Iterated and double limits

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Abstract. The paper deals with sequences of functions $f_n: E \rightarrow M$ where M is a complete metric space and E is any metric space. Necessary and sufficient conditions are given for the existence (in a certain sense) and equality of the two iterated limits, $\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$, $\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t)$, and the double limit $\lim_{\substack{n \rightarrow \infty \\ t \rightarrow x}} f_n(t)$, even in those cases where the inner limits may not exist. By

further weakening of the conditions involving the (modified) inner limits, necessary and sufficient conditions for the existence and equality of the two iterated limits are also given. As a special case we obtain the 1969 criterion of R. Nielsen for the differentiability at a point of the limit of a sequence of real-valued functions each of which is differentiable at that point.

1. Introduction. Analysts are continually confronted with the problem of interchanging the order of two limit operations. According to a famous theorem of E. H. Moore (cf. [6], p. 28), if both of the inner limits exist (pointwise) and at least one of these is uniform then both iterated limits, as well as the corresponding double limit, exist and all these limits are equal. Various versions of this important theorem appear in most advanced calculus texts, often without mention of the double limit — see for example Apostol [1], p. 221, and Rudin [9], p. 135. It is, of course, possible for both iterated limits to exist when the double limit does not, and for neither iterated limit to exist (in the ordinary sense) when the double limit does exist. For example, both iterated limits at $(x, y) = (0, 0)$ of $xy(x^2 + y^2)^{-1}$ exist and equal zero while the double limit does not exist. However, $(x + y)\sin x^{-1} \cdot \sin y^{-1}$ has the double limit 0 at $(x, y) = (0, 0)$, but neither iterated limit exists. It is also well known (cf. Apostol [1], p. 199) that if the double limit and either iterated limit exists, then these limits must be equal.

In the case of real-valued functions necessary and sufficient conditions for the existence and equality of the two iterated limits or for the existence and equality of all three limits, have been known for a long time. Such results are given in E. W. Hobson [7], vol. I, p. 408–414, and [7], vol. II, p. 45–49, and involve the limit superior ($\limsup$) and limit inferior ($\liminf$) of the inner limits in question. In Section 2 we shall give criteria (= necessary and sufficient conditions) for the existence (in a certain sense) and equality of all three limits for functions with values in a complete metric space. Our criteria

will involve a type of convergence which we call *uniform convergence near a point* (or near ∞) and which, according to Bromwich [5], A.3, p. 139, was first formulated by W. H. Young in 1903. As with Hobson the pointwise existence of the inner limits need not be assumed. It may be of interest to note that this type of convergence is only slightly weaker than one of the original kinds (Bromwich [5], A.2) of uniform convergence used by Weierstrass as early as 1842. (See Bromwich [5], p. 138–140, and Hobson [7], vol. II, p. 130–131, for a historical discussion of some of the modes of uniform convergence.)

In case the functions are continuous (or differentiable, integrable, etc.) the existence of the corresponding double limit is usually not of interest, and it is only the question of interchanging the order of the two iterated limits which concerns the analyst. Again for real-valued functions a number of criteria for the validity of such interchanges have long been known. For example, see Hobson [7], vol. II, § 83, who gives two different criteria for the limit of a sequence of continuous functions to be continuous at a point. Hobson says that both conditions are essentially due to Dini; these appear to date back to 1878 (cf. Dunford and Schwartz [6], p. 383, and J. W. Brace [3], p. 286). The second of these conditions is now called *simple uniform convergence* at a point (for example in Bourbaki [2], Dictionnaire) but was earlier called *quasi-uniform convergence* at a point by G. H. Hardy and others (cf. Bromwich [5], B.3, p. 140). The first condition of Dini seems to have submerged until 1965 when Brace [3], Definition 1.2, apparently unaware of this form of Dini's result, introduced the general notion of *convergence on a filter*. He showed [3], Theorem 1.3, that such convergence is necessary and sufficient for preservation of continuity at a point of the limit of a filter of functions each of which is continuous at that point. A special case of this kind of convergence (*convergence on a family of neighbourhoods*) was used in 1969 by R. Nielsen [8] to obtain criteria for the differentiability at a point of the limit of a sequence of real-valued functions each of which is differentiable at that point. In Section 3 we shall use this special kind of convergence (Dini's first condition, essentially) to formulate criteria for the existence (in a certain sense) and equality of iterated limits, even in cases where not both of the inner limits exist. Again we consider only functions with values in a complete metric space. Our chief aim, unlike that of Nielsen and Brace, is to find criteria of the above kind without making additional assumptions about the functions. We shall, however, obtain the relevant results of Brace and Nielsen as corollaries.

All of our results will be stated for sequences of functions $\{f_n(t)\}$ although it will be clear that analogous results can be easily formulated for double sequences $\{a_{m,n}\}$ or for general functions $f(x, y)$ of two variables. Moreover, one could also replace convergence of sequences by Moore–Smith convergence of nets in a complete metric space. In fact, in [4], Definition 1.1,

J. W. Brace introduced a type of "repeated" limit for double nets and, in the case of the existence of the inner limits, obtained some of the results of this paper. See Remarks 1.0 and 2.1 (iii) below.

