thus

$$\|\sum_{k=1}^{n} f_{kk}\|_r \leqslant C\|B\|(\sum\|x_j\|^p)^{1/p}(\sum\|y_j\|^q)^{1/q}.$$

Now for any $\alpha_j > 0$

$$\|\sum_{k=1}^{n} f_{kk}\|_r \leqslant C\|B\|(\sum\alpha_j^p\|x_j\|^p)^{1/p}(\sum\alpha_j^{-q}\|y_j\|^q)^{1/q}.$$

Let $\alpha_j = \|y_j\|^{s/p}\|x_j\|^{-s/q}$. The result then follows.

Corollary 4.7. *Suppose X is a quasi-Banach space of type p, Y is a quasi-Banach space of type q and Z is a natural quasi-Banach space. Then there is a constant $C = C(X, Y, Z)$ so that if $B\colon X\times Y\to Z$ is a bilinear form then*

$$\|\sum_{i=1}^{n} B(x_i, y_i)\| \leqslant C\|B\|(\sum_{i=1}^{n}\|x_i\|^s\|y_i\|^s)^{1/s}$$

where $1/s = 1/p+1/q$.

Proof. This is an immediate consequence of Theorem 4.2 and Theorem 4.6.

Now let H be a separable infinite-dimensional Hilbert space and let S_p denote the Schatten p-class where $0<p<1$. Thus $T\in S_p$ if and only if $T\in\mathscr{L}(H)$ is a compact operator whose singular values $s_n(T)$ satisfy

$$\|T\|_{(p)} = (\sum_n s_n(T)^p)^{1/p} < \infty.$$

S_p is a p-normable operator-ideal (cf. Pietsch [18], pp. 216, 255).

Theorem 4.8. *Suppose that $0<p<1$ and that Z is a natural quasi-Banach space. Then there is a constant C so that if $T\colon S_p\to Z$ is a bounded linear operator then*

$$\|T(A)\| \leqslant C\|T\|\,\|A\|_{(1)}$$

and hence T factors in the form $T = T_0 J$ where $T_0\colon S_1\to Z$ is bounded and $J\colon S_p\to S_1$ is the inclusion map.

Proof. Consider the bilinear form $B: H\times H^*\to Z$ given by

$$B(h, h^*) = T(h^*\otimes h).$$

Then $\|B\|\leqslant\|T\|$. Hence if $A\in S_p$ then we can write

$$A=\sum\sigma_j e_j^*\otimes f_j$$

where $\sum|\sigma_i|^p=\|A\|_{(p)}^p$, $\|e_n^*\|=\|f_n\|=1$. Thus

$$T(A)=\sum_{j=1}^{\infty}\sigma_j B(f_j, e_j^*)$$

and hence

$$\|T(A)\|\leqslant C\|B\|\sum_{j=1}^{\infty}|\sigma_j|\leqslant C\|T\|\,\|A\|_{(1)}.$$

COROLLARY 4.9. *For $0<p<1$, S_p is not natural.*

THEOREM 4.10. *S_p is A-convex for $0<p<1$; in fact $\|\cdot\|_{(p)}$ is plurisubharmonic.*

Proof. It will suffice to show that if $A, B\in S_p$ are of finite rank and $F(z)=A+zB$, then $\|F(z)\|_{(p)}^p$ is subharmonic on C.

For any $z_0\in C$ there is an isometry U of H so that $UF(z_0)$ is a positive hermitian operator. Let $H_0=\mathscr{R}(UF(z_0))$ and suppose $\dim H_0=m=\operatorname{rank}F(z_0)$. Let P be the orthogonal projection of H onto H_0 and define $G: C\to\mathscr{L}(H_0)$ by

$$G(z)(h)=PUF(z)(h),\qquad h\in H_0.$$

$G(z_0)$ is invertible on H_0 and has eigenvalues $\sigma_1\geqslant\sigma_2\geqslant\ldots\geqslant\sigma_m>0$. There exists $\delta>0$ so that if $|z-z_0|<\delta$ then the spectrum of $G(z)$ is contained in some compact subset C of $\{z: \operatorname{Re}z>0\}$. The function $\varphi(z)=z^p$ can be defined to be analytic on this half-plane with $\varphi(x)=x^p$ if $x>0$.

