GEOMETRIC AND ALGEBRAIC TOPOLOGY
BANACH CENTER PUBLICATIONS, VOLUME 18
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1986

ON TOTAL HOMOLOGY *

L. D. MDZINARISHVILI

Tbilisi, U.S.S.R.

1. Introduction

The total homology [9] began to be applied in the homology theory of topological spaces for two reasons. The first one (see [3], [5], [10], [11], [12], [27]) was the desire to define new algebraic invariants for arbitrary spaces. The second reason (see [13]–[15], [21]–[25]) was the creation of the strong shape theory and of its invariants. In both cases the aim was to define homology functors on the full category of topological spaces which satisfy all Eilenberg–Steenrod axioms [7], including the exactness axiom, so that they coincide with the Steenrod homology on compact metric spaces.

When constructing such a total homology of a space two methods have been employed. The first one was to consider the hyperhomology of a certain cochain complex, whose cohomology was isomorphic to the Alexandrov–Čech cohomology of the given space ([3], [5], [10]–[12]).

In the second method one considers the total homology of the second type [9] of a certain chain complex ([24], [25], [15], [23]) or of its fragments ([13], [14], [21], [22], [27]) associated with the given space. This approach was related to the appearance of a sequence of strong shape theories ([1], [2], [6], [13], [16]–[18], [21]–[23]).

For total homologies obtained in the first way the universal coefficient formula is valid which makes possible an easy verification of the Eilenberg–Steenrod axioms. For the total homologies obtained in the second way this formula can be false, as is e.g. for the homology functors defined in [15] for infinite polyhedra. Therefore the direct approach towards the verification of the Eilenberg–Steenrod axioms can lead here to considerable technical difficulties.

In [23] Z. P. Miminoshvili has derived two exact sequences for a total homology he defined earlier. Those sequences, one of which is short, relate his total homology to its "fragments" as well as to the derived inverse limit functor when applied to the homology inverse system, which is induced by some inverse system of polyhedra associated with a given space. In parti-

* This paper is in final form and no version of it will be submitted for publication elsewhere.

cular, they allow to prove that this total homology coincides on compacta with the Steenrod homology. On the other hand, those exact sequences also allow an easy verificiation of the Eilenberg–Steenrod axioms for arbitrary topological spaces.

Now a few words concerning the terminology. In [15] a certain total homology is called "Steenrod–Sitnikov homology". In our opinion the fact that this homology is known to be isomorphic to the Steenrod homology for compact metric spaces is not a sufficient reason for such a name; after all, the homology defined in [10] has all desired properties and coincides on compacta with the Steenrod homology, but differs from the homology defined in [15] on infinite polyhedra. The name "total homology" better reflects the structure of its construction.

The purpose of this paper is to investigate, in the most general context of a double complex of abelian groups (modules), the basic properties of the second type total homology and of its fragments. To achieve this goal, the horizontal homology of a complex obtained from the vertical homology is employed. As examples, we consider the homologies defined in [10] and [23]. We also define total homology of a space over a module.

2. Construction of total homology

Let $\underline{CC}$ be the category, whose objects are chain complexes of abelian groups (modules)

$$C_* = \ldots \leftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \leftarrow \ldots \tag{1}$$

and morphisms $f = \{f_n\}\colon C_* \to \bar{C}_*$ are chain mappings.

Consider a cochain complex in the category $\underline{CC}$:

$$C_*^* = \ldots \to C_*^{i-1} \xrightarrow{\delta^{i-1}} C_*^i \xrightarrow{\delta^i} C_*^{i+1} \to \ldots \tag{2}$$

We define the chain complex $R_*^\infty = \{R_n^\infty, \Delta_n\} \in \underline{CC}$ by $R_n^\infty = \prod_{i \in \mathbf{Z}} C_{n+i}^i$ with the differentials $\Delta_n = \{\Delta_n^i\}\colon R_n^\infty \to R_{n-1}^\infty$ given by $\Delta_n^i = \partial_{n+i}^i + (-1)^n \delta_{n+i}^i$. It is clear that $\Delta_{n-1}\Delta_n = 0$.

The chain complex R_*^∞ will be called the cone of the cochain complex C_*^* and the homology $H_*(R_*^\infty) = H_*^\infty$ will be called the total homology of the second kind.

For each $p \in \mathbf{Z}$ the cone of a cochain complex C_*^* in which $C_*^i = 0$ for $i > p$ is denoted by R_*^p. The cones R_*^p, $p \in \mathbf{Z}$, and the conical projections $s^p\colon R_*^p \to R_*^{p-1}$ form the inverse system

$$\{R_*^p, s^p\}. \tag{3}$$

Each projection s^p generates an exact sequence of chain complexes

$$0 \to C_*^p \xrightarrow{t^p} R_*^p \xrightarrow{s^p} R_*^{p-1} \to 0 \tag{4}$$

where t^p is a chain mapping of degree $(-p)$.