2. Double and iterated limits. Throughout this paper we consider functions $f, f_n: E \rightarrow M$, where E is a subset of any metric space and M is a complete metric space. We also let x denote a limit point of E and, for $\delta > 0$, let $S_\delta(x)$ denote the open ball of radius δ centered at x and $S_\delta^*(x) = S_\delta(x) \setminus \{x\}$. We will never require any notation for distance in E , and write $|y - z|$ for the distance in M between points y and z .

Let $\{n_k(t)\}$ always denote a strictly increasing sequence of positive integers, in general depending on the point $t \in E$, and for $n \geq 1$ let E_n denote a subset of E (possibly discrete) which also has x as a limit point. In this section we shall be concerned with the following convergence conditions:

- (a) $\lim_{k \rightarrow \infty} f_{n_k(t)}(t) = f(t)$ pointwise for $t \in E$;
- (b) $\lim_{t \rightarrow x, t \in E_n} f_n(t) = a_n$ for each $n \geq 1$;
- (c) $\lim_{t \rightarrow x, t \in E} f(t) = L_1$;
- (d) $\lim_{n \rightarrow \infty} a_n = L_2$;
- (e) $\lim_{\substack{n \rightarrow \infty \\ t \rightarrow x, t \in E}} f_n(t) = L$;
- (α) $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ uniformly near x (on E);
- (β) $\lim_{t \rightarrow x, t \in E} f_n(t) = a_n$ uniformly near ∞ .

The convergence in (a)–(d) is ordinary convergence and requires no further explanation but we give explicit definitions for the last three convergence conditions. Condition (e) is the *double limit* condition and holds iff $\forall \varepsilon > 0 \exists N_\varepsilon \geq 1$ and $\delta_\varepsilon > 0$ such that

$$(1) \quad |f_n(t) - L| < \varepsilon \quad \forall n \geq N_\varepsilon, \quad t \in E \cap S_{\delta_\varepsilon}^*(x).$$

The limits in (α), (β) are called *uniform near limits* and the convergence is called *uniform near convergence*. (α) holds iff $\forall \varepsilon > 0 \exists N_\varepsilon \geq 1$ and $\delta_\varepsilon > 0$ such that

$$(2) \quad |f_n(t) - f(t)| < \varepsilon \quad \forall n \geq N_\varepsilon, \quad t \in E \cap S_{\delta_\varepsilon}^*(x)$$

and (β) holds iff $\forall \varepsilon > 0 \exists \delta_\varepsilon > 0$ and $N_\varepsilon \geq 1$ such that

$$(3) \quad |f_n(t) - a_n| < \varepsilon \quad \forall n \geq N_\varepsilon, \quad t \in E \cap S_{\delta_\varepsilon}^*(x).$$

In case (2) also holds for $t = x$ we say that $\{f_n\}$ converges *uniformly to f at x* (cf. Bromwich [5], A.3, p. 139) and note that in Weierstrass' original definition of uniform convergence of $\{f_n\}$ to f at x this condition was further

strengthened by requiring that δ_ε be independent of ε . (In all of (1)–(3), $\delta_\varepsilon = \delta_\varepsilon(x)$ depends on x , of course.)

We state, without proof, the following generalization of the theorem of W. H. Young:

If K is a compact set and $\{f_n\}$ converges uniformly to f at x for each $x \in K$, then $\lim_{n \rightarrow \infty} f_n = f$ uniformly on K .

The proof is easy (cf. Bromwich [5], p. 139) and we will make no use of this result. The following examples show that uniform convergence near x has certain peculiarities: uniform near limits are not unique, and uniform near convergence neither implies nor is implied by pointwise convergence. It clearly is, however, implied by uniform convergence on any $S_{\delta_0}(x)$.

EXAMPLE 1.1. Take $f_n(t) = t$ for $t \in E = \mathbb{R}$, $x = 0$ and let f be any function which is continuous at 0 with $f(0) = 0$. Then $\lim f_n = f$ uniformly near (and at) $x = 0$, but $\lim f_n(t) \neq f(t)$ pointwise for any $t \neq 0$ provided $f(t) \neq t$ for $t \neq 0$.

EXAMPLE 1.2. If $f_n(t) = nt(1+nt)^{-1}$ for $t \in E = (0, \infty)$, $f(t) \equiv 1$, $x = 0$, then $\lim f_n = f$ pointwise on E but this convergence is not uniform near $x = 0$.

THEOREM 1.1. Suppose (e) holds so the double limit exists, and that the pointwise limit (a) holds for some increasing sequences $\{n_k(t)\}$. Then (c) holds with $L_1 = L$ and, moreover, the convergence of $\{f_n\}$ to f is uniform near x , so (α) holds. In addition, (e) and (α) together imply (c) with $L_1 = L$ while (α) and (c) imply (e) with $L = L_1$.

Proof. If (e) holds then given $\varepsilon > 0 \exists N_\varepsilon \geq 1$, $\delta_\varepsilon > 0$ such that

$$(*) \quad |f_n(t) - L| < \frac{1}{2}\varepsilon \quad \forall n \geq N_\varepsilon, \quad t \in E \cap S_{\delta_\varepsilon}^*(x).$$

Fix any such t and take $n = n_k(t)$ in (*). Letting $k \rightarrow \infty$ it follows from (a) that we have

$$|f(t) - L| \leq \frac{1}{2}\varepsilon < \varepsilon \quad \forall t \in E \cap S_{\delta_\varepsilon}^*(x),$$

so (c) holds with $L_1 = L$. To prove that (α) holds, note that

$$|f_n(t) - f(t)| \leq |f_n(t) - L| + |L - f(t)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

for $n \geq N_\varepsilon$, $t \in E \cap S_{\delta_\varepsilon}^*(x)$ by the last two inequalities.