Now consider $\varphi(G(z))$ for $|z-z_0|<\delta$. Precisely,

$$\varphi(G(z))=(2\pi i)^{-1}\int_{\gamma}\varphi(w)\big(w-G(z)\big)^{-1}\,dw$$

where γ is any contour in the right half-plane around C. The map $z\to\varphi(G(z))$ is analytic and hence so is the map $z\to\operatorname{tr}\varphi(G(z))$. If $0<r<\delta$ then

$$\operatorname{tr}\varphi(G(z_0))=\int_{T}\operatorname{tr}\varphi(G(z_0+rw))\,d\lambda(w).$$

Now $\operatorname{tr}\varphi(G(z_0+rw))=\sum_{i=1}^{m}\alpha_i^p$ where $\alpha_1,\ldots,\alpha_m$ are the eigenvalues of $G(z_0+rw)$. Now

$$\sum|\alpha_i|^p\leqslant\|G(z_0+rw)\|_{(p)}^p\quad\text{(see Gohberg–Krein [9], p. 41)},$$
$$\leqslant\|F(z_0+rw)\|_{(p)}^p$$

and hence

$$\operatorname{tr}\varphi(G(z_0))=\|G(z_0)\|_{(p)}^p=\|F(z_0)\|_{(p)}^p\leqslant\int_{T}\|F(z_0+rw)\|_{(p)}^p\,d\lambda(w)$$

so that $\|\cdot\|_{(p)}^p$ is plurisubharmonic.

5. Applications to algebras. We begin with a slight strengthening of a theorem of Coifman and Rochberg on Bergman spaces. Let us suppose $0<q<p$, $\sigma=1/q-1$ and $\nu=[\sigma]$. $B_{q,p}$ is the Bergman space of all analytic functions φ on Δ so that

$$\|\varphi\|_{q,p}^p=\int_{\Delta}|\varphi(z)|^p(1-|z|^2)^{p/q-2}\,dm(z)<\infty$$

where m denotes the planar measure on Δ.

LEMMA 5.1. *If $0<r<1$ there is a constant $C=C(p,q,r)$ so that if $\varphi\in B_{q,p}$ then there exist z_k, $k\geqslant 1$, with $r\leqslant|z_k|<1$ and α_k so that*

$$\varphi(w)=\sum_{k=1}^{\infty}\alpha_k(1-|z_k|^2)^{\nu+1-\sigma}(1-wz_k)^{-(\nu+2)}$$

and $\left(\sum|\alpha_k|^p\right)^{1/p}\leqslant C\|\varphi\|_{q,p}$.

Proof. According to Theorem 2 of [4], there exist $(\xi_k: k\geqslant 1)$ with $|\xi_k|\to 1$, and $C=C(p,q)$ so that if $\varphi\in B_{q,p}$ then we can write

$$\varphi(w)=\sum_{k=1}^{\infty}\alpha_k(1-|\xi_k|^2)^{\nu+1-\sigma}(1-w\xi_k)^{-(\nu+2)}$$

with $\left(\sum|\alpha_k|^p\right)^{1/p}\leqslant C\|\varphi\|_{q,p}$. The ξ_k are independent of φ.

Let Γ denote the set of $z\in\Delta$ with $r\leqslant|z|<1$. We define a map $S: l_p(\Gamma)\to B_{q,p}$ by

$$S\{\alpha_z\}=\sum_{z\in\Gamma}\alpha_z(1-|z|^2)^{\nu+1-\sigma}(1-wz)^{-(\nu+2)}.$$

Let E be the range of S. Let F be the finite-dimensional space spanned by $(1-w\xi_k)^{-(\nu+2)}$ for $|\xi_k|<r$. Then $E+F=B_{q,p}$.

Consider the map $S\oplus I: l_p(\Gamma)\oplus F\to B_{q,p}$. This map is an open mapping. Let ϱ be any linear functional on $B_{q,p}$ so that $\varrho(E)=0$. Then $\varrho(S\oplus I)(\{\alpha_z\},f)=\varrho(f)$ so that $\varrho\circ(S\oplus I)$ is continuous. Hence ϱ is continuous on $B_{q,p}$. Now let

$$h(z)=\varrho\big((1-wz)^{-(\nu+2)}\big).$$

h is analytic on Δ and vanishes for $r\leqslant|z|<1$. Then $h\equiv 0$ and so $\varrho=0$.

Thus $E = B_{q,p}$ and S is onto. The lemma follows by the Open Mapping Theorem.

