Suppose that $|\nu_n^0(f)| > M$ for some $f \in \mathfrak{F}$. Then for any such f denote by $f_j(S)$ the function $f_{ji} \in \mathfrak{F}_j$ for which the LHS of (6) achieves its minimum. Notice that for any fixed integer s, $s < j(n)$,

$$\nu_n^0(f) - \nu_n^0(f_s(S)) = \sum_{j=s+1}^{j(n)} [\nu_n^0(f_j(S)) - \nu_n^0(f_{j-1}(S))] + \tau(S),$$

where $|\tau(S)| \leqslant |\nu_n^0(f - f_{j(n)}(S))| \leqslant n^{1/2} 2^{-j(n)} \leqslant 1/2$, by definition of $j(n)$. From now on, suppress the S in $f_k(S)$ and just write f_k.

Using $\sum_{j=r+1}^{\infty} \eta_j < 1/2$, it follows that either $|\nu_n^0(f_r)| > M-1$ or that there is a $j > r$ such that $|\nu_n^0(f_j - f_{j-1})| > \eta_j$.

Using a chaining argument similar to that in the proof of Theorem 1, it may be shown (with the help of Facts 3.2 and 4.1 and Hoeffding's inequality) for the fixed S and for $\omega \in A_{2n}^c$ that

$$Q^\infty \{\sup |\nu_n^0(f)| > M\} \leqslant 6 \exp\left(-(M+2)^2/5\right).$$

Combining this with Lemma 3.1 and replacing $M+2$ by M gives the desired result. ■

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The Sobolev spaces of harmonic functions

by

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Abstract. Let H^s be the subspace of harmonic functions in the Sobolev space $W^s(D)$ for a smooth bounded domain D in $\boldsymbol{R}^n$. In this paper we prove that the dual of H^s can be represented as the space $L^2 H(D, \varrho^{2s})$ of harmonic functions which are square integrable on D with weight ϱ^{2s}, where ϱ is a defining function for D, i.e. $\varrho(x) = -\operatorname{dist}(x, \partial D) e^{h(x)}$, $h \in C^\infty(\bar{D})$.

I. Introduction and statement of results. Let D be a bounded domain in $\boldsymbol{R}^n$ defined by $D = \{x \in \boldsymbol{R}^n \colon \varrho(x) < 0\}$, where $\varrho \in C^\infty(\boldsymbol{R}^n)$ and grad $\varrho \neq 0$ on ∂D. Denote by $W^s(D)$, $s \geqslant 0$, the usual Sobolev space on D. Now we shall recall the definition of the negative Sobolev spaces. Let $\mathring{W}^s(D)$ denote the closure of $C_0^\infty(D)$ in $W^s(D)$. The negative Sobolev space $W^{-s}(D)$ is the completion of $L^2(D)$ with respect to the norm

$$\|u\|_{-s} = \sup_{\substack{v \in \mathring{W}^s(D) \\ \|v\|_s \leqslant 1}} |\langle u, v\rangle|.$$

The space $W^{-s}(D)$ is a representation of the space dual to $\mathring{W}^s(D)$ via the $L^2(D)$-scalar product $\langle\ ,\ \rangle$. The expression "a function f vanishes on ∂D up to order k" means that f and all its derivatives $D^\alpha f$, $|\alpha| \leqslant k$, are identically zero on ∂D. Note that a function $f \in C^\infty(\bar{D})$ belongs to $\mathring{W}^s(D)$ iff f vanishes on ∂D up to order $s-1$. For each integer s, let $H^s(D)$ denote the space of harmonic functions belonging to the Sobolev space $W^s(D)$. For each s, $H^s(D)$ is a closed subspace of $W^s(D)$. We shall also denote by $H^\infty(\bar{D})$ the subspace of $C^\infty(\bar{D})$ consisting of harmonic functions. We shall prove the following.

