STUDIA MATHEMATICA, T. LXXXIV. (1986)

Uniform exponential bounds for the normalized empirical process

by

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Abstract. Let $(X, \mathfrak{A}, P)$ be a probability space and $\mathfrak{F}$ a collection of real-valued measurable functions on X. Let $\xi_1, \xi_2, \ldots$ be i.i.d. X-valued random variables with distribution P. Let $P_n := n^{-1}(\delta_{\xi_1} + \ldots + \delta_{\xi_n})$ be the nth empirical measure for P, and let $\nu_n := n^{1/2}(P_n - P)$. Using an entropy condition for $\mathfrak{F}$ we obtain exponential bounds for $\sup_{\mathfrak{F}} |\nu_n(f)|$ which hold uniformly for all $n \geqslant 1$. We show that the entropy condition is essentially the best possible and cannot be significantly weakened. Applications to classes of bounded Lipschitz functions and to $\mathfrak{F} := \{g \cdot 1_C\colon C \in \mathfrak{C} \subset \mathfrak{A},\ g \in L^2(X, \mathfrak{A}, P) \text{ and fixed}\}$ are considered.

§ 1. **Introduction.** Let $(X, \mathfrak{A}, P)$ be a probability space and $\mathfrak{F}$ a collection of real-valued measurable functions on $(X, \mathfrak{A}, P)$. For each $x \in X$ let $F := F_{\mathfrak{F}}(x) := \sup\{|f(x)|\colon f \in \mathfrak{F}\}$. F is called the *envelope function* for $\mathfrak{F}$. Let $(X^\infty, \mathfrak{A}^\infty, P^\infty)$ be a countable product of copies of $(X, \mathfrak{A}, P)$ with coordinates $\xi_j = \xi(j)$ so that the ξ_j are independent identically distributed random variables with values in X and distribution P.

Let $P_n := n^{-1}(\delta_{\xi(1)} + \ldots + \delta_{\xi(n)})$, where δ_x is the unit mass at x, be the nth empirical measure for P. For each $f \in \mathfrak{F}$ let $\nu_n(f) := n^{1/2} \int f\,(dP_n - dP)$; ν_n is called the *normalized empirical process* and this article will be concerned with the suprema of $|\nu_n|$ over the class $\mathfrak{F}$. The main results of this article center around the following concept of entropy for the class $\mathfrak{F}$.

Definition. Given $\mathfrak{F}$, F, and a finite subset $S \subset X$, let

$$N(\delta, S, \mathfrak{F}) := \inf\{m\colon \exists f_1, \ldots, f_m \in \mathfrak{F} \text{ such that } \min_i \sum_{x \in S} (f(x) - f_i(x))^2 < \delta^2 \sum_{x \in S} (F(x))^2 \text{ for every } f \in \mathfrak{F}\}.$$

Let $N(\delta, \mathfrak{F}) := \sup_S N(\delta, S, \mathfrak{F})$ and $\|F\|_{2n} := \{(2n)^{-1} \sum_{i=1}^{2n} F(\xi_i)^2\}^{1/2}$; i.e., $\|F\|_{2n}$ denotes the $L^2(P_{2n})$ seminorm of F.

Key words and phrases: empirical measure, entropy, exponential bounds, envelope function.

1980 *AMS Subject Classification*: Primary 60 F10; Secondary 60 E15.

Note that $N(\delta, \mathfrak{F}) = \inf\{m\colon \forall n$ and $\forall$ values of P_{2n} there are $f_1, \ldots, f_m \in \mathfrak{F}$ such that $\forall f \in \mathfrak{F}\ \exists i \leqslant m$ such that $\int (f-f_i)^2\, dP_{2n} < \delta^2 \|F\|_{2n}^2\}$. As in Pollard [6], $N(\delta, \mathfrak{F})$ will be called the *δ-entropy* of $\mathfrak{F}$ for the $L^2(P_{2n})$ seminorm.