Our next theorem is an annular maximum modulus principle valid for all quasi-Banach spaces.

We recall first that if $\sigma > 0$ and $\nu = [\sigma]$ then $A_\sigma(X)$ is defined to be the space of analytic functions $f\colon \Delta \to X$ satisfying

$$\sup_{|z|\leqslant 1} (1-|z|^2)^{\nu+1-\sigma} \|f^{(\nu+1)}(z)\| < \infty.$$

It is shown in [14], Theorem 5.1, that if $f \in A_\sigma(X)$ and X is p-normed then there exists $T \in \mathscr{L}(B_{q,p}, X)$ so that $T((1-wz)^{-1}) = f(z)$.

THEOREM 5.2. *Let X be a p-normed quasi-Banach space, and suppose $0 < r < 1$. Then there exists a constant $C = C(r, X)$ so that if $f \in A_0(X)$ then*

$$\|f(0)\| \leqslant C \max_{r\leqslant |z| \leqslant 1} \|f(z)\|.$$

Proof. As above suppose $0 < q < p$, $\sigma = 1/q - 1$ and $\nu = [\sigma]$. For any $f \in A_0(X)$, let

$$f(z) = \sum_{n=0}^{\infty} x_n z^n, \qquad z \in \Delta$$

(this is permissible, see [21]), and

$$F(z) = \sum_{n=0}^{\infty} x_n \frac{n!}{(n+\nu+1)!} z^{n+\nu+1}.$$

Then $F \in A_\sigma(X)$ and so there is a bounded linear operator $T\colon B_{q,p} \to X$ so that

$$T((1-wz)^{-1}) = F(z), \qquad z \in \Delta$$

(Theorem 5.1 of [14]). If $\varphi \in B_{q,p}$ suppose

$$\varphi(w) = \sum_{n=0}^{\infty} a_n w^n.$$

Note that $T(w^k) = 0$, $0 \leqslant k \leqslant \nu$, and hence $T\varphi = T(w^{\nu+1}\psi)$ where

$$\psi(w) = \sum_{n=\nu+1}^{\infty} a_n w^{n-\nu-1}.$$

Note that $\|\psi\|_{q,p} \leqslant C\|\varphi\|_{q,p}$ where $C = C(p, q)$. Now by Lemma 5.1 we can write

$$\psi(w) = \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2)^{\nu+1-\sigma} (1-wz_k)^{-(\nu+2)}$$

where $r \leqslant |z_k| < 1$ and

$$\left(\sum |\alpha_k|^p\right)^{1/p} \leqslant C\|\psi\|_{q,p} \leqslant C\|\varphi\|_{q,p}.$$

Hence

$$w^{\nu+1}\psi(w) = ((\nu+1)!)^{-1} \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2)^{\nu+1-\sigma} \frac{\partial^{\nu+1}}{\partial z^{\nu+1}} (1-wz)\Big|_{z=z_k}$$

and so

$$T\varphi = ((\nu+1)!)^{-1} \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2)^{\nu+1-\sigma} F^{(\nu+1)}(z_k)$$
$$= ((\nu+1)!)^{-1} \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2)^{\nu+1-\sigma} f(z_k).$$

Hence since $\nu+1-\sigma \geqslant 0$,

$$\|T\varphi\| \leqslant C\big(\max_{r\leqslant |z| \leqslant 1} \|f(z)\|\big) \|\varphi\|_{q,p}$$

where $C = C(p, q, r)$. In particular

$$\|f(0)\| = \big\|T((\nu+1)!\, w^{\nu+1})\big\| \leqslant C \max_{r\leqslant |z| \leqslant 1} \|f(z)\|.$$

Now let us suppose that B is a quasi-Banach algebra with identity. We recall that the spectral radius formula

$$\varrho(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$$

is valid in B (cf. [24]).

We now generalize a result of Vesentini [23] (see also [2]).

