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THE MINIMAL INTEGRAL WHICH INCLUDES LEBESGUE INTEGRABLE FUNCTIONS AND DERIVATIVES

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Both the Perron and the Denjoy integrals provide integration processes which include the Lebesgue integral and also integrate the derivatives of differentiable functions. It is natural to ask whether one needs the generality of these integrals to do this; that is, is there a weaker integral which suffices for this purpose?

In the present paper we provide an affirmative answer to this question by giving a descriptive definition of an integral which integrates all Lebesgue integrable functions and all derivatives. We call f *integrable in this sense* iff f can be decomposed into a sum of a Lebesgue integrable function and a derivative.

Theorem 2 provides a comparison of this integral with the Lebesgue integral and Example 1 shows that this integral is strictly smaller than the Denjoy–Perron integral. In Theorem 4 we give a simple condition which suffices for a function to be an integral in this new sense.

Recall the descriptive definitions of the Lebesgue and Denjoy–Perron integrals [3].

A function f is *Lebesgue integrable* if there exists an absolutely continuous function F such that $F' = f$ almost everywhere.

A function f is *Denjoy–Perron integrable* if there exists a function F which is generalized absolutely continuous in the restricted sense and $F' = f$ almost everywhere.

In what follows, functions will be considered to be real valued and defined on $[0, 1]$. Throughout,

$|X|$ will refer to the Lebesgue measure of X,

$\mathrm{Var}(F)$ the total variation of F,

$\mathrm{Var}(F; I)$ the total variation of F on the interval I,

$\mathrm{O}(F; I)$ the oscillation of F on I,

AC the class of absolutely continuous functions,

Δ the class of differentiable functions,

ACG* the set of primitives for the Perron integral,

ACG the set of primitives for the wide sense Denjoy integral,

$\langle \mathrm{AC}, \Delta \rangle$ the set of functions $H = F + G$ with $F \in \Delta$, $G \in \mathrm{AC}$,

(N) the set of functions F which satisfy Lusin's condition (N); i.e., the image under F of each set measure 0 is of measure 0.

Clearly, the primitives for the smallest integral which includes the AC functions and the differentiable functions and is closed under addition is $\langle \mathrm{AC}, \Delta \rangle$.

The following theorem is well known and will be used for the theorems which follow.

THEOREM 1. *A function F is absolutely continuous if and only if there exists a sequence of continuously differentiable functions $\{G_n\} \subseteq C_1$ such that* $\lim_{n \to \infty} \mathrm{Var}(F - G_n) = 0$.

Theorem 1 can either be viewed as an analogue of a theorem of Goffman and Liu [2] or, as pointed out by the editor, to follow easily from the fact that every L^1 function is the L^1 limit of trigonometric polynomials and that the variation norm for integrals corresponds exactly to the L^1 norm for integrable functions.

THEOREM 2. *A function H belongs to $\langle \mathrm{AC}, \Delta \rangle$ if and if H is the limit in variation of a sequence of differentiable functions.*

Proof. If $H \in \langle \mathrm{AC}, \Delta \rangle$, $H = F + G$ where $F \in \Delta$, $G \in \mathrm{AC}$. By the above result, G is the limit in variation of a sequence $\{G_n\}$ of C_1 functions. Then H is the limit in variation of $H_n = F + G_n$ since $\mathrm{Var}(H - H_n) = \mathrm{Var}(G - G_n)$ approaches 0. If, on the other hand, $H = \lim F_n$ where $\mathrm{Var}(H - F_n)$ approaches 0 and each $F_n \in \Delta$, then $H = F_1 + (H - F_1)$. It remains to show that $H - F_1$ is in AC. Since for each n, $H - F_n$ is of bounded variation, $F_n - F_1$ is both of bounded variation and in Δ and hence is in AC. Given $\varepsilon > 0$ choose n so that $\mathrm{Var}(H - F_n) < \varepsilon/2$. Choose $\delta > 0$ so that whenever $\{I_k\}$ is a sequence of non-overlapping intervals with $\sum_k |I_k| < \delta$ it follows that $\sum_k |(F_n - F_1)(I_k)| < \varepsilon/2$. But then

$$\sum_k |(H - F_1)(I_k)| \leqslant \sum_k |(H - F_n)(I_k)| + \sum_k |(F_n - F_1)(I_k)| < \varepsilon.$$

Thus $H - F_1$ is in AC.