THEOREM 1. *Let $T_k(u) = \varrho^k u$ where k is a positive integer. Then for each integer s, $-\infty < s < \infty$, the operator T_k maps continuously $H^s(D)$ into $W^{s+k}(D)$.*

In [2] S. Bell constructed a family of operators $L^s \colon C^\infty(\bar{D}) \to C^\infty(\bar{D})$ such that for every $s > 0$ and $f \in C^\infty(\bar{D})$, $L^s f$ vanishes up to order $s-1$ on ∂D and $f - L^s f \perp H^0(D)$ in $L^2(D)$. Bell used these operators to construct a nondegenerate sesquilinear pairing

$$\langle f, g\rangle_0 = \int_D L^s f \cdot \bar{g} \quad \text{for } f \in H^\infty \text{ and } g \in H^{-s} \subset H^{-\infty} = \lim_{s \to \infty} \operatorname{ind} H^{-s}.$$

He proved that H^∞ and $H^{-\infty}$ are mutually dual via this pairing. (Note that if $g \in H^0(D)$ then $\langle f, g\rangle_0 = \langle f, g\rangle$). The statement of Theorem 1 for $s > 0$ and the construction of L^s permit us to prove that for every $s > 0$, L^s maps continuously H^s into $\mathring{W}^s(D)$. In fact, L^s is an isomorphic imbedding of H^s into $\mathring{W}^s$. This implies the following

THEOREM 2. *For each integer $s > 0$ the spaces $H^s(D)$ and $H^{-s}(D)$ are mutually dual via the nondegenerate sesquilinear pairing*

$$\langle f, g\rangle_0 = \int L^s f \cdot \bar{g}, \quad f \in H^s(D),\ g \in H^{-s}(D).$$

Boas in [5] proved that $\|\varrho^s h\|_{L^2(D)} \leqslant c\|h\|_{-s}$ for holomorphic functions h. This estimate can be proved in the same way for harmonic h; it permits us to prove Theorem 1 for $s < 0$ and the following:

THEOREM 3. *For integer $s > 0$, the space $H^{-s}(D)$ is equal to the space $L^2 H(D, \varrho^{2s})$ and the norm $\|\ \|_{-s}$ is equivalent to the norm $\|\ \|_{L^2 H(D,\varrho^{2s})}$; thus we have the representation of the space $(H^s)^*$ as the space $L^2 H(D, \varrho^{2s})$ with an equivalent norm.*

Bell's duality theory was originally invented to study the Sobolev spaces of holomorphic functions in $\boldsymbol{C}^n$ (see [3]). We shall now outline the similarities and differences between these two cases.

Let D be a bounded smooth domain in $\boldsymbol{C}^n$. Denote by $\mathrm{Hol}^s(D)$ the closed subspace of $W^s(D)$ consisting of holomorphic functions, and by $L^2\mathrm{Hol}(D, \varrho^{2s})$ the space of holomorphic functions square-integrable with weight ϱ^{2s}.

It follows immediately from Theorem 3 that for every (integer) $s > 0$

$$\mathrm{Hol}^{-s}(D) = L^2\mathrm{Hol}(D, \varrho^{2s}), \qquad \|\ \|_{-s} = \|\ \|_{L^2\mathrm{Hol}(D,\varrho^{2s})}.$$

The pairing $\langle f, g\rangle_0 = \int L^s f \cdot \bar{g}$ is a sesquilinear pairing between $\mathrm{Hol}^s(D)$ and $\mathrm{Hol}^{-s}(D)$. It is equal to the pairing introduced by S. Bell in [3]. The main difference between the harmonic and the holomorphic case is that $\langle\cdot,\cdot\rangle_0$ can be degenerate or the spaces Hol^s and Hol^{-s} can be not mutually dual via this pairing. Let B denote the Bergman projection from $L^2(D)$ onto $L^2\mathrm{Hol}(D)$ (that means the orthogonal projection from $L^2(D)$ onto $L^2\mathrm{Hol}(D)$). Bell and Boas proved in [4] that $\mathrm{Hol}^\infty(\bar{D})$ and $\mathrm{Hol}^{-\infty}(\bar{D})$ are mutually dual via $\langle\cdot,\cdot\rangle_0$ iff B is continuous from $C^\infty(\bar{D})$ onto $\mathrm{Hol}^\infty(\bar{D})$. (This fact was also proved independently by G. Komatsu [6]). We can now prove the following.