THEOREM 5.3. *If B is a unital quasi-Banach algebra then $\log \varrho$ and ϱ are plurisubharmonic.*

Proof. For any r, $0 < r < 1$, there exists a constant $C = C(r)$ so that if $f \in A_0(B)$ then

$$\|f(0)\| \leqslant C(r) \max_{r\leqslant |z| \leqslant 1} \|f(z)\|.$$

Hence

$$\|f(0)^n\| \leqslant C(r) \max_{r\leqslant |z| \leqslant 1} \|(f(z))^n\|$$

since $z \to f(z)^n$ is also in $A_0(B)$. Thus

$$\|f(0)^n\|^{1/n} \leqslant C(r)^{1/n} \max_{r\leqslant |z| \leqslant 1} \|(f(z))^n\|^{1/n}.$$

Letting $n \to \infty$ we obtain

$$\varrho(f(0)) \leqslant \max_{r \leqslant |z| \leqslant 1} \|(f(z))^n\|^{1/n}$$

for all $n \in \mathbf{N}$. Hence

$$\varrho(f(0)) \leqslant \max_{|z|=1} \|f(z)\|.$$

It follows that $\varrho(x) \leqslant \|x\|_A$ for all $x \in B$ by Theorem 3.7.

Now if f is any polynomial then

$$\log \varrho(f(0)) \leqslant \int_{\mathbf{T}} \log \|f(w)\| \, d\lambda(w)$$

and hence applying the theorem to $f(z)^n$ we obtain

$$\log \varrho(f(0)) \leqslant \int_{\mathbf{T}} n^{-1} \log \|(f(w))^n\| \, d\lambda(w).$$

Letting $n = 2^k$ and $k \to \infty$ we use the Monotone Convergence Theorem to deduce

$$\log \varrho(f(0)) \leqslant \int_{\mathbf{T}} \log \varrho(f(w)) \, d\lambda(w),$$

i.e. $\log \varrho$ is plurisubharmonic. It follows easily that ϱ is also plurisubharmonic.

The main use we make of Theorem 5.3 is to show that on a quasi-Banach algebra $\|\cdot\|_A$ cannot be trivial.

THEOREM 5.4. *Let B be a unital quasi-Banach algebra. Then on B*

(i) $\|xy\|_A \leqslant \|x\|_A \|y\|_A$, $\|1\|_A = 1$.

(ii) *If $\|x\|_A = 0$ then x is in the Jacobson radical of B.*

(iii) *If B is semisimple then $\|\cdot\|_A$ is an algebra quasi-norm on B.*

Proof. (i) For $\varepsilon > 0$ let $f, g \colon \mathbf{C} \to B$ be polynomials so that $f(0) = x$, $g(0) = y$ and

$$\|f(w)\| \leqslant \|x\|_A + \varepsilon, \quad w \in \mathbf{T},$$
$$\|g(w)\| \leqslant \|y\|_A + \varepsilon, \quad w \in \mathbf{T}.$$

Then $f(z) \cdot g(z)$ is a polynomial with $f(0) g(0) = xy$. Hence

$$\|xy\|_A \leqslant (\|x\|_A + \varepsilon)(\|y\|_A + \varepsilon)$$

and as $\varepsilon > 0$ is arbitrary, the first part follows.

Now as $\varrho(x) \leqslant \|x\|_A$ we have $\|1\|_A = 1$.

(ii) Suppose $\|x\|_A = 0$ and I is a maximal right ideal. If $x \notin I$ then there exist $y_1, y_2 \in I$ so that

$$xy_1 + y_2 = 1.$$

Hence $\varrho(xy_1) \geqslant 1$ and $\|x\|_A \|y_1\|_A \geqslant 1$ so that $\|x\|_A > 0$. This contradiction shows $x \in I$.

(iii) is immediate.

Remark. Note in particular that a quasi-Banach algebra with identity cannot be A-trivial. In fact, if an A-trivial space is embedded in a quasi-Banach algebra, it embeds into the radical. It is asked in [11] whether a quasi-Banach algebra with identity can have trivial dual; this question is related to the question of whether L_p is prime for $0 < p < 1$.

A quasi-Banach space X is called *boundedly transitive* ([15], p. 151) if there exists a constant M so that if $x, y \in X$ with $\|x\| = \|y\| = 1$ then there exists $T \in \mathscr{L}(X)$ with $Tx = y$ and $\|T\| \leqslant M$. L_p is boundedly transitive if $0 < p < 1$ ([15], p. 126 or [20], p. 253) and there is a space universal for separable quasi-Banach spaces which is boundedly transitive ([10], Theorem 4.3).