PROBLEM 1. (**P 1290**) If H is the limit in variation of a sequence of functions satisfying (N), must H satisfy (N)?

Generally, the primitives of an integral are closed under multiplication. This is the case for Riemann integral and for the classes AC, ACG* and ACG (cf. [1]). Theorem 3 shows that it is also the case for $\langle \mathrm{AC}, \Delta \rangle$.

THEOREM 3. *If H_1, H_2 belong to $\langle AC, \Delta\rangle$ then so does $H_1 \cdot H_2$.*

Proof. Let $H_i = F_i + G_i$ where $F_i \in \Delta$ and $G_i \in AC$. Then $G_1 \cdot G_2 \in AC$ and $F_1 \cdot F_2 \in \Delta$. It remains to show that the product of a function $F \in \Delta$ with a function $G \in AC$ belongs to $\langle AC, \Delta\rangle$. The function $F \cdot G$ is differentiable a.e. and

$$(FG)' = F'G + FG' \text{ a.e.}$$

Since the product of an AC function and a derivative is a derivative (cf. [1]), $F'G$ is a derivative and there is $H_1 \in \Delta$ with $H_1' = F'G$. Since F is continuous and G' is Lebesgue integrable, FG' is Lebesgue integrable and there is $H_2 \in AC$ with $H_2' = FG'$ a.e. Since $F \cdot G \in ACG$ and $H_1 + H_2 \in ACG$, it follows that $F \cdot G$ differs from $H_1 + H_2$ by a constant function. Thus $F \cdot G \in \langle AC, \Delta\rangle$.

The following example shows that there are functions in ACG* which are not in $\langle AC, \Delta\rangle$.

EXAMPLE 1. Let $H(x) = x\sin(x^{-2})$, $H(0) = 0$. Then $H \in ACG^*$.

Let $c_n = (\pi/2 + 2n\pi)^{-1/2}$ and note that $H(c_n) = c_n$. That H is not of bounded variation follows from the fact that $\sum c_n = \infty$, since H assumes the value zero in between c_n and c_{n+1}. Suppose $H = F + G$ with $F \in \Delta$, $G \in AC$. Clearly, by addition of a linear function, F can be chosen so that $F(0) = F'(0) = 0$. Then $\{x \in (0, c_1) | \ H(x) \geqslant x/6\}$ is the union of a sequence of intervals $\{I_n\}$ with $c_n \in I_n = [a_n, b_n]$. Note that $O(H; I_n) = c_n - \frac{1}{6}a_n > \frac{5}{6}c_{n+1}$. Choose N so that for $n > N$, $c_{n+1} > \frac{2}{3}c_{n-1}$ and $|F(x)| < \frac{1}{6}x$ when $x < c_n$. Then for $n > N$, $O(F; I_n) \leqslant \frac{1}{3}b_n < \frac{1}{3}c_{n-1} < \frac{1}{2}c_{n+1}$.

Consequently,

$$O(H-F; I_n) > \tfrac{1}{3}c_{n+1} \quad \text{and} \quad \operatorname{Var}(H-F) > \sum_{n=N}^{\infty} O(H-F; I_n) = \infty .$$

Thus $G = H - F$ is of unbounded variation, a contradiction.

A sufficient condition for a function H to belong to $\langle AC, \Delta\rangle$ is given by the following

THEOREM 4. *Given a continuous function H, let U be the set of x such that H is not of bounded variation in any neighborhood of x and let N be the set of x where $H'(x)$ does not exist. Suppose $U \cap N = \emptyset$. Then if H satisfies* (N), *$H \in \langle AC, \Delta\rangle$.*

Proof. Note that U is closed. Since H satisfies (N), H is differentiable on a dense set of points (cf. [3], pp. 280–284). Since $U \cap N = \emptyset$, U is nowhere dense. Let $\{I_n\}$ be the set of intervals contiguous to U and for each integer k choose $a_{n,k}$ such that if $I_n = [a_n, b_n]$

$$a_n < a_{n,k} < a_{n,k+1} < b_n, \quad a_{n,k+1} - a_{n,k} \leqslant a_{n,k} - a_n,$$