Similarly, if (e) and (α) hold, then from

$$|f(t) - L| \leq |f(t) - f_n(t)| + |f_n(t) - L|$$

it follows that (c) holds with $L_1 = L$. For, given $\varepsilon > 0$ we have (*) as above, and, by (α), $\exists \hat{N}_\varepsilon \geq 1$ and $\hat{\delta}_\varepsilon > 0$ such that

$$|f_n(t) - f(t)| < \frac{1}{2}\varepsilon \quad \forall n \geq \hat{N}_\varepsilon, \quad t \in E \cap S_{\hat{\delta}_\varepsilon}^*(x).$$

Hence if we fix any $n \geq \max(N_\varepsilon, \hat{N}_\varepsilon)$, it follows that

$$|f(t) - L| < \varepsilon \quad \forall t \in E \cap S_{\delta_{1,\varepsilon}}^*(x),$$

where $\delta_{1,\varepsilon} = \min(\delta_\varepsilon, \hat{\delta}_\varepsilon)$.

Finally, for the partial converse, suppose (c) and (α) hold. Then as above,

$$|f_n(t) - L_1| \leq |f_n(t) - f(t)| + |f(t) - L_1| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

will hold for all $n \geq \hat{N}_\varepsilon$, $t \in E \cap S_{\delta_{2,\varepsilon}}^*(x)$ for an appropriate $\delta_{2,\varepsilon} = \min(\hat{\delta}_\varepsilon, \delta_\varepsilon)$. Hence (e) holds with $L = L_1$.

Remark 1.0. In a paper published in 1962, J. W. Brace [4], introduced a "repeated" limit which coincides with an ordinary iterated limit whenever the (pointwise) inner limit exists. For the restricted case ($B_\alpha \equiv B$) dealt with in [4], his Lemma 1.5 includes (in the context of nets rather than sequences of functions) the final sentence of Theorem 1.1 above. Similarly his Theorem 1.6 gives conditions for the equality of the iterated and double limits which are the special cases $n_k(t) \equiv k$, $E_n \equiv E$ of Corollary 1.1 below. The condition $\lim_n \lim_x \{f_n(t) - f(t)\} = 0$ used by Brace reduces, in this case, to uniform near convergence as defined at (1).

Remark 1.1. The second example noted in the introduction (with y replaced by $1/n$) shows that when (e) holds, (a) may not hold with $n_k(t) \equiv k$. On the other hand we have shown that

$$\lim_{t \rightarrow x, t \in E} (\lim_{k \rightarrow \infty} f_{n_k(t)}(t)) = \lim_{\substack{n \rightarrow \infty \\ t \rightarrow x, t \in E}} f_n(t)$$

holds for any increasing sequences $\{n_k(t)\}$ such that the inner limit in (4) exists, provided the double limit exists. In case the metric space M is also *boundedly compact* (the compact sets are the closed, bounded sets) then the existence of the double limit in (4) implies the existence of the inner limit in (4) for some sequences $\{n_k(t)\}$ and hence the truth of (4) for all such $\{n_k(t)\}$. For, corresponding to $\varepsilon' = 1$ there is an integer $N \geq 1$ and $\delta > 0$ such that $|f_n(t) - L| < 1 \quad \forall n \geq N$ and $t \in E \cap S_\delta^*(x)$. For each such t it follows that there is sequence $\{n_k(t)\}$ for which the subsequence $\{f_{n_k(t)}(t)\}$ converges. Thus (a) holds, but with E replaced by $E \cap S_\delta(x)$.

Remark 1.2. An examination of the proof of the last sentence of Theorem 1.1 shows that it does not depend on the interpretation of $f(t)$ as a pointwise limit given by (a). In fact it shows that in the presence of (α) — with any f — (c) and (e) are equivalent and the two limits are equal. That is, if $f: E \rightarrow M$ is any function such that $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ uniformly near x , then the existence of either of the two limits (c), (e) implies that of the other together with their equality.

In exactly the same way we can obtain

THEOREM 1.2. Suppose (e) holds and that for each $n \geq 1$ the inner limit (b) exists for some $E_n \subset E$. Then (d) holds with $L_2 = L$ and, moreover, the convergence of $f_n(t)$ to a_n (as $t \rightarrow x$ in E) is uniform near ∞ , so (β) holds. In addition, (e), (β), imply (d) with $L_2 = L$, while (β), (d) imply (e) with $L = L_2$.

We now have

$$(5) \quad \lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E_n} f_n(t) \right) = \lim_{\substack{n \rightarrow \infty \\ t \rightarrow x, t \in E}} f_n(t)$$

for any $E_n \subset E$ having x as a limit point, such that the inner limit as well as the double limit in (5) both exist. Moreover, if the inner limit in (5) is interpreted as a uniform near ∞ limit, then (5) holds whenever either (hence both) of the other two limits in (5) exist.