THEOREM 5.5. *If X is a boundedly transitive quasi-Banach space then $\mathscr{L}(X)$ is A-convex.*

Proof. If $T \in \mathscr{L}(X)$ there exists $x \in X$ with $\|x\| = 1$ and $\|Tx\| \geqslant \frac{1}{2}\|T\|$. If $T \neq 0$ we can find $S \in \mathscr{L}(X)$ with $\|S\| \leqslant M$ and $STx = \|Tx\| x$. Thus $\varrho(ST) \geqslant \|Tx\|$ and hence

$$\|S\|_A \|T\|_A \geqslant \|ST\|_A \geqslant \tfrac{1}{2}\|T\|$$

so that $\|T\|_A \geqslant \dfrac{1}{2M}\|T\|$.

6. Applications to tensor products. We shall say that two quasi-Banach spaces X and Y admit a *p-normable weak tensor product* if there exists a p-normable space Z and a bilinear form $B \colon X \times Y \to Z$ with $\|B\| \leqslant 1$ so that for some $k > 0$

$$k\|x\| \, \|y\| \leqslant \|B(x, y)\| \leqslant \|x\| \, \|y\|$$

for $x \in X$, $y \in Y$. B induces a linear map $B \colon X \otimes Y \to Z$. If B can be chosen to be one-one then X and Y admit a *p-normable tensor product*.

In [22] it is shown that if $0 < p \leqslant 1$ and $0 < q \leqslant 1$, X is p-normable and Y is q-normable then X and Y always admit an r-normable tensor product where $1/r = 1/p + 1/q - 1$. This is best possible (cf. [14]).

LEMMA 6.1. *In order that X and Y admit a weak p-normable tensor product it is necessary and sufficient that for some $k > 0$ and every $x \in X$, $y \in Y$ there exists a bilinear form $B \colon X \times Y \to Z$ where Z is p-normed with $\|B\| = 1$ and*

$$\|B(x, y)\| \geqslant k\|x\| \, \|y\|.$$

Proof. For each $x \in X$, $y \in Y$ there exists $T_{x,y}: X \otimes Y \to Z_{x,y}$ with

$$\|T_{x,y}(x \otimes y)\| \geqslant k\|x\|\,\|y\|,$$
$$\|T_{x,y}(\xi \otimes \eta)\| \leqslant \|\xi\|\,\|\eta\|, \qquad \xi \in X,\ \eta \in Y.$$

Quasi-norm $X \otimes Y$ by

$$\|u\| = \sup_{x,y} \|T_{x,y} u\|.$$

Then the natural bilinear form $(x, y) \to x \otimes y$ into $X \otimes Y$ satisfies our definition.

We shall say that a quasi-Banach space X is *p-tensorial* if for every p-normable space Y, X and Y admit a weak p-normable tensor product. We remark that L_p, for $0 < p < 1$, is p-tensorial ($L_p(Y)$ is a p-normable tensor product if Y is p-normable) and hence every subspace of L_p is p-tensorial.

THEOREM 6.2. *Let X be a p-normable quasi-Banach space. Then*

(i) *If X is natural then X is q-tensorial for any q, $0 < q < p$.*

(ii) *If X is q-tensorial for some q, $0 < q < p$, then X is A-convex.*

Proof. (i) There exists $M = M(q)$ so that if $x \in X$ with $x \neq 0$ there exists a probability measure space (Ω, Σ, P) and a linear operator $T_x: X \to L_q(\Omega, \Sigma, P)$ with $\|T_x\| \leqslant M\|x\|^{-1}$ and $T_x x = 1_\Omega$ (Theorem 4.2).

If Y is q-normable define $B_x: X \times Y \to L_q(Y)$ by $B_x(\xi, \eta) = M^{-1}\|x\|(T_x\xi \otimes \eta)$. Then $\|B_x\| \leqslant 1$ and $\|B_x(x, y)\| = M^{-1}\|x\|\,\|y\|$ so that X and Y are q-tensorial by Lemma 6.1.

(ii) It clearly suffices to show that every separable subspace of X is A-convex. To do this suppose that X is separable. Let U be a boundedly transitive q-normed separable universal space ([10]). Then there exists a bilinear $B: X \times U \to U$ so that

$$k\|x\|\,\|u\| \leqslant \|B(x, u)\| \leqslant \|x\|\,\|u\|, \qquad x \in X,\ u \in U,$$

where $k > 0$. Now B induces a linear map $T: X \to \mathscr{L}(U)$ defined by

$$(Tx)(u) = B(x, u)$$

and $k\|x\| \leqslant \|Tx\| \leqslant \|x\|$. Thus X is isomorphic to a subspace of $\mathscr{L}(U)$ which is A-convex by Theorem 5.5.