$$a_{n,k+1} - a_{n,k} \leqslant b_n - a_{n,k+1},$$

$$\lim_{k\to\infty} a_{n,k} = b_n, \quad \lim_{k\to-\infty} a_{n,k} = a_n$$

and H is differentiable at each $a_{n,k}$. Let $F_{n,k}$ be a C_1 function defined on $[a_{n,k}, a_{n,k+1}]$ such that $F_{n,k}(a_{n,k}) = H(a_{n,k})$ and

$$\mathrm{Var}(H - F_{n,k};\ [a_{n,k}, a_{n,k+1}]) < (a_{n,k+1} - a_{n,k})^2.$$

This is possible because H is AC on each closed subinterval of intervals contiguous to U and thus can be so approximated by a C_1 function. Let

$$F(x) = \begin{cases} F_{n,k}(x) & \text{if } x \in [a_{n,k}, a_{n,k+1}], \\ H(x) & \text{if } x \in U. \end{cases}$$

Then $(H - F)(x) = 0$ at each $x \in U$. Since

$$\mathrm{Var}(H - F) = \sum_{n,k} \mathrm{Var}(H - F;\ [a_{n,k}, a_{n,k+1}]) < \infty$$

and $H - F$ satisfies (N), $H - F \in \mathrm{AC}$. But F is differentiable at each point of the complement of U. So it remains to show that F is differentiable at each point of U. Let $x \in U$. Then

$$\frac{F(x+h) - F(x)}{h} = \frac{F(x+h) - H(x+h)}{h} + \frac{H(x+h) - H(x)}{h}$$

since $F(x) = H(x)$ for $x \in U$. As $h \to 0$ $(H(x+h) - H(x))/h$ approaches $H'(x)$ and

$$\left|\frac{F(x+h) - H(x+h)}{h}\right| \leqslant \frac{\mathrm{Var}(F - H;\ [x, x+h])}{|h|} \leqslant \frac{\sum'(a_{n,k+1} - a_{n,k})^2 + h^2}{|h|} \leqslant 2|h|$$

where the sum $\sum'$ is taken over all $[a_{n,k}, a_{n,k+1}] \subset [x, x+h]$ and the h^2 following $\sum'$ allows for the possibility that $x+h$ belongs to an interval $(a_{n,k} - a_{n,k+1})$. In this case the variation on that interval is less than $(a_{n,k+1} - a_{n,k})^2$ which is in turn less than both $(a_{n,k} - a_n)^2$ and $(b_n - a_{n,k+1})^2$ and thus less than h^2. Thus $(F(x+h) - H(x+h))/h$ approaches 0 as $h \to 0$ and F is also differentiable at all $x \in U$.

PROBLEM 2. **(P 1291)** Let $N_{\mathrm{ap}} = \{x \mid F'_{\mathrm{ap}} \text{ does not exist}\}$. If F is continuous and satisfies (N) and $N_{\mathrm{ap}} \cap U = \emptyset$, must F be ACG?

The following example shows that the condition in Theorem 4 is not necessary for membership in $\langle \mathrm{AC}, \Delta \rangle$.

EXAMPLE 2. Let $F(x)$ satisfy the following: for each natural number n, $F(x) = 0$ if $x \in [(2n+1)^{-2}, (2n)^{-2}]$, $0 \leqslant F(x) \leqslant x^2$, $F(x)$ is differentiable and $\mathrm{Var}(F;\ [(2n+2)^{-2}, (2n+1)^{-2}]) = 1$. Let G be defined as follows: $G(0) = 0$, $G(x) = 0$ if $x \in [(2n+2)^{-2}, (2n+1)^{-2}]$, $G(x) = x$ if $x = \frac{1}{2}((2n+1)^{-2} + (2n)^{-2})$. Then $G(x)$ is defined to be linear on all remaining intervals so that it is continuous on $[0, 1]$. Let $H = F + G$. Then $H'(0)$ does not exist and H is not of bounded variation in any neighborhood of 0. But $F \in \Delta$ and $G \in \mathrm{AC}$. This same behavior could clearly be created on any closed nowhere dense set.

Tolstov [4] showed that the Perron integral can be defined using majorants which are everywhere differentiable in the extended sense. However, if h is Perron integrable and h has differentiable majorant F (i.e., $F'(x) \geqslant h$ a.e.), then $F'-h$ is nonnegative and is therefore Lebesgue integrable and has an AC primitive G. Then $F-G$ is a primitive for h and hence h has a primitive in $\langle \mathrm{AC}, \Delta \rangle$.

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