THEOREM 1.3. Suppose that for some $f: E \rightarrow M$ and $\{a_n\}$, (α) holds as well as either (β) or (b) with $E_n \equiv E$. Then (c), (d), (e) hold with $L_1 = L_2 = L$. Moreover, if also $f_{n_k}(t) \rightarrow g(t)$ pointwise for $t \in E$ as $k \rightarrow \infty$, or if $f_n(t) \rightarrow b_n$ for $n \geq 1$ as $t \rightarrow x$ in some $E_n \subset E$, then we have

$$\lim_{t \rightarrow x, t \in E} g(t) = L, \quad \text{or} \quad \lim_{n \rightarrow \infty} b_n = L.$$

Proof. To prove that the limit in (c) exists, we use the Cauchy criterion, with

$$(6) \quad |f(t) - f(t')| \leq |f(t) - f_n(t)| + |f_n(t) - a_n| + |a_n - f_n(t')| + |f_n(t') - f(t')|.$$

Given $\varepsilon > 0$, by (α), $\exists N_\varepsilon \geq 1$, $\delta_\varepsilon > 0$ such that

$$|f_n(s) - f(s)| < \frac{1}{4}\varepsilon \quad \forall n \geq N_\varepsilon, \quad s \in E \cap S_{\delta_\varepsilon}^*(x),$$

whence

$$(7) \quad |f(t) - f(t')| < \frac{1}{2}\varepsilon + |f_n(t) - a_n| + |a_n - f_n(t')| \quad \forall n \geq N_\varepsilon$$

if $t, t' \in E \cap S_{\delta_\varepsilon}^*(x)$. Now if (β) holds $\exists \hat{N}_\varepsilon \geq 1$, $\hat{\delta}_\varepsilon > 0$ such that

$$(8) \quad |f_n(s) - a_n| < \frac{1}{4}\varepsilon \quad \forall n \geq \hat{N}_\varepsilon, \quad s \in E \cap S_{\hat{\delta}_\varepsilon}^*(x).$$

Fix any $n \geq \max(N_\varepsilon, \hat{N}_\varepsilon)$ in (7) and we obtain

$$(9) \quad |f(t) - f(t')| < \varepsilon \quad \forall t, t' \in E \cap S_{\delta_{1,\varepsilon}}^*(x),$$

where $\delta_{1,\varepsilon} = \min(\delta_\varepsilon, \hat{\delta}_\varepsilon)$. The existence of the limit in (c) follows from the completeness of M . If (b) holds with all $E_n = E$, fix $n = N_\varepsilon$ in (7). By (b), $\exists \hat{\delta}_\varepsilon > 0$ such that (8) again holds for $n = N_\varepsilon$ whence (9) again follows.

Since (c), as well as (α), holds it follows from the last part of Theorem 1.1 that (e) holds with $L = L_1$. If either (β) holds or (b) holds, then (d) follows from Theorem 1.2 with $L_2 = L$. The last assertions of the theorem follow from (e) and the first parts of Theorems 1.1, 1.2.

Again there is an analogous result involving the other inner limit, namely,

THEOREM 1.4. *Suppose that for some $f: E \rightarrow M$ and $\{a_n\}$, (β) holds as well as either (α) or (a) with $n_k(t) \equiv k$. Then (c), (d), (e), hold with $L_1 = L_2 = L$.*

The final assertion of Theorem 1.3 also remains valid under these hypothesis.

The proof proceeds as before except that (6), (9) are replaced by

$$|a_n - a_{n'}| \leq |a_n - f_n(t)| + |f_n(t) - f(t)| + |f(t) - f_{n'}(t)| + |f_{n'}(t) - a_{n'}|,$$

and

$$|a_n - a_{n'}| < \varepsilon \quad \forall n, \quad n' \geq \max(N_\varepsilon, \hat{N}_\varepsilon).$$

As corollaries of the preceding four theorems we have the following necessary and sufficient conditions for the existence and equality of the three limits in

$$(+)$$

$$\lim_{\substack{n \rightarrow \infty \\ t \rightarrow x, t \in E}} f_n(t), \quad \lim_{t \rightarrow x, t \in E} (\lim_{k \rightarrow \infty} f_{n_k(t)}(t)), \quad \lim_{n \rightarrow \infty} (\lim_{t \rightarrow x, t \in E_n} f_n(t)).$$

COROLLARY 1.1. *If the two inner limits in (+) exist, then all three limits, in (+) exist and are equal iff both (α) $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ uniformly near x , and (β) $\lim_{t \rightarrow x, t \in E} f_n(t) = a_n$ uniformly near ∞ , where $f(t)$ denotes the inner limit in the second of (+), and a_n denotes the inner limit in the third of (+).*

COROLLARY 1.2. *If the two inner limits in (+) exist, the first for $n_k(t) \equiv k$, then all three limits exist and are equal iff condition (β) holds.*

COROLLARY 1.3. *If the two inner limits in (+) exist, the last for $E_n \equiv E$, then all three limits exist and are equal iff condition (α) holds.*

3. Iterated limits. It is clear from the results of the last section that the existence of the double limit is a very strong condition. (See, in particular, Remark 1.1.) In this section we will not require the existence of a double limit. To compensate for this we shall strengthen our assumptions concerning the pointwise existence of the inner limits. Here we always let $\{n_k\}$ denote a strictly increasing sequence of positive integers (independent of $t \in E$), and let E_1 denote a subset of E (possibly discrete) having x as a limit point. We consider the following kinds of convergence on families of neighbourhoods of x or of ∞ :