PROPOSITION 1. *Let Hol_0^{-s} denote the closure of $L^2\mathrm{Hol}(D)$ in Hol^{-s}. For every integer $s > 0$ the following conditions on D are equivalent:*

(1) *$\mathrm{Hol}^s(D)$ and $\mathrm{Hol}_0^{-s}(D)$ are mutually dual via the pairing $\langle\ ,\ \rangle_0$.*

(2) *B is a continuous projection from $W^s(D)$ onto $\mathrm{Hol}^s(D)$.*

(3) *B extends to a continuous projection from $H^{-s}(D)$ onto $\mathrm{Hol}_0^{-s}(D)$.*

(4) *There exists $c > 0$ such that for every harmonic function from $L^2(D)$*

$$\|Bu\|_{L^2(D,\varrho^{2s})} \leqslant c\|u\|_{L^2(D,\varrho^{2s})}.$$

Barrett in [1] constructed an example of a bounded smooth domain (not pseudoconvex) for which (2) fails for every $s > 0$.

Note that the orthogonal projection P from $L^2(D)$ onto the space $L^2 H(D)$ of square-integrable harmonic functions is continuous from $W^s(D)$ onto $H^s(D)$ for every smooth bounded D in $\boldsymbol{R}^n$ and each $s > 0$ (see [2]).

II. Proofs.

1) Proof of Theorem 1 for $s > 0$. Let $s \geqslant k$. If $u \in H^s(D)$ then $\Delta^k \varrho^k u \in W^{s-k}$ and there exists a constant c such that $\|\Delta^k \varrho^k u\|_{s-k} \leqslant c\|u\|_s$. Denote by G_k the operator which solves the Dirichlet problem $\Delta^k G_k(f) = f$, $G_k f$ vanishes on ∂D up to order $k-1$. It is well known that for every $s \geqslant 0$, G_k maps continuously W^s into $W^{s+2k} \cap \mathring{W}^k$. ($\mathring{W}^k = \overline{C_0^\infty(D)} \subset W^k$.) Let $F(u) = G_k(\Delta^k(\varrho^k u))$. Then $F(u) \in W^{s+k} \cap \mathring{W}^k$ and $\|F(u)\|_{s+k} \leqslant c\|u\|_s$, $u \in H^s(D)$. For $u \in H^\infty(\bar{D})$ the function $F(u) - \varrho^k u \in C^\infty(\bar{D})$ vanishes on ∂D up to order $k-1$ and $\Delta^k(F(u) - \varrho^k u) = 0$. Thus $F(u) = \varrho^k u$ for $u \in H^\infty(\bar{D})$. The orthogonal projection P from $L^2(D)$ onto $L^2 H(D) = H^0(D)$ is continuous from $W^s(D)$ onto $H^s(D)$ for each $s > 0$, since $Pf = I - \Delta G_2 \Delta f$. This implies that P maps $C^\infty(\bar{D})$ onto $H^\infty(\bar{D})$ and that $H^\infty(\bar{D})$ is dense in $H^s(D)$. The operators $F(u)$ and $T_k(u) = \varrho^k u$ are continuous from $H^s(D)$ into $W^s(D)$ and equal on the dense subset of $H^s(D)$. Thus for every $u \in H^s(D)$, $\varrho^k u = F(u) \in W^{s+k}(D)$ and $\|\varrho^k u\|_{s+k} \leqslant c\|u\|_s$.

Let now $u \in H^0(D)$. We have $\Delta^k \varrho^k u \in W^{-k}(D)$ and $\|\Delta^k \varrho^k u\|_{-k} \leqslant c\|u\|_0$. We shall prove that there exists a continuous operator G_{-k} from W^{-k} to $\mathring{W}^k$ such that $\Delta^k(G_{-k} f) = f$. The scalar product

$$\langle\langle f, g\rangle\rangle_k = \sum_{|\alpha|=k} \langle D^\alpha f, D^\alpha g\rangle$$