Under "mild restrictions" on the δ-entropy, Theorem 1, our main result, provides exponential bounds for $\sup_{\mathfrak{F}} |\nu_n(f)|$ when $F = 1$; these bounds hold *uniformly* for all $n \geqslant 1$. The uniformity in n is entirely new and is the most significant aspect, especially since it provides possible applications to estimation problems in nonparametric statistics. An example shows that the "mild restrictions" on the δ-entropy are essentially the weakest possible.

Applications of our main result to classes of Lipschitz functions and to $\mathfrak{F} = \{g \cdot 1_C\colon C \in \mathfrak{C} \subset \mathfrak{A},\ g \in L^2(X, \mathfrak{A}, P)$ and fixed$\}$ are considered.

In § 3 we prove the main result. In § 4 we use a slightly different form of δ-entropy, essentially δ-entropy for the $L^1(P_{2n})$ seminorm, and obtain exponential bounds for $\sup_{\mathfrak{F}} |\nu_n(f)|$ for n sufficiently large. For both results the method of proof uses new symmetrization techniques, extending those developed by Pollard [6].

§ 2. **Exponential bounds: the main theorem.** The following is the main result of this article.

THEOREM 1. *Let $\mathfrak{F}$ be a class of real-valued measurable functions on the probability space $(X, \mathfrak{A}, P)$. Assume that $F = 1$ and suppose that there are constants $0 < \delta_0 \leqslant 1$, $0 < \varepsilon < 1$, and $C \geqslant 1$ such that*

$$N(\delta, \mathfrak{F}) \leqslant \exp(C/\delta^{2-\varepsilon}) \qquad \forall \delta,\ 0 < \delta \leqslant \delta_0. \tag{1}$$

Then

$$\Pr\{\sup_{f\in\mathfrak{F}} |\nu_n(f)| > M\} \leqslant 8\ \exp(-M^2/5) \qquad \forall n \geqslant 1, \tag{2}$$

where

$$M \geqslant M(\varepsilon, C, \delta_0) := 2 + \max\bigl(37,\ (5C)^{1/2}(120(6C)^{1/2}/\varepsilon)^{2/\varepsilon},\ 3(C/2)^{1/2}\delta_0^{(-2+\varepsilon)/2}\bigr).$$

Remark. It is the uniformity in n which makes these exponential bounds most significant. Clearly, (2) also provides a bounded law of the iterated logarithm, i.e.,

$$\limsup_{n\to\infty} \sup_{f\in\mathfrak{F}} \frac{|\nu_n(f)|}{(\log\log n)^{1/2}} < \infty \qquad \text{a.s. (Pr).}$$

COROLLARY 2. *Let $\mathfrak{F}$ be as above and let $N(\delta, \mathfrak{F}, \sup) := \inf\{m\colon \exists f_1, \ldots, f_m \in \mathfrak{F}$ such that $\forall f \in \mathfrak{F}\ \exists f_i,\ i \leqslant m$ with $\|f-f_i\|_{\sup} < \delta\}$. Suppose that there are constants $0 < \delta_0 \leqslant 1$, $0 < \varepsilon < 1$, and $C \geqslant 1$ such that (1) holds with $N(\delta, \mathfrak{F})$ replaced by $N(\delta, \mathfrak{F}, \sup)$. Then (2) holds.*

Proof. Immediate. ∎

The following example shows that the δ-entropy condition (1) is essentially the weakest possible; i.e., the exponent for δ cannot be as large as $2+\varepsilon$, $\varepsilon > 0$.