$\lim_{n \rightarrow \infty} f_n(t) = f(t)$ on $\mathcal{N}_E(x) \Leftrightarrow \forall \varepsilon > 0 \exists N_\varepsilon \geq 1$ such that for $n \geq N_\varepsilon$
 $\exists \delta_{n,\varepsilon} > 0$ such that

$$(10) \quad |f_n(t) - f(t)| < \varepsilon \quad \forall t \in E \cap S_{\delta_{n,\varepsilon}}^*(x);$$

$\lim_{t \rightarrow x, t \in E} f_n(t) = a_n$ on $\mathcal{N}(\infty) \Leftrightarrow \forall \varepsilon > 0 \exists \delta_\varepsilon > 0$ such that for $t \in E \cap S_{\delta_\varepsilon}^*(x)$

$\exists N_{\epsilon, t} \geq 1$ such that

$$(11) \quad |f_n(t) - a_n| < \epsilon \quad \forall n \geq N_{\epsilon, t}.$$

A comparison of (2) with (10) shows that convergence on neighbourhoods of x in E is implied by uniform convergence near x , but in general not conversely. The same examples 1.1, 1.2 suffice to show the lack of uniqueness of limits for convergence on $\mathcal{N}_E(x)$ and that such convergence neither implies nor is implied by pointwise convergence on E . The example $f_n(t) = nt(1+n^2t^2)^{-1}$, $f(t) \equiv 0$, $E = (0, \infty)$, $x = 0$ shows that convergence on $\mathcal{N}_E(x)$ need not imply uniform convergence near x . Similar comments apply to convergence on $\mathcal{N}(\infty)$.

As noted in the introduction, convergence on $\mathcal{N}_E(x)$ appears to have been first formulated by Dini, and is the special case of convergence on filters introduced by Brace [3] which was used by Nielsen [8]. It may be of interest to note that the concept of simple uniform convergence of $\{f_n\}$ to f near x (Bromwich [5], B.3, p. 140; Brace [3], Definition 1.1; Hobson [7], vol. II, p. 124, second theorem) is an even weaker type of convergence than convergence on $\mathcal{N}_E(x)$. We give a brief discussion of this relationship now since it has a bearing on one of our later results (Corollary 2.5). For convenience of comparison we assume $x \in E$ in what follows, and also assume that (10) holds for $t = x$. Then we say that $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ *simply uniformly at x* iff

$$(12) \quad \lim f_n(x) = f(x), \text{ and } \forall \epsilon > 0 \text{ and } n \geq 1 \exists \delta_{\epsilon, n} > 0 \text{ and } m_{\epsilon, n} > n \text{ such that} \\ |f_m(t) - f(t)| < \epsilon \quad \text{for } m = m_{\epsilon, n}, t \in E \cap S_{\delta_{\epsilon, n}}(x).$$

If now $\lim f_n = f$ on $\mathcal{N}_E(x)$ so $\exists N_\epsilon, \delta_{\epsilon, n}$ as in (10) with $x \in E$, then it is clear that $\lim f_n(x) = f(x)$ holds. Moreover, corresponding to ϵ and $n \geq 1$ one may choose $\delta_{\epsilon, n}$ and $m_{\epsilon, n} > n$ satisfying (12) as follows: If $1 \leq n < N_\epsilon$, take $\delta_{\epsilon, n} = \delta_{N_\epsilon, \epsilon}$ and $m_{\epsilon, n} = N_\epsilon$, while if $n \geq N_\epsilon$ take $\delta_{\epsilon, n} = \delta_{N_\epsilon+1, \epsilon}$ and $m_{\epsilon, n} = N_\epsilon + 1$. Thus convergence on $\mathcal{N}_E(x)$ implies simple uniform convergence at x . For an example to show the converse need not hold, let $E = [0, 1]$, $x = 0$ and take $g_n(t) = nt/(1+nt)$ on E , so $g_n \rightarrow g$ pointwise on E where $g(0) = 0$, $g(t) = 1$, $0 < t \leq 1$. Then $\lim(g_n - g) \neq 0$ on $\mathcal{N}_E(0)$. For an arbitrary function h on E , define $\{f_n\}$ on E by $f_{2k-1}(t) = h(t)$, $f_{2k}(t) = h(t) + g_k(t) - g(t)$. Then $\lim f_n = h$ simply uniformly at 0, but since $|f_{2k}(t) - h(t)| = |g_k(t) - g(t)|$ we do not have $\lim f_n = h$ on $\mathcal{N}_E(0)$.

However, in case all f_n are *continuous* at x the two modes of convergence are equivalent. To see this we use the known (and easily proved) result (cf. Bromwich [5], p. 140) of Dini that in this case if $\lim f_n = f$ simply uniformly at x , then f is also continuous at x (and conversely). But then the fact that $\lim f_n = f$ on $\mathcal{N}_E(x)$ follows at once from

$$|f_n(t) - f(t)| \leq |f_n(t) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(t)|,$$

since $\lim f_n(x) = f(x)$ by simple uniform convergence at x . It now follows from the result of Dini that if all f_n are continuous at x and $\lim f_n(x) = f(x)$, then f is continuous at x iff $\lim f_n = f$ on $\mathcal{U}_E(x)$. This is the (alternative) version of Corollary 2.5 noted following (22).