defines on $\mathring{W}^k$ a Hilbert norm equivalent to the usual Sobolev norm. Let $f \in W^{-k}(D)$; then $\langle\ , f\rangle_0$ defines a continuous linear functional on $\mathring{W}^k$. By the Riesz theorem there exists an element $G_{-k}(f) \in \mathring{W}^k$ such that for every $g \in \mathring{W}^k$, $\langle g, f\rangle_0 = (-1)^k \langle\langle g, G_{-k}(f)\rangle\rangle_k$. The operator G_{-k} is an isomorphism between W^{-k} and $\mathring{W}^k$. If $g \in C_0^\infty(D)$ then integration by parts implies that $\langle g, f\rangle_0 = \langle\langle \Delta^k g, G_{-k}(f)\rangle\rangle_k$ which means that $\Delta^k G_{-k}(f) = f$.

Let $F(u) = G_k \Delta^k(\varrho^k u)$. We have $Fu \in \mathring{W}^k(D)$ and $\|Fu\|_k < c\|u\|_0$ for each $u \in L^2 H(D)$. The same consideration as above shows that $F(u) = \varrho^k u$ for $u \in L^2 H(D)$. The standard interpolation shows now that $\varrho^k u$ maps $H^s(D)$ into $W^{s+k}(D) \cap \mathring{W}^k(D)$ for $s \geqslant 0$ and

$$\|\varrho^k u\|_{s+k} \leqslant c_{sk}\|u\|_s.$$

2) Proof of Theorem 2. It follows from the proved part of Theorem

1 that if $u\in H^s(D)$ for $s\geqslant 0$ then for every multiindex α, $D^\alpha(u\varrho^{|\alpha|})\in W^s(D)$ and $|D^\alpha(u\varrho^{|\alpha|})|\leqslant C\|u\|_s$ (D^α is the partial derivative corresponding to α). This implies that also $D^\alpha(u)\varrho^{|\alpha|}\in W^s$ and $\|D^\alpha(u)\varrho^{|\alpha|}\|_s\leqslant C\|u\|_s$. Now, let us recall the construction of $L^s u$:

$$L^s u = u-\Delta\Big(\sum_{k=0}^{s-1}\theta_k\varrho^{k+2}\Big),$$

$$\theta_t=\frac{\varphi}{(t+2)!}|\nabla\varrho|^{-2}\Big(\frac{\partial}{\partial\eta}\Big)^t L^t u,\qquad \frac{\partial}{\partial\eta}=\sum_{i=1}^{n}\frac{\frac{\partial\varrho}{\partial x_i}\cdot\frac{\partial}{\partial x_i}}{|\nabla\varrho|^2},$$

$$L^1 u = u-\Delta(\theta_0\varrho^2),\qquad \theta_0=\frac{\varphi u}{2|\nabla\varrho|^2}$$

where φ is an arbitrarily chosen C^∞-function equal to 1 in a neighborhood of ∂D and equal to 0 in a neighborhood of the set $\{\nabla\varrho=0\}$.

The above construction and harmonicity of u imply that for each $r>1$, $L^r u$ consists of terms of the type $h_\beta D^\beta(u)\varrho^{|\alpha|}$ where $0\leqslant|\beta|\leqslant|\alpha|\leqslant r(r+1)/2$. Thus for all $r>1$, $s\geqslant 0$, L^r extends to a continuous operator from $H^s(D)$ into $\mathring{W}^s(D)$. In particular, L^s maps continuously H^s into $\mathring{W}^s$. To prove Theorem 2 it suffices now to repeat the construction from the proof of Theorem 1 from [2]. We shall outline it briefly: since $\|L^s f\|\leqslant C\|f\|_s$, $f\in H^s$, we have for every $f\in H^s$ and $g\in H^{-s}$

$$|\langle f,g\rangle_0| = |\langle L^s f, g\rangle_0|\leqslant c\|f\|_s\|g\|_{-s}.$$

Thus for every $f\in H^s$, $\overline{\langle f,\cdot\rangle}_0\in(H^{-s})^*$ and for every $g\in H^{-s}$, $\langle\cdot,g\rangle_0\in(H^s)^*$. Every continuous functional on H^s has the form $\langle\cdot,v\rangle_s$. If u and v are in $C^\infty(\bar D)$ then

$$\langle u,v\rangle_s=\sum_{|\alpha|\leqslant s}\langle D^\alpha u, D^\alpha v\rangle_0=\langle u, E^s v\rangle_0$$

where $E^s v = P\big(\sum_{|\alpha|\leqslant s}(-1)^{|\alpha|}D^\alpha L^s D^\alpha v\big)$.