EXAMPLE 2.1. Let $\alpha > 0$, $K > 0$, and let β be the greatest integer $< \alpha$. Let

$$D^p := \partial^{[p]}/\partial x_1^{p_1} \ldots \partial x_d^{p_d}, \qquad [p] := p_1 + \ldots + p_d,$$

for p_i integers $\geqslant 0$, $p = (p_1, \ldots, p_d)$. For a function f on $\boldsymbol{R}^d$ such that $D^p f$ is continuous whenever $[p] \leqslant \beta$, let

$$\|f\|_\alpha := \max_{[p]<\beta} \sup\{|D^p f(x)|\colon x \in \boldsymbol{R}^d\} + \max_{[p]=\beta} \sup_{x\neq y} \{|D^p f(x) - D^p f(y)|/|x-y|^{\alpha-\beta}\},$$

where $|u| := (u_1^2 + \ldots + u_d^2)^{1/2}$, $u \in \boldsymbol{R}^d$. Let I^d be the unit cube $\{x \in \boldsymbol{R}^d\colon 0 \leqslant x_j \leqslant 1,\ j = 1, \ldots, d\}$. As in [3] let $\mathfrak{F}_{d,\alpha,K} := \{f$ on $I^a\colon \|f\|_\alpha \leqslant K$, $\alpha = q + r$, $0 < r \leqslant 1$, q some integer$\}$. Let $N_{[]}(\delta, \mathfrak{F}, \sup) := \inf\{m\colon \exists f_1, \ldots, f_m \in \mathfrak{F}$: for all $f \in \mathfrak{F}$ there are i, j: $f_i \leqslant f \leqslant f_j$ and $\|f_j - f_i\|_{\sup} < \delta\}$. In [5] bounds on δ-entropy in the sup norm are established and from this it follows immediately that there are constants $m_{\alpha,d}$ and $M_{\alpha,d}$ such that

$$m_{\alpha,d}/\delta^{d/\alpha} \leqslant \log N_{[]}(\delta, \mathfrak{F}_{d,\alpha,K}, \sup) \leqslant M_{\alpha,d}/\delta^{d/\alpha}.$$

Choose $K = d = 1$. Given $\varepsilon > 0$, choose $\alpha = (2+\varepsilon)^{-1}$. Then $d/\alpha = 2+\varepsilon$; i.e., the exponent for δ in Theorem 1 and Corollary 2 is $2+\varepsilon$.

However, Theorem 1 in [2] implies that there is a $\gamma = \gamma(1, \alpha) > 0$ such that for all possible values of P_n,

$$\sup_{\mathfrak{F}_{1,\alpha,1}} \{\textstyle\int f\, d(P_n - P)\} \geqslant \gamma n^{-\alpha/d}, \text{ and}$$

$$\sup_{\mathfrak{F}_{1,\alpha,1}} \{\textstyle\int f\, d\nu_n\colon \|f\|_\alpha \leqslant 1\} \geqslant \gamma n^{-\alpha+1/2} \to \infty,$$

completing the example.

We now give examples of classes of functions satisfying the δ-entropy hypothesis (1).

EXAMPLE 2.2. Let $\mathfrak{F} := \{$all bounded Lipschitz functions f on $[0, 1]$ satisfying $\|f\|_{\sup} \leqslant 1$ and $|f(x) - f(y)| \leqslant |x-y|\}$. Then

$$N(\delta, \mathfrak{F}) \leqslant (2/\delta + 1)\, 3^{1+1/\delta},$$

which may be seen as follows. Consider a grill on $[0, 1] \times [-1, 1]$ with grill width δ. Considering the intersection points of the verticals and the horizontals, construct the class $\mathfrak{F}_\delta$ of piecewise linear functions f passing through the intersection points, linear between these points, and satisfying the Lipschitz condition $|f(x)-f(y)| \leqslant |x-y|$. The cardinality of $\mathfrak{F}_\delta$ cannot exceed $(2/\delta+1)3^{1+1/\delta}$. Clearly, for any $f \in \mathfrak{F}$ there is an $f_i \in \mathfrak{F}_\delta$ such that $\|f-f_i\|_{\sup} < \delta$ for some $i \leqslant (2/\delta+1)3^{1+1/\delta}$.