Throughout this section we shall be concerned with the following convergence conditions:

$$(a') \lim_{k \rightarrow \infty} f_{n_k}(t) = f(t) \text{ pointwise for } t \in E;$$

$$(b') \lim_{t \rightarrow x, t \in E_1} f_n(t) = a_n \text{ for each } n \geq 1;$$

$$(c) \lim_{t \rightarrow x, t \in E} f(t) = L_1;$$

$$(c') \lim_{t \rightarrow x, t \in E_1} f(t) = \hat{L}_1;$$

$$(d) \lim_{n \rightarrow \infty} a_n = L_2;$$

$$(d') \lim_{k \rightarrow \infty} a_{n_k} = \hat{L}_2;$$

$$(\alpha') \lim_{n \rightarrow \infty} f_n(t) = f(t) \text{ on } \mathcal{N}_{E_1}(x);$$

$$(\beta') \lim_{t \rightarrow x, t \in E} f_{n_k}(t) = a_{n_k} \text{ on } \mathcal{N}(\infty).$$

THEOREM 2.1. *If (a'), (b') hold, then a necessary and sufficient condition that (c') and (d) hold with $\hat{L}_1 = L_2$ is that (α') holds.*

Proof. Suppose that (a'), (b'), (c') and (d) hold with $\hat{L}_1 = L_2 = L$ say. Given $\varepsilon > 0$ it follows from (c') and (d) that $\exists \delta_\varepsilon > 0$ and $N_\varepsilon \geq 1$ such that

$$|f(t) - L| < \frac{1}{3}\varepsilon \quad \text{if } t \in E_1 \cap S_{\delta_\varepsilon}^*(x), \quad |a_n - L| < \frac{1}{3}\varepsilon \quad \text{if } n \geq N_\varepsilon.$$

Hence,

$$(13) \quad |f_n(t) - f(t)| \leq |f_n(t) - a_n| + |a_n - L| + |L - f(t)| \\ < \frac{2}{3}\varepsilon + |f_n(t) - a_n| \quad \text{for } t \in E_1 \cap S_{\delta_\varepsilon}^*(x), \quad n \geq N_\varepsilon.$$

Fix any $n \geq N_\varepsilon$. By (b') $\exists \delta_{n,\varepsilon} > 0$ such that

$$|f_n(t) - a_n| < \frac{1}{3}\varepsilon \quad \text{if } t \in E_1 \cap S_{\delta_{n,\varepsilon}}^*(x),$$

whence

$$|f_n(t) - f(t)| < \varepsilon \quad \forall n \geq N_\varepsilon, \quad t \in E_1 \cap S_{\delta_{n,\varepsilon}}^*(x) \quad \text{for } \delta_{n,\varepsilon} = \min(\delta_\varepsilon, \delta_{n,\varepsilon}),$$

so that (α') holds.

Conversely, suppose that (a'), (b') and (α') all hold. We shall first prove that $\{a_n\}$ is a Cauchy sequence so that by completeness of M , (d) holds for some $L_2 \in M$. By (α') given $\varepsilon > 0 \exists N_\varepsilon \geq 1$ and $\forall n \geq N_\varepsilon \exists \delta_{n,\varepsilon} > 0$ such that

$$|f_n(t) - f(t)| < \frac{1}{4}\varepsilon \quad \forall t \in E_1 \cap S_{\delta_{n,\varepsilon}}^*(x).$$

Hence

$$(14) \quad |f_n(t) - f_m(t)| \leq |f_n(t) - f(t)| + |f(t) - f_m(t)| < \frac{1}{2}\varepsilon$$

holds for all $m, n \geq N_\varepsilon$, $t \in E_1 \cap S_{\delta_\varepsilon}^*(x)$, where $\delta_\varepsilon = \min(\delta_{n,\varepsilon}, \delta_{m,\varepsilon})$. Fix any $m, n \geq N_\varepsilon$ and take limits at (14) as $t \rightarrow x$ in E_1 . By (b') we obtain $|a_n - a_m| \leq \frac{1}{2}\varepsilon < c \quad \forall m, n \geq N_\varepsilon$, proving that (d) holds.

To show that (c') holds with $\hat{L}_1 = L_2$, consider

$$(15) \quad |f(t) - L_2| \leq |f(t) - f_n(t)| + |f_n(t) - a_n| + |a_n - L_2|.$$

Using (d) and (b') $\exists N_\varepsilon \geq 1$ and $\delta_{n,\varepsilon} > 0$ such that

$$|f(t) - L_2| < \frac{2}{3}\varepsilon + |f_n(t) - f(t)| \quad \forall n \geq N_\varepsilon, \quad t \in E_1 \cap S_{\delta_{n,\varepsilon}}^*(x).$$

By (α'), $\exists N_{1,\varepsilon} \geq 1$ such that for $n \geq N_{1,\varepsilon}$ $\exists \delta_{n,\varepsilon} > 0$ such that

$$|f_n(t) - f(t)| < \frac{1}{3}\varepsilon \quad \text{if} \quad t \in E_1 \cap S_{\delta_{n,\varepsilon}}^*(x).$$

Now fix $n = n_\varepsilon = \max(N_\varepsilon, N_{1,\varepsilon})$ and set $\delta_\varepsilon = \min(\delta_{n_\varepsilon,\varepsilon}, \delta_{n_\varepsilon,\varepsilon})$. Then we have

$$|f(t) - L_2| < \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon \quad \forall t \in E_1 \cap S_{\delta_\varepsilon}^*(x)$$

proving that (c') holds with $\hat{L}_1 = L_2$.