The operator E^s extends to a continuous operator from H^s into H^{-s} and $\langle\cdot,v\rangle_s=\langle\cdot,E^s v\rangle_0$ on H^s. Every continuous functional φ on H^{-s} extends by the Hahn–Banach theorem to a continuous functional on W^{-s} and therefore can be represented as $\langle\cdot,\Phi\rangle_0$. Then it can be proved that $\varphi(u)=\langle u,P\Phi\rangle_0$ and $P\Phi_1=P\Phi_2$ for two different representations of φ.

To prove this we shall consider the operator L^{s*} acting from $\mathring{W}^s$ into H^s. We have $\langle L^{s*}u,v\rangle_s=\langle u,L^s v\rangle_s$. There exists an isomorphism F^s of W^{-s} onto $\mathring{W}^s$ such that

$$I+\sum_{k=0}^{s}\Delta^k F^s(u)(-1)^k=u.$$

Thus the operator $L^{s*}F^s$ maps H^{-s} into H^s and for all $h\in H^{-s}$ and $g\in H^s$

$$\langle h,g\rangle_0=\langle h,L^s g\rangle_0=\langle F^s h, L^s g\rangle_s=\langle L^{s*}F^s h, g\rangle_s.$$

Thus $\langle h-E^sL^{s*}F^s h, g\rangle_0=0$ for every $g\in H^s$. Since the pairing $\langle\cdot,\cdot\rangle_0$ is nondegenerate on $H^{-\infty}\times H^\infty$ ([2]) we have $h=E^sL^{s*}F^s h$. This means that E^s is an isomorphism between H^s and H^{-s}. Since $E^s(H^\infty)\subset H^\infty$, it follows that H^∞ is dense in H^{-s} for every $s>0$. Note that it $\varphi(u)$ is a functional on H^{-s} and Φ_1, Φ_2 its two representations in $\mathring{W}^s$ then $\varphi(u)=\langle u,P\Phi_1\rangle_0 = \langle u,P\Phi_2\rangle_0$ for all $u\in H^\infty$. Thus $P\Phi_1=P\Phi_2$ and $\varphi(u)=\langle u,P\Phi_1\rangle_0$ for all $u\in H^{-s}$.

3) Proof that $T_s(u)=\varrho^s u$ maps continuously H^{-s} into $L^2(D)$ (see Boas [5]). Let $u\in H^\infty(D)$. By the Sobolev inequality we have

$$\|\varrho^s u\|_0^2\leqslant\|u\|_{-s}\|\varrho^{2s}u\|_s\leqslant c\|u\|_{-s}\|\Delta^s\varrho^{2s}u\|_{-s}.$$

By the generalized Leibniz formula and the harmonicity of u

$$\begin{aligned}\|\Delta^s\varrho^{2s}u\|_{-s} &= \sup_{\substack{\Phi\in\mathring{W}^s\\ \|\Phi\|_s\leqslant 1}}|\langle\Delta^s\varrho^{2s}u,\Phi\rangle_0| = \sup_{\substack{\Phi\in\mathring{W}^s\\ \|\Phi\|_s\leqslant 1}}\Big|\sum_{|\alpha|\leqslant s}\langle f_\alpha\varrho^{|\alpha|}D^\alpha u,\Phi\rangle_0\Big|\\ &\leqslant\sum_{|\alpha|\leqslant s}\sup_{\|\Phi\|_s\leqslant 1}|\langle D^\alpha(f_\alpha\varrho^{|\alpha|})u,\Phi\rangle_0|\\ &\quad+\sum_{|\alpha|\leqslant s}\sup_{\|\Phi\|_s\leqslant 1}|\langle f_\alpha\varrho^{|\alpha|}u, D^\alpha\Phi\rangle_0|\leqslant c\|\varrho^s u\|_0.\end{aligned}$$

This last inequality follows from the fact that $D^\alpha\Phi\in\mathring{W}^{s-|\alpha|}$, and from the following inequality valid for every $v\in\mathring{W}^r$, $r\geqslant 1$:

$$\|v/\varrho^r\|_0\leqslant c\|v\|_r.$$

This implies that $\|\varrho^s u\|_0\leqslant c\|u\|_{-s}$ for $u\in H^\infty$. Since H^∞ is dense in H^{-s}, this ends the proof.