EXAMPLE 2.3. For this example we first recall

DEFINITION. Given a class $\mathfrak{C}$ of subsets of a set X and a finite set $T \subset X$, let $\Delta^{\mathfrak{C}}(T)$ be the number of different sets $T \cap C$ for $C \in \mathfrak{C}$. For $n = 1, 2, \ldots$ let $m^{\mathfrak{C}}(n) := \max\{\Delta^{\mathfrak{C}}(T)\colon T \text{ has } n \text{ elements}\}$. Let

$$v := v(\mathfrak{C}) := \begin{cases} \inf\{n\colon m^{\mathfrak{C}}(n) < 2^n\} \\ +\infty \quad \text{if } m^{\mathfrak{C}}(n) = 2^n \quad \forall n. \end{cases}$$

If $v < \infty$ then $\mathfrak{C}$ is called a *Vapnik–Červonenkis* class (VCC).

Let $g \in L^2(X, \mathfrak{A}, P)$, $\mathfrak{C}$ a VCC, and $\mathfrak{F} := \{g \cdot 1_C\colon C \in \mathfrak{C}\}$. As shown in [6], $N(\delta, \mathfrak{F}) \leqslant A\delta^{-W}$ where A and W are constants depending only upon v. If $\|g\|_2 =: L$ and if $\|g\|_{\sup} =: D < \infty$ then a slight modification of Theorem 1 gives

$$\Pr\{\sup_{\mathfrak{F}} |\nu_n(f)| > M+2L\} \leqslant K \exp\{-(M+2)^2/5D^2\},$$

for all $n \geqslant 1$ for $M \geqslant M_0$, where K and M_0 are constants depending only upon v and D.

§ 3. Proof of Theorem 1. The techniques of the proof center around those developed by Pollard [6] and generalize those used by Alexander [1] for the case when $\mathfrak{F}$ is a class of sets. It will be convenient to use the following method of randomization.

Given $n = 1, 2, \ldots$ and $\xi_1, \ldots, \xi_{2n}$ as the coordinates on $(X^{2n}, \mathfrak{A}^{2n}, P^{2n})$, let $\sigma(1), \ldots, \sigma(n)$ be random variables independent of each other and the ξ_i with $Q(\sigma(i) = 2i) = Q(\sigma(i) = 2i-1) = 1/2$, $i = 1, 2, \ldots, n$. Let $\tau(i) = 2i$ if $\sigma(i) = 2i-1$ and $\tau(i) = 2i-1$ otherwise. Let $\xi(i) = \xi_i$. Then the $\xi(\sigma(j))$ are i.i.d. P. Let

$$P'_n := n^{-1} \sum_{j=1}^{n} \delta_{\xi(\sigma(j))} \quad \text{and} \quad P''_n := n^{-1} \sum_{j=1}^{n} \delta_{\xi(\tau(j))}.$$

Finally, define $\nu'_n := n^{1/2}(P'_n - P)$, $\nu''_n := n^{1/2}(P''_n - P)$ and $\nu^0_n := \nu'_n - \nu''_n$. Note that ν'_n and ν''_n are two independent copies of ν_n and that ν^0_n is the symmetrized empirical process. Let $\Pr := P^\infty \times Q^\infty$ and $\boldsymbol{P} := P^\infty$. Throughout we shall assume that $\sup_{\mathfrak{F}} |\nu^0_n(f)|$ and $\sup_{\mathfrak{F}} |\nu''_n(f)|$ are measurable. Before presenting the proof, we collect a lemma and two facts. The following is adapted from [6], Lemma 2.3.

LEMMA 3.1. *For all $M > 0$ we have*

$$\Pr\{\sup_{\mathfrak{F}} |\nu''_n(f)| > M+2\} \leqslant \tfrac{4}{3} \boldsymbol{P}\{\sup_{\mathfrak{F}} |\nu^0_n(f)| > M\}.$$