7. Towards a theory of holomorphic functions. Throughout this section X and Y will denote fixed quasi-Banach spaces both of which are p-normed. We define a bounded *homogeneous polynomial* of degree $n \geqslant 1$, $P: X \to Y$, to be a map of the form

$$P(x) = \hat{P}(x, \ldots, x)$$

where $\hat{P}: X^n \to Y$ is a bounded symmetric n-linear form. The identification $P \to \hat{P}$ between bounded homogeneous polynomials and symmetric n-linear forms is bijective as shown by the polarization formula ([6], p. 4). We set

$$\|P\| = \sup_{\|x\| \leqslant 1} \|P(x)\|.$$

For convenience a polynomial of degree zero is a constant map.

LEMMA 7.1. *Let P_n be a sequence of homogeneous polynomials and suppose for some open set U we have*

$$\sup_{n \in N} \|P_n(x)\| < \infty$$

whenever $x \in U$. Then there exists $\eta > 0$ so that

$$\sup_{n \in N} \sup_{\|x\| \leqslant \eta} \|P_n(x)\| < \infty.$$

Proof (cf. [6], p. 9). By the Baire Category Theorem there exists $M < \infty$ and $x_0 \in U$, and $\nu > 0$ so that if $\|x - x_0\| < \nu$ then

$$\|P_n(x)\| \leqslant M, \qquad n \in N.$$

Let $\eta = 2^{-1/p}\nu$ and choose r_0, $0 < r_0 < 1$, so that $(1 - r_0)\|x_0\| < \eta$.

By Theorem 5.2 there exists a constant $C = C(r_0, Y)$ so that if $f \in A_0(Y)$ then

$$\|f(0)\| \leqslant C \max_{r_0 \leqslant z \leqslant 1} \|f(z)\|.$$

If $\xi \in X$ with $\|\xi\| < \eta$ then for $n \in N$

$$\|P_n(\xi)\| \leqslant C \max_{r_0 \leqslant |z| \leqslant 1} \|P_n(\xi + z x_0)\|$$

since $f(z) = P_n(\xi + z x_0)$ is analytic. Now if $z = re^{i\theta}$ with $r_0 \leqslant r \leqslant 1$ and $0 \leqslant \theta \leqslant 2\pi$, then

$$\|P_n(\xi + z x_0)\| = \|P_n(e^{-i\theta}\xi + r x_0)\| = \|P_n(e^{-i\theta}\xi - (1 - r)x_0 + x_0)\| \leqslant M.$$

Hence $\|P_n(\xi)\| \leqslant CM$ for $\|\xi\| \leqslant \eta$.

Now if $(P_n)_{n=0}^{\infty}$ is a sequence of polynomials where $\deg P_n = n$, we say that $\sum P_n$ is a *power series* if for every $x \in X$

$$\sup_{n \in N} \|P_n(x)\|^{1/n} < \infty.$$

LEMMA 7.2. *If $\sum P_n$ is a power series then there exists $\eta > 0$ so that the series $\sum P_n(x)$ converges uniformly for $\|x\| \leqslant \eta$.*

Proof. It follows from the Baire Category Theorem that there exists an open set V and $M > 0$ so that

$$\|P_n(x)\|^{1/n} \leqslant M, \qquad x \in V,\ n \in N.$$

Let $U=(1/2M)V$. For $x\in U$,

$$\|P_n(x)\|\leqslant(\tfrac{1}{2})^n\leqslant 1.$$

By Lemma 7.1 there exists $\eta_1>0$ and $C<\infty$ so that

$$\|P_n(x)\|\leqslant C,\quad \|x\|\leqslant\eta_1,\ n\in N.$$

Hence $\|P_n\|\leqslant C\eta_1^{-n}$. Let $\eta=\frac{1}{2}\eta_1$. Then for $\|x\|\leqslant\eta$

$$\|P_n(x)\|\leqslant C(\tfrac{1}{2})^n$$

and so $\sum P_n(x)$ converges uniformly.