4) Proof of Theorem 3 and Theorem 1 for $s<0$. In [5] (estimates 2.6) we can find the following estimate: $\|u\|_{-r}\leqslant c\|\varrho^r u\|_0$ for all real $r<0$, $r\neq(2k-1)/2$, and all smooth u. It implies that for each integer $s>0$, $\|\varrho^s u\|_0$ is equivalent to the norm $\|u\|_{-s}$ on H^{-s}. The norm $\|\varrho^s u\|_0$ is equal to the norm $\|u\|_{L^2(D,\varrho^{2s})}$. This implies that $H^{-s}=L^2H(D,\varrho^{2s})$ and Theorem 2 shows that H^s is dual to the space $L^2H(D,\varrho^{2s})$ via the pairing $\langle\ ,\ \rangle_0$. Now we shall return to the proof of Theorem 1. If $s>k$ then for every $u\in H^{-s}$

$$\|\varrho^k u\|_{-s+k}\leqslant c\|\varrho^k u\varrho^{s-k}\|_0=c\|\varrho^s u\|_0\leqslant c_1\|u\|_{-s}.$$

If $k>s$ then for every $u\in H^\infty$, $\varrho^k u\in W^{k-s}$ and

$$\begin{aligned}\|\varrho^k u\|_{k-s}\leqslant c\|\Delta^{k-s}\varrho^k u\|_{-k+s} &= c\sup_{\substack{\Phi\in\mathring{W}^{k-s}\\ \|\Phi\|\leqslant 1}}\Big\langle\sum_{|\alpha|<k-s}f_\alpha\varrho^{s+|\alpha|}D^\alpha u,\Phi\Big\rangle_0\\ &\leqslant c_1\|\varrho^s u\|_0\leqslant c_2\|u\|_{-s}\end{aligned}$$

by the same reason as in the proof in 3). Thus if $k > s$ then $T_k(u) = \varrho^k u$ maps continuously H^{-s} into $\mathring{W}^{k-s}$. This ends the proof of Theorem 1.

5) Proof of Proposition 1.

$(1) \Rightarrow (2)$. Let $u \in W^s$. (1) implies that

$$\|Bu\|_s \leqslant c \sup_{\substack{h \in L^2\mathrm{Hol}(D) \\ \|h\|_{-s} \leqslant 1}} |\langle Bu, h\rangle_0| = c \sup_{\substack{h \in L^2\mathrm{Hol}(D) \\ \|h\|_{-s} \leqslant 1}} |\langle u, h\rangle_0| \leqslant c\,\|u\|_s$$

by the Sobolev inequality.

$(2) \Rightarrow (3)$. Let $h \in H^{-s} \cap L^2(D)$. Then

$$\|Bh\|_{-s} \leqslant c \sup_{\substack{u \in H^s \\ \|u\|_s \leqslant 1}} |\langle Bh, u\rangle_0| = c \sup_{\substack{u \in H^s \\ \|u\|_s \leqslant 1}} |\langle h, Bu\rangle_0| \leqslant c \sup_{\substack{u \in H^s \\ \|u\|_s \leqslant 1}} \|h\|_{-s}\|Bu\|_s \leqslant c_1 \|h\|_{-s}$$

since B is continuous from H^s onto Hol^s. H^∞ is dense in H^{-s} for each s (see [2]) and thus B extends to a continuous projection from H^{-s} onto Hol_0^{-s}.

$(3) \Rightarrow (2)$. For every $f \in W^s$, $Pf \in H^s$ and $Bf = BPf$. Thus, it suffices to prove that B is a continuous projection from H^s onto Hol^s.