Proof. For $x \in X^\infty$, let $\omega_1(x) := (x_1, \ldots, x_n)$ and $\omega_2(x) := (x_{n+1}, \ldots, x_{2n})$. By Chebyshev's inequality and since $E(\nu''_n(f))^2 = \int f^2\,dP - (\int f\,dP)^2 \leqslant 1$, we have $\boldsymbol{P}\{|\nu''_n(f)| < 2\} \geqslant 3/4$ for all f in $\mathfrak{F}$. Therefore,

$$\begin{aligned} \boldsymbol{P}\{\sup |\nu^0_n(f)| > M\} &= \int_{X^{2n}} P^{2n}\{\sup |\nu^0_n(f)| > M \,|\, \sigma(\xi_1, \ldots, \xi_n)\}\, dP^{2n} \\ &= \int_{X^n} \int_{X^n} 1_{\sup|\nu^0_n(f)| > M}\, dP^n(\omega_1)\, dP^n(\omega_2). \end{aligned}$$

Suppose $\omega_2 \in \{|\nu''_n(f)| > M+2\}$ and $f \in \mathfrak{F}$ is arbitrary. Then

$$\int_{X^n} 1_{\sup|\nu^0_n(f)| > M}\, dP^n(\omega_1) \geqslant \int_{X^n} 1_{|\nu_n(f)| < 2}\, dP^n(\omega_1) \geqslant 3/4.$$

Since f is arbitrary, this implies that

$$\boldsymbol{P}\{\sup |\nu^0_n(f)| > M\} \leqslant \int_{\sup|\nu''_n(f)| > M+2} \tfrac{3}{4}\, dP^n(\omega_2),$$

and the lemma follows. ∎

FACT 3.2. If $\gamma = 9/8$ and if $M \geqslant 37$ then

$$M^2/4\gamma - (M-1)^2/2 \leqslant -(M+2)^2/4\gamma \quad \text{and} \quad M^2/4\gamma \geqslant (M+2)^2/5.$$

FACT 3.3. Let γ be as above and $r := r(M) := [(2-\varepsilon)^{-1} \log_2 M^2/4\gamma C]$ where $[\cdot]$ denotes integer part. If $M \geqslant M(\varepsilon, C, \delta_0)$ then

$$(1-2^{-\varepsilon/2})^{-1} 2^{-\varepsilon(r+1)/2} (216C)^{1/2} < 1 \quad \text{and} \quad 2^{-r} \leqslant \delta_0.$$

DEFINITION. Let $\eta_j > 0$ be such that $\eta_j^2 = 2^{-\varepsilon j}(216C)$ and let $m_j := N(2^{-j}, \mathfrak{F}) \leqslant \exp(C2^{(2-\varepsilon)j})$.

We are now ready for the

Proof of Theorem 1. Suppose that we are given the realization of $\xi_1, \ldots, \xi_{2n}$; i.e., we are given the values $\langle x_1, \ldots, x_{2n}\rangle$ which we will call S. Then $\forall j \geqslant 1$ we may find $\mathfrak{F}_j := \{f_{j1}, \ldots, f_{jm_j}\}$ such that

$$\min_{i \leqslant m_j} \left(\int (f-f_{ji})^2\, dP_{2n}\right)^{1/2} \leqslant 2^{-j} \tag{3}$$

for all $f \in \mathfrak{F}$. From now on consider any M such that $M \geqslant M(\varepsilon, C, \delta_0)$. Fact 3.3 and the definition of r and η_j show that $\sum_{j=r+1}^{\infty} \eta_j < 1$.

Suppose that $|\nu^0_n(f)| > M$ for some $f \in \mathfrak{F}$. Then for any such f denote by $f_j(S)$ a function $f_{ji} \in \mathfrak{F}_j$ for which the LHS of (3) achieves its minimum. For any integer s

$$\nu^0_n(f) - \nu^0_n(f_s(S)) = \sum_{j=s+1}^{\infty} [\nu^0_n(f_j(S)) - \nu^0_n(f_{j-1}(S))].$$

From now on, suppress the S in $f_k(S)$ and just write f_k. Using $\sum_{j=r+1}^{\infty} \eta_j < 1$ it follows that either $|v_n^0(f_j)-v_n^0(f_{j-1})| > \eta_j$ for some $j > r$ or $|v_n^0(f_r)| > M-1$.