Now if $U\subset X$ is an open set we say that a map $F\colon U\to X$ is *holomorphic* if for every $x_0\in U$, there exists a power series $\sum_{n=0}^{\infty} P_n$ and $\delta>0$ so that if $\|\xi\|<\delta$ then $x_0+\xi\in U$ and

$$F(x_0+\xi)=\sum_{n=0}^{\infty}P_n(\xi).$$

In the notation of [6], $P_0(\xi)=F(x_0)$ and

$$P_n(\xi)=\frac{d^nF(x_0)}{n!}(\xi).$$

We now return to the consideration of power series. If $\sum P_n$ is a power series we set

$$\varrho_n(x)=\sup_{k\geqslant n}\|P_k(x)\|^{1/k}$$

and then let ϱ_n^* be the upper-semicontinuous regularization of ϱ_n, i.e.

$$\varrho_n^*(x)=\limsup_{y\to x}\varrho_n(y).$$

Let $\varrho(x)=\lim_{n\to\infty}\varrho_n^*(x)$. Then ϱ is upper-semicontinuous and $\sum P_n$ converges on the open set $\{x\colon \varrho(x)<1\}$. We call this set the *domain of convergence* of the series $\sum P_n$.

THEOREM 7.3. *Let* $F(x)=\sum_{n=0}^{\infty}P_n(x)$ *for* $\varrho(x)<1$. *Then* F *is holomorphic on* $\{x\colon \varrho(x)<1\}$.

Proof. Let K be a compact subset of $V=\{x\colon \varrho(x)<1\}$. Then since $\varrho(x)<1$ on K, there exists, by Dini's theorem, N so that if $x\in K$ then

$$\varrho_N^*(x)\leqslant\alpha<1.$$

Thus

$$\|P_k(x)\|\leqslant\alpha^k,\quad k\geqslant N,$$

and $\sum P_k$ converges uniformly on K.

Now suppose $x_0\in V$. Pick $\delta>0$ so that if $\|\xi\|\leqslant\delta$ then $x_0+\xi\in V$. Now suppose $\|\xi\|<\delta$. Then $\xi=\alpha\xi_1$ where $\|\xi_1\|=\delta$ and $\alpha=\|\xi\|/\delta$. We define

$$f_n(z)=\sum_{k=0}^{n}P_k(x_0+z\xi_1)$$

for $z\in C$. Then f_n converges uniformly on $|z|\leqslant 1$ to f where $f(z)=F(x_0+z\xi_1)$. Each f_n is a polynomial and hence $f\in A_0(Y)$ ([14], Theorem 6.3). Furthermore for each m

$$\lim_{n\to\infty}f_n^{(m)}(0)=f^{(m)}(0)$$

([12], Theorem 6.1).

Now for fixed $m\leqslant k$, let $Q_{k,m}$ be the homogeneous polynomial of degree m,

$$Q_{k,m}(x)=\frac{k!}{(k-m)!}\hat{P}_k(x_0,\ldots,x,\ldots,x)$$

where x_0 is repeated $k-m$ times and x is repeated m times. Then

$$f_n^{(m)}(0)=\sum_{m\leqslant k\leqslant n}Q_{k,m}(\xi_1).$$

We conclude that

$$\sum_{k=m}^{\infty}Q_{k,m}(\xi_1)=f^{(m)}(0).$$

In particular the sequence of polynomials (each of degree m) $\sum_{m\leqslant k\leqslant n}Q_{k,m}$ is pointwise convergent. By Lemma 7.1

$$\sup_n\Big\|\sum_{m\leqslant k\leqslant n}Q_{k,m}\Big\|<\infty$$

and hence if we set $Q_m(\xi)=\sum_{k=m}^{\infty}Q_{k,m}(\xi)$ then Q_m is a (continuous) polynomial of degree m. Now

$$F(x_0+\xi)=f(\alpha)=\sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}\alpha^m=\sum_{m=0}^{\infty}\frac{Q_m(\xi_1)}{m!}\alpha^m$$
$$=\sum_{m=0}^{\infty}\frac{Q_m(\xi)}{m!}$$

provided $\|\xi\| < \delta$. Thus in particular

$$\limsup_{m\to\infty} \|Q_m(\xi)/m!\|^{1/m} \leqslant 1$$

if $\|\xi\| < \delta$, so that $\sum (1/m!)\,Q_m$ is a power series and F is holomorphic on V.

Now we come to our main results.