Remark 2.1. (i) When (α') holds the conclusion of Theorem 2.1 is that

$$(16) \quad \lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E_1} f_n(t) \right) = \lim_{t \rightarrow x, t \in E_1} \left(\lim_{k \rightarrow \infty} f_{n_k}(t) \right)$$

provided the two inner limits exist. The left side of (16) appears to be independent of the sequence $\{n_k\}$ appearing on the right side. However, the equality condition (α') for (16) involves $\{n_k\}$ since f is defined in terms of $\{n_k\}$ by (a'). Hence when (16) holds both sides, in general, depend on both the set $E_1 \subset E$ and the subsequence $\{n_k\}$.

(ii) An examination of the proof shows that (a') was only used to produce a function $f: E \rightarrow M$ to which (c) or (α') applied; f need not be a pointwise limit of any kind. Thus we always have

$$(16') \quad \lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E_1} f_n(t) \right) = \lim_{t \rightarrow x, t \in E_1} \left(\lim_{n \rightarrow \infty} f_n(t) \text{ on } \mathcal{N}_{E_1}(x) \right)$$

whenever both inner limits exist. More precisely, if the inner limit on the left side of (16') exists, then a necessary and sufficient condition for the existence and equality of both outer limits in (16') is the existence of the inner neighbourhood limit on the right side of (16').

(iii) If the "repeated" limit used by Brace [9], Definition 1.1, is not restricted to the case $B_\alpha \equiv B$ (see Remark 1.0), then the Brace condition $\lim_n \lim_x \{f_n(t) - f(t)\} = 0$ is equivalent to my condition (α') of convergence on

$\mathcal{N}_{E_1}(x)$. Although Brace did not, in fact, deal with this unrestricted case in his Lemma 1.5 and Theorem 1.6, he could have. In this case his Lemma 1.5 would coincide with the above result concerning (16'), and his Theorem 1.6 would coincide with Corollary 2.4 below.

There is a corresponding theorem in which the roles played by the inner limits are interchanged.

THEOREM 2.2. *If (a'), (b') hold, then a necessary and sufficient condition that (c) and (d') hold with $\hat{L}_2 = L_1$ is that (β') holds.*

The proof is the same as before with the obvious interchanges. For example (13), (14), (15) are replaced by

$$\begin{aligned} |f_{n_k}(t) - a_{n_k}| &\leq |f_{n_k}(t) - f(t)| + |f(t) - L| + |L - a_{n_k}|, \\ |f_{n_k}(t) - f_{n_k}(t')| &\leq |f_{n_k}(t) - a_{n_k}| + |a_{n_k} - f_{n_k}(t')|, \\ |a_{n_k} - L_1| &\leq |a_{n_k} - f_{n_k}(t)| + |f_{n_k}(t) - f(t)| + |f(t) - L_1|. \end{aligned}$$

Remark 2.2. (i) When (β') holds the conclusion of Theorem 2.2 is that

$$(17) \quad \lim_{k \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E_1} f_{n_k}(t) \right) = \lim_{t \rightarrow x, t \in E} \left(\lim_{k \rightarrow \infty} f_{n_k}(t) \right)$$

provided the two inner limits exist. Now, because the equality condition (β') involves E_1 since $\{a_n\}$ is defined in terms of E_1 by (b'), it follows that the common limit value in (17) depends on both $\{n_k\}$ and E_1 (and not just on $\{n_k\}$).

(ii) Precisely as at (16') we always have

$$(17') \quad \lim_{k \rightarrow \infty} \left\{ \lim_{t \rightarrow x, t \in E} f_{n_k}(t) \text{ on } \mathcal{N}(\infty) \right\} = \lim_{t \rightarrow x, t \in E} \left(\lim_{k \rightarrow \infty} f_{n_k}(t) \right)$$

whenever both inner limits exist, even if the inner limit on the left side is not also a pointwise limit.

COROLLARY 2.1. *If the two inner limits of*

$$(18) \quad \lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E_1} f_n(t) \right) = \lim_{t \rightarrow x, t \in E} \left(\lim_{k \rightarrow \infty} f_{n_k}(t) \right)$$

both exist, then (18) holds if and only if both (α') , (β') , hold.

This follows from Theorems 2.1, 2.2. Although the left side of (18) appears to be independent of $\{n_k\}$ and the right side independent of $E_1 \subset E$, the equality conditions (α') , (β') involve both $\{n_k\}$ and E_1 so the common limit value in (18) depends on both of these factors.

Bearing in mind the negative implications of this remark as well as of Remarks 2.1 (i), 2.2 (i), the following corollaries are of interest.

COROLLARY 2.2. *If the two inner limits of*

$$(19) \quad \lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E_1} f_n(t) \right) = \lim_{t \rightarrow x, t \in E_1} \left(\lim_{k \rightarrow \infty} f_k(t) \right)$$

both exist, then (19) holds iff $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ on $\mathcal{N}_{E_1}(x)$, where $f(t)$ denotes the inner pointwise limit on the right side of (19).

This is just the case $n_k \equiv k$ of Theorem 2.1.

COROLLARY 2.3. *If the two inner limits of*

$$(20) \quad \lim_{k \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E} f_{n_k}(t) \right) = \lim_{t \rightarrow x, t \in E} \left(\lim_{k \rightarrow \infty} f_{n_k}(t) \right)$$

both exist, then (20) holds iff $\lim_{t \rightarrow x, t \in E} f_{n_k}(t) = a_{n_k}$ on $\mathcal{N}(\infty)$, where a_{n_k} denotes the inner limit on the left side of (20).