Using the standard chaining arguments, it follows for our fixed S that

$$(4)\qquad Q^{\infty}\{\sup|v_n^0(f)| > M\} \leqslant m_r \max_{i\leqslant m_r} Q^{\infty}\{|v_n^0(f_{ri})| > M-1\}$$
$$+\sum_{j=r+1}^{\infty} m_j m_{j-1} \max_{i\leqslant m_j, k\leqslant m_{j-1}}{}' \; Q^{\infty}\{|v_n^0(f_{ji}-f_{(j-1)k})| > \eta_j\},$$

where max′ denotes max subject to $\|f_{ji}-f_{(j-1)k}\|_{2n}^2 \leqslant 9\cdot 2^{-2j}$.

Now $v_n^0(f_{ji}-f_{(j-1)k})$ can be written as $n^{-1/2}\sum_{\lambda=1}^{n} h_\lambda$ where

$$h_\lambda := (f_{ji}-f_{(j-1)k})(\xi_{2\lambda})-(f_{ji}-f_{(j-1)k}(\xi_{2\lambda-1}).$$

By Theorem 2 of Hoeffding [4],

$$Q^{\infty}\{|v_n^0(f_{ji}-f_{(j-1)k}| > \eta_j\} \leqslant 2\exp\big(-2n\eta_j^2(4\sum_{\lambda=1}^{n} h_\lambda^2)^{-1}\big).$$

By (3) we have $\sum_{\lambda=1}^{n} h_\lambda^2 \leqslant 36\, n\, 2^{-2j}$ and therefore the second term on the RHS of (4) is

$$\leqslant 2\sum_{j=r+1}^{\infty} \exp(2C\, 2^{(2-\varepsilon)j})\exp(-\eta_j^2\, 2^{2j}/72)$$
$$= 2\sum_{j=r+1}^{\infty} \exp(-C\, 2^{(2-\varepsilon)j}).$$

Applying Hoeffding [4] to the first term on the RHS of (4) we get

$$Q^{\infty}\{|v_n^0(f_{ri})| > M-1\} \leqslant 2\exp(-(M-1)^2/2),$$

which holds for all i, $i\leqslant m_r$. Now $m_r \leqslant \exp(C2^{(2-\varepsilon)r}) \leqslant \exp(M^2/4\gamma)$ by the way r was chosen. Using Fact 3.2, the first term on the RHS of (4) is thus bounded by $2\exp(-(M+2)^2/5)$. The second term is bounded by

$$2\sum_{j=r+1}^{\infty} \exp(-C\, 2^{(2-\varepsilon)j}) \leqslant 4\exp(-C2^{(2-\varepsilon)(r+1)}) \leqslant 4\exp(-M^2/4\gamma)$$
$$\leqslant 4\exp(-(M+2)^2/5).$$

So on S we have $Q^{\infty}\{\sup|v_n^0(f)| > M\} \leqslant 6\exp(-(M+2)^2/5)$. Integrating over X^{2n} and applying Lemma 3.1 gives the desired result. ■

§ 4. **A variation of the main result.** If the metric entropy condition (1) holds for all values of P_{2n} except those in a set A_{2n} with $P^{\infty}(A_{2n}) = p_n$, then

(2) takes the form

$$(5)\qquad \Pr\{\sup_{\mathfrak{F}}|v_n(f)| > M\} \leqslant 8\exp(-M^2/5)+4p_n/3.$$

The importance of (5) lies in the fact that if p_n satisfies $\sum_{k=1}^{\infty} p_{2^k} < \infty$, then the standard subsequence techniques show that $\sup_{\mathfrak{F}}|v_n(f)|$ satisfies a bounded law of the iterated logarithm. The following theorem, which uses an $L^1(P_{2n})$ seminorm to define δ-entropy, embodies the above ideas.