Let $h \in H^s$. Then

$$\|Bh\|_s \leqslant c \sup_{\substack{u \in H^{-s} \\ \|u\|_{-s} \leqslant 1}} |\langle Bh, u\rangle_0| \leqslant c \sup_{\substack{u \in H^{-s} \\ \|u\|_{-s} \leqslant 1}} \|h\|_s\|Bu\|_{-s} \leqslant c_1 \|h\|_s.$$

(2) and $(3) \Rightarrow (1)$. We have $\mathrm{Hol}_0^{-s} = B(H^{-s})$. Every continuous functional φ on Hol_0^{-s} can be extended by the Hahn–Banach theorem to a continuous functional on H^{-s} and therefore for every $f \in \mathrm{Hol}_0^{-s}$

$$\varphi(f) = \langle f, u\rangle_0 = \langle f, Bu\rangle_0.$$

Analogously every continuous functional on Hol^s can be written in the form

$$\varphi(f) = \overline{\langle u_\varphi, f\rangle_0} = \overline{\langle Bu_\varphi, f\rangle_0}, \quad u_\varphi \in H^{-s}.$$

The equivalence $(3) \Leftrightarrow (4)$ follows immediately from Theorem 2.

III. Remarks.

Remark 1. Bell's operator L^s can be extended to a continuous operator from W^s into $\mathring{W}^s$ via $\tilde{L}^s u = L^s Pu$. We have $P(\tilde{L}^s u) = P(L^s Pu) = Pu$ and thus $u - L^s u \perp H^0 = L^2 H(D)$. It is possible also to extend the operator L^s to the continuous *projection* $\tilde{\tilde{L}}^s$ from W^s onto $\mathring{W}^s$. To do this we need the following

THEOREM. *Let ϱ denote a fixed defining function for a smooth bounded domain D in $\boldsymbol{R}^n$. Then every $f \in W^s(D)$ has a uniquely determined decomposition in the form*

$$f = h_0 + \varrho h_1 + \ldots + \varrho^{s-1} h_{s-1} + u,$$

where

$$h_k \in H^{s-k}, \quad k = 0, \ldots, s-1,$$

and $u \in \mathring{W}^s(D)$. The operators $S_k(f) = h_k$ map continuously W^s onto H^{s-k} and the operator $R(f) = u$ is a continuous projection of W^s onto $\mathring{W}^s$. In other words, W^s is a direct sum of $\mathring{W}^s$ and the spaces $T_k(H^{s-k})$, $k = 0, \ldots, s-1$, where $T_k(h) = \varrho^k h$.

Proof. Let $f \in C^\infty(\bar{D})$. Let u_0 be a solution of the Dirichlet problem $\Delta v = \Delta f$, $v = 0$ on ∂D. Then $h_0 = f - u_0$ and $\|h_0\|_s \leqslant c\,\|f\|_s$ for every $s \geqslant 1$. We have $u_0 = \varrho f_1$, $f_1 \in C^\infty(\bar{D})$.

As before we take the harmonic function h_1 on D s.t. $h_1 = f_1$ on ∂D, and obtain $f = h_0 + \varrho h_1 + \varrho^2 f_2 = h_0 + \varrho h_1 + u_1$. We also have

$$h_1 = \frac{\sum_{i=1}^n \frac{\partial \varrho}{\partial x_i}\left(\frac{\partial}{\partial x_i} u_0\right)}{\sum_{i=1}^n \left(\frac{\partial \varrho}{\partial x_i}\right)^2} \quad \text{on } \partial D$$

and thus $\|h_1\|_{s-1} \leqslant c\,\|f\|_s$ for $s \geqslant 2$. Theorem 1 yields $\|\varrho h_1\|_s \leqslant c\,\|f\|_s$ for $s \geqslant 2$ and therefore $\|u_1\|_s \leqslant c\,\|f\|_s$. Now we can continue this process inductively. We have

$$f = h_0 + \varrho h_1 + \ldots + \varrho^r h_r + u_r, \quad \|h_r\|_{s-r} \leqslant c_r \|f\|_s \quad \text{for } s \geqslant r+1$$