This is the case $E_1 = E$ of Theorem 2.2. By a relabelling we see that the last two corollaries can be combined into the single

COROLLARY 2.4. *If the two inner limits of*

$$(21) \quad \lim_{k \rightarrow \infty} \left(\lim_{t \rightarrow x, t \in E} g_k(t) \right) = \lim_{t \rightarrow x, t \in E} \left(\lim_{k \rightarrow \infty} g_k(t) \right)$$

both exist, then (21) holds iff either the inner limit on the right side is also a limit on $\mathcal{N}_E(x)$, or that on the left side is a limit on $\mathcal{N}(\infty)$.

COROLLARY 2.5. *Let f and f_n ($n = 1, 2, \dots$) all be defined in some neighbourhood $E = S_\delta(x)$, suppose all f_n are continuous at x and that $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ pointwise for $t \in E$. Then f is continuous at x iff $\lim_{n \rightarrow \infty} f_n = f$ on $\mathcal{N}_E(x)$. Moreover, in this case $\{f_n\}$ has property (β') : $\forall \varepsilon > 0 \exists \delta_\varepsilon > 0$ such that for $t \in S_{\delta_\varepsilon}(x) \exists N_{\varepsilon, t} \geq 1$ such that*

$$|f_n(t) - f_n(x)| < \varepsilon \quad \forall n \geq N_{\varepsilon, t}.$$

The first assertion follows from Corollary 2.4 plus the fact that f is continuous at x iff

$$(22) \quad \lim_{t \rightarrow x} \left(\lim_{n \rightarrow \infty} f_n(t) \right) = \lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow x} f_n(t) \right),$$

since each f_n is continuous at x . The last part also follows from Corollary 2.4 since property (β') is just the condition that the inner limit on the right side of (21), namely $f_n(x)$, is a limit on $\mathcal{N}(\infty)$.

There is a better version of this preservation of continuity theorem, the one first given for convergence on filters by J. W. Brace [3], Theorem 1.3:

Let f_n, f be as in Corollary 2.5 with all f_n continuous at x , and suppose only that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$. Then f is continuous at x iff $\lim_{n \rightarrow \infty} f_n = f$ on $\mathcal{N}_E(x)$.

This version follows from Remark 2.1 (ii), (16') with $E_1 = E = S_\delta(x)$. As noted earlier this result is implied by Dini's theorem on preservation of continuity under simple uniform convergence at x , and in fact the two results are equivalent.

COROLLARY 2.6 (Nielsen [8]). *Let f and f_n ($n = 1, 2, \dots$) be real or complex-valued functions of a real variable defined in some $S_\delta(x) = E$, and suppose that*

$$(23) \quad \lim_{n \rightarrow \infty} f_n = f \quad \text{pointwise on } E,$$

$$(24) \quad f'_n(x) = \lim_{t \rightarrow x} \frac{f_n(t) - f_n(x)}{t - x} \equiv \lim_{t \rightarrow x} \varphi_n(t)$$

exists for each $n \geq 1$. Then $f'(x)$ exists, $\lim_{n \rightarrow \infty} f'_n(x)$ exists, and these are equal iff $\lim_{n \rightarrow \infty} \varphi_n = \varphi$ on $\mathcal{N}_E(x)$, where $\varphi(t) \equiv [f(t) - f(x)]/(t - x)$ for $t \in S_\delta^(x)$.*

Proof. By (23), (24) both inner limits in

$$\lim_{t \rightarrow x} (\lim_{n \rightarrow \infty} \varphi_n(t)), \quad \lim_{n \rightarrow \infty} (\lim_{t \rightarrow x} \varphi_n(t))$$

exist, and have the values $\varphi(t)$, $f'_n(x)$ respectively. By Corollary 2.4 both (outer) limits above exist and are equal iff $\lim_{n \rightarrow \infty} \varphi_n = \varphi$ uniformly on $\mathcal{N}_E(x)$, proving the result.

We conclude by stating another version of this corollary using Corollary 2.1, a version which requires only pointwise convergence of a subsequence of $\{f_n\}$ and does not require the existence of any ordinary derivative $f'_n(x)$.

COROLLARY 2.7. *Suppose that for some increasing sequence $\{n_k\}$ of positive integers and some (possibly discrete) set $E_1 \subset E$ having x as a limit point, we have:*

$$\lim_{k \rightarrow \infty} f_{n_k}(t) = f(t) \quad \text{exists, pointwise on } E;$$

$$\lim_{t \rightarrow x, t \in E_1} \frac{f_n(t) - f_n(x)}{t - x} \equiv f'_{n, E_1}(x) \quad \text{exists, } n \geq 1.$$

Then $f'(x)$ exists, $\lim_{k \rightarrow \infty} f'_{n_k, E_1}(x)$ exists, and these are equal iff both (α') $\lim_{n \rightarrow \infty} \varphi_n(t) = \varphi(t)$ on $\mathcal{N}_{E_1}(x)$ and (β') $\lim_{t \rightarrow x, t \in E} \varphi_{n_k}(t) = f'_{n_k, E_1}(x)$ on $N(\infty)$, where φ_n and φ are as defined in Corollary 2.6.

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