THEOREM 3. *Let $(X, \mathfrak{A}, P)$ be a probability space and $\mathfrak{F}$ a class of real-valued functions on $(X, \mathfrak{A}, P)$. Assume that $\mathfrak{F}$ has envelope $F = 1$. Given n, let $j(n) := [(\log_2 n)/2]+2$. Suppose that for all values of P_{2n} (except those in a set A_{2n} with $P^{\infty}(A_{2n}) =: p_n \downarrow 0$) and for all j, $1\leqslant j\leqslant j(n)$, there exist functions $f_{j1}, \ldots, f_{jm} \in \mathfrak{F}$, $m = m(j)$, such that for all $f\in\mathfrak{F}$ there is an $i\leqslant m$ such that*

$$\int |f-f_{ji}|\,dP_{2n} < 2^{-j}.$$

Assume for some ε, $0<\varepsilon<1/2$, and $C\geqslant 1$ that

$$m = m(j) \leqslant \exp(C2^{(1-\varepsilon)j}),\qquad j = 1, 2, \ldots$$

If $M \geqslant M(\varepsilon, C) := 2+\max\big(37, (5C)^{1/2}(480(C)^{1/2}/\varepsilon)^{1/\varepsilon}\big)$, then for all n sufficiently large

$$\Pr\{\sup|v_n(f)| > M\} \leqslant 8\exp(-M^2/5)+\tfrac{4}{3}p_n.$$

Proof. Define v_n', v_n'', v_n^0, Pr, and $\boldsymbol{P}$ as in the proof of Theorem 1. We will need the analog of Fact 3.3:

FACT 4.1. Let $\gamma = 9/8$ and $r := r(M) := [(1-\varepsilon)^{-1}\log_2 M^2/4\gamma C]$. If $M \geqslant M(\varepsilon, C)$ then

$$(1-2^{-\varepsilon/2})^{-1}\, 2^{-\varepsilon(r+1)/2}\, 12(C)^{1/2} < 1/2.$$

DEFINITION. Let $\eta_j > 0$ be such that $\eta_j^2 = 2^{-\varepsilon j}\, 144\, C$ and let $m_j := m(j)$.

With these preliminaries we proceed with the proof, following the general method of the proof of Theorem 1.

For $\omega\in A_{2n}^c$ suppose that we are given the realization of $\xi_1, \ldots, \xi_{2n}$, i.e., we are given $\langle x_1, \ldots, x_{2n}\rangle$, which we call S. Then for all j, $1\leqslant j\leqslant j(n)$, find $\mathfrak{F}_j := \{f_{j1}, \ldots, f_{jm_j}\}$ such that

$$(6)\qquad \min_{i\leqslant m_j} \int |f-f_{ji}|\,dP_{2n} < 2^{-j}$$

for all $f\in\mathfrak{F}$. From now on consider any M such that $M\geqslant M(\varepsilon, C)$. Fact 4.1 and the definition of r and η_j imply that $\sum_{j=r+1}^{\infty}\eta_j < 1/2$.

Henceforth, as we still have $\omega\in A_{2n}^c$, all equations will hold except with probability $P^{\infty}(A_{2n}) =: p_n$.

Suppose that $|\nu_n^0(f)| > M$ for some $f \in \mathfrak{F}$. Then for any such f denote by $f_j(S)$ the function $f_{ji} \in \mathfrak{F}_j$ for which the LHS of (6) achieves its minimum. Notice that for any fixed integer s, $s < j(n)$,

$$\nu_n^0(f) - \nu_n^0(f_s(S)) = \sum_{j=s+1}^{j(n)} [\nu_n^0(f_j(S)) - \nu_n^0(f_{j-1}(S))] + \tau(S),$$

where $|\tau(S)| \leqslant |\nu_n^0(f - f_{j(n)}(S))| \leqslant n^{1/2} 2^{-j(n)} \leqslant 1/2$, by definition of $j(n)$. From now on, suppress the S in $f_k(S)$ and just write f_k.

Using $\sum_{j=r+1}^{\infty} \eta_j < 1/2$, it follows that either $|\nu_n^0(f_r)| > M-1$ or that there is a $j > r$ such that $|\nu_n^0(f_j - f_{j-1})| > \eta_j$.