and by Theorem 1, $\|u_r\|_s \leqslant c_r \|f\|_s$. Since $f \in C^\infty(\bar{D})$ we have $u_r = \varrho^{r+1} f_{r+1}$, $f_{r+1} \in C^\infty(\bar{D})$. We can take as h_{r+1} the harmonic function on D such that $h_{r+1} = f_{r+1}$ on ∂D. We also have

$$h_{r+1} = \frac{\sum_i \left(\frac{\partial \varrho}{\partial x_i}\right)^{r+1} \frac{\partial^{r+1}}{\partial x_i^{r+1}}(u_r)}{\sum_i \left(\frac{\partial \varrho}{\partial x_i}\right)^{2r+2}}$$

on ∂D and thus

$$\|h_{r+1}\|_{s-r-1} \leqslant c_{r+1} \|f\|_s, \quad \|u_{r+1}\|_s \leqslant c_{r+1} \|f\|_s, \quad s \geqslant r+2.$$

This implies that our operators S_k and R are well defined and continuous on the space of smooth functions in W^s. Since smooth functions are dense in W^s, S_k and R can be extended to continuous operators on W^s. Now let

$$f \in W^s, \quad f = \sum_{k=0}^{s-1} h_k \varrho^k + u.$$

We can define

$$\tilde{\tilde{L}}^s f = \sum_{k=0}^{s-1} L^s(h_k \varrho^k) + u.$$

We have $P\tilde{\tilde{L}}^s f = Pf$. If $h \in H^s$ then $\tilde{\tilde{L}}^s h = L^s h$ and if $u \in \mathring{W}^s$ then $\tilde{\tilde{L}}^s u = u$. Theorem 1 implies that

$$\|L^s(h_k \varrho^k)\|_s \leqslant c_k \|h_k\|_{s-k} \leqslant c_k \|f\|_s.$$

Thus $\tilde{\tilde{L}}^s$ is a continuous projection from W^s onto $\mathring{W}^s$.

Remark 2. The operator $Q(u) = P(\varrho^s u)$ maps continuously $L^2(D)$ onto $H^s(D)$, because

$$\|P(\varrho^s u)\|_s = \sup_{\substack{h \in H^0 \\ \|h\|_{-s} \leqslant 1}} |\langle P(\varrho^s u), h\rangle| = \sup_{\substack{h \in H^0 \\ \|h\|_{-s} \leqslant 1}} |\langle \varrho^s u, h\rangle|$$

$$\leqslant \sup_{\substack{h \in H^0 \\ \|h\|_{-s} < 1}} \|u\|_{L^2(D)} \|\varrho^s h\|_{L^2(D)} \leqslant c\|u\|_{L^2(D)}.$$

Since $\frac{1}{\varrho^s} L^s(f) \in L^2(D)$ for $f \in H^s(D)$, it follows that $\operatorname{Im} Q = H^s(D)$. In the same way we can prove the following

Proposition. *Let D be a smooth bounded domain in C^n. The Bergman projection B maps continuously $W^s(D)$ onto* $\mathrm{Hol}^s(D)$ *if and only if for every defining function ϱ of the domain D the operator $Qu = P(\varrho^s u)$ maps continuously $L^2(D)$ onto* $\mathrm{Hol}^s(D)$.

Remark 3. The operator P cannot be extended to a continuous operator from W^{-s} into H^{-k} for any $s, k > 0$.

Suppose that P is continuous from W^{-s} into H^{-k}. That implies that the operator $P(\Delta^s u)$ maps continuously $\mathring{W}^s$ into H^{-k}. For every $u \in C_0^\infty$, $\Delta^s u$ is orthogonal to the harmonic functions. Since $C_0^\infty(D)$ is dense in $\mathring{W}^s$, $P(\Delta^s u) \equiv 0$ on $\mathring{W}^s$. Δ^s maps $\mathring{W}^s$ onto W^{-s} and thus $P \equiv 0$. Contradiction. For the same reason the Bergman projection B cannot be bounded in negative Sobolev norms.

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