THEOREM 7.4. (i) *There exists $\gamma > 0$ so that*

$$\varrho(x) \leqslant \gamma \|x\|_A, \qquad x \in X.$$

(ii) *ϱ is plurisubharmonic.*

Proof. By Lemma 7.2 we have

$$\|P_n\| \leqslant C_0 \gamma^n$$

for some constants C_0 and γ.

Now let $f \in A_0(X)$, and suppose $0 < r < 1$. Then

$$\|P_n(f(0))\| \leqslant C(r) \max_{r \leqslant |z| \leqslant 1} \|P_n(f(z))\|$$

where $C = C(r, Y)$ is determined by Theorem 5.2. Thus

$$\|P_n(f(0))\|^{1/n} \leqslant C(r)^{1/n} \max_{r \leqslant |z| \leqslant 1} \|P_n(f(z))\|^{1/n}$$

and hence

$$\varrho_n(f(0)) \leqslant C(r)^{1/n} \sup_{r \leqslant |z| \leqslant 1} \varrho_n(f(z)).$$

If $y_k \to 0$, there exist z_k, $r \leqslant |z_k| \leqslant 1$,

$$\varrho_n(f(0)+y_k) \leqslant C(r)^{1/n}\big(\varrho_n(f(z_k)+y_k)+2^{-k}\big).$$

If $\lim_{k\to\infty} \varrho_n(f(0)+y_k) = \varrho_n^*(f(0))$ we may pass to a subsequence so that $z_k \to z$ and deduce that

$$\varrho_n^*(f(0)) \leqslant C(r)^{1/n} \max_{r \leqslant |z| \leqslant 1} \varrho_n^*(f(z)).$$

Thus

$$\varrho(f(0)) \leqslant C(r)^{1/n} \max_{r \leqslant |z| \leqslant 1} \varrho_n^*(f(z)).$$

Using the upper semicontinuity of each ϱ_n^* we deduce

$$\varrho(f(0)) \leqslant \max_{r \leqslant |z| \leqslant 1} \varrho(f(z)),$$

and hence letting $r \to 1$

$$\varrho(f(0)) \leqslant \max_{|z|=1} \varrho(f(z)).$$

Thus ϱ is plurisubharmonic. Note that $\varrho_n(x) \leqslant C_0^{1/n}\gamma\|x\|$ and hence $\varrho(x) \leqslant \gamma\|x\|$. Thus

$$\varrho(x) \leqslant \gamma \|x\|_A.$$

COROLLARY 7.5. *Let X be A-trivial and suppose U is a connected open subset of X. Let $F\colon U \to Y$ be a holomorphic function. Then there is an entire holomorphic function $G\colon X \to Y$ so that $G(x) = F(x)$ for $x \in U$.*

Proof. Fix $x_0 \in U$. Let $P_n = (1/n!)\,d^n F(x_0)$ $(n \geqslant 0)$. Then the power series $\sum P_n(\xi)$ converges for all $\xi \in X$ by Theorem 7.4.

Set $G(x) = \sum_{n=0}^{\infty} P_n(x-x_0)$. Then by Theorem 7.3, G is an entire holomorphic function. Let $H(x) = F(x) - G(x)$ for $x \in U$. Let W be the set of $x \in U$ so that $H(x) = 0$ on a neighbourhood of x. Then W is open in U. Let x_1 be a boundary point of W. Pick $\delta > 0$ so that if $\|x - x_1\| < \delta$ then $x \in U$. Pick $y \in W$ with $\|x_1 - y\| < \delta$ and define

$$f(z) = H\big(x_1 + z(y - x_1)\big).$$

Then f is analytic on a disc $(z\colon |z| < 1+\varepsilon)$. Since f vanishes on an open set, we conclude $f \equiv 0$ and hence $H(x_1) = 0$. Thus W is open and closed in U. Since $x_0 \in W$, $W = U$.

Remark. If $X = L_q/H_q$ where $p \leqslant q < 1$ we can go further. Every holomorphic function is a polynomial. In fact if $1/p < n/q - n$ then every homogeneous polynomial of degree n, $P\colon L_q/H_q \to Y$ is identically zero (see [14]).

In a similar vein, if $T\colon L_q/H_q \to A$ is a bounded linear operator and A is a quasi-Banach algebra then there exists $n \in \mathbf{N}$ so that if $x_1, \ldots, x_n \in L_q/H_q$ then

$$(Tx_1)\ldots(Tx_n) = 0.$$

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Received June 18, 1985 (2063)