Using a chaining argument similar to that in the proof of Theorem 1, it may be shown (with the help of Facts 3.2 and 4.1 and Hoeffding's inequality) for the fixed S and for $\omega \in A_{2n}^c$ that

$$Q^\infty \{\sup |\nu_n^0(f)| > M\} \leqslant 6 \exp\left(-(M+2)^2/5\right).$$

Combining this with Lemma 3.1 and replacing $M+2$ by M gives the desired result. ■

Acknowledgement. I would like to thank my thesis advisor, Professor R. Dudley, for his guidance and help in preparing this article, originally a part of my Ph.D. thesis.

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Received January 14, 1985 (2027)

STUDIA MATHEMATICA, T. LXXXIV. (1986)

The Sobolev spaces of harmonic functions

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Abstract. Let H^s be the subspace of harmonic functions in the Sobolev space $W^s(D)$ for a smooth bounded domain D in $\boldsymbol{R}^n$. In this paper we prove that the dual of H^s can be represented as the space $L^2H(D, \varrho^{2s})$ of harmonic functions which are square integrable on D with weight ϱ^{2s}, where ϱ is a defining function for D, i.e. $\varrho(x) = -\operatorname{dist}(x, \partial D) e^{h(x)}$, $h \in C^\infty(\bar{D})$.

I. Introduction and statement of results. Let D be a bounded domain in $\boldsymbol{R}^n$ defined by $D = \{x \in \boldsymbol{R}^n\colon \varrho(x) < 0\}$, where $\varrho \in C^\infty(\boldsymbol{R}^n)$ and grad $\varrho \neq 0$ on ∂D. Denote by $W^s(D)$, $s \geqslant 0$, the usual Sobolev space on D. Now we shall recall the definition of the negative Sobolev spaces. Let $\mathring{W}^s(D)$ denote the closure of $C_0^\infty(D)$ in $W^s(D)$. The negative Sobolev space $W^{-s}(D)$ is the completion of $L^2(D)$ with respect to the norm

$$\|u\|_{-s} = \sup_{\substack{v \in \mathring{W}^s(D) \\ \|v\|_s \leqslant 1}} |\langle u, v\rangle|.$$

The space $W^{-s}(D)$ is a representation of the space dual to $\mathring{W}^s(D)$ via the $L^2(D)$-scalar product $\langle\ ,\ \rangle$. The expression "a function f vanishes on ∂D up to order k" means that f and all its derivatives $D^\alpha f$, $|\alpha| \leqslant k$, are identically zero on ∂D. Note that a function $f \in C^\infty(\bar{D})$ belongs to $\mathring{W}^s(D)$ iff f vanishes on ∂D up to order $s-1$. For each integer s, let $H^s(D)$ denote the space of harmonic functions belonging to the Sobolev space $W^s(D)$. For each s, $H^s(D)$ is a closed subspace of $W^s(D)$. We shall also denote by $H^\infty(\bar{D})$ the subspace of $C^\infty(\bar{D})$ consisting of harmonic functions. We shall prove the following.

THEOREM 1. *Let $T_k(u) = \varrho^k u$ where k is a positive integer. Then for each integer s, $-\infty < s < \infty$, the operator T_k maps continuously $H^s(D)$ into $W^{s+k}(D)$.*

In [2] S. Bell constructed a family of operators $L^s\colon C^\infty(\bar{D}) \to C^\infty(\bar{D})$ such that for every $s > 0$ and $f \in C^\infty(\bar{D})$, $L^s f$ vanishes up to order $s-1$ on ∂D and $f - L^s f \perp H^0(D)$ in $L^2(D)$. Bell used these operators to construct a nondegenerate sesquilinear pairing

$$\langle f, g\rangle_0 = \int_D L^s f \cdot \bar{g} \quad \text{for } f \in H^\infty \text{ and } g \in H^{-s} \subset H^{-\infty} = \lim_{s\to\infty} \operatorname{ind} H^{-s}.$$