Therefore for each $p \in \boldsymbol{Z}$ we have the exact homology sequence

$$\ldots \to H_{n+p}(C^p_*) \xrightarrow{t^p} {}_n(R^p_*) \xrightarrow{s^p} H_n(R^{p-1}_*) \xrightarrow{\omega^p_n} H_{n+p-1}(C^p_*) \to \ldots \tag{5p}$$

write $K^p_n = \operatorname{Ker} \omega^p_n$.

LEMMA 1. *The composition*

$$H_n(R^{p-1}_*) \xrightarrow{\omega^p_n} H_{n+p-1}(C^p_*) \xrightarrow{\delta^p} H_{n+p-1}(C^{p+1}_*)$$

is a trivial homomorphism.

LEMMA 2. *The composition*

$$H_{n+p}(C^{p-1}_*) \xrightarrow{\delta^{p-1}} H_{n+p}(C^p_*) \xrightarrow{t^p} H_n(R^p_*)$$

is a trivial homomorphism.

LEMMA 3. *The diagram*

$$\begin{tikzcd}
H_{n+p-1}(C^{p-1}_*) \arrow[rr, "\delta^{p-1}"] \arrow[dr, "t^{p-1}"'] & & H_{n+p-1}(C^p_*) \\
& H_n(R^{p-1}_*) \arrow[ur, "\omega^p_n"'] &
\end{tikzcd}$$

is commutative if n is even, and anticommutative if n is odd.

Proof. Let z be a representative of $h \in H_{n+p-1}(C^{p-1}_*)$. Then $(-1)^n \delta^{p-1} z$ is a representative of the class $\omega^p_n t^{p-1} h$ and $\delta^{p-1} z$ is a representative of the class $\delta^{p-1} h$. The lemma is proved.

The cochain complex C^*_* (2) induces the cochain complex

$$\ldots \to H(C^{i-1}_*) \xrightarrow{\delta^{i-1}} H(C^i_*) \xrightarrow{\delta^i} H(C^{i+1}_*) \to \ldots \tag{6}$$

the homology groups of which are denoted by $E^* = \{E^p\}_{p \in \boldsymbol{Z}}$, where

$$E^p = \{E^p_q\}_{q \in \boldsymbol{Z}} = \{\operatorname{Ker} \delta^p_q / \operatorname{Im} \delta^{p-1}_q\}_{q \in \boldsymbol{Z}}.$$

LEMMA 4. *We have the exact sequence*

$$K^p_n \to K^{p-1}_n \to E^p_{n+p-1}.$$

Proof. Consider the diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & \\
& K^p_n \arrow[r, "\tilde{s}"] \arrow[d, "i"] & \operatorname{Ker} \tilde{\omega} \arrow[d, "\tilde{i}"] & \\
H_{n+p-1}(C^{p-1}_*) \arrow[r, "t^{p-1}"] \arrow[d, equal] & H_n(R^{p-1}_*) \arrow[r, "s^{p-1}"] \arrow[d, "\omega^p_n"] & K^{p-1}_n \arrow[r] \arrow[d, "\tilde{\omega}"] & 0 \\
H_{n+p-1}(C^{p-1}_*) \arrow[r, "\delta^{p-1}"] & \operatorname{Ker} \delta^p_{n+p-1} \arrow[r, "r"] & E^p_{n+p-1} \arrow[r] & 0
\end{tikzcd} \tag{7}$$

in which the exactness of the middle row is provided by the sequence (s_{p-1}) and the exactness of the last row by the definition of the cohomology groups E^p_{n+p-1}; the exactness of the middle column follows from Lemma 1; the homomorphisms $\tilde{\omega}$ and $\tilde{s}$ are induced by the homomorphisms ω^p_n and s^{p-1}, respectively.

Let us consider the composition

$$K^p_n \xrightarrow{\tilde{s}} K^{p-1}_n \xrightarrow{\tilde{\omega}} E^p_{n+p-1}, \tag{8}$$

where

$$\bar{s} = s^{p-1} \cdot i = \tilde{i} \cdot \tilde{s}. \tag{9}$$

It will be shown that $\tilde{s}$ is an epimorphism. Let $x \in \operatorname{Ker} \tilde{\omega}$. Then $r\omega^p_n(s^{p-1})^{-1}\tilde{i}(x) = 0$. Taking into consideration the exactness of the last row, we have $\delta^{p-1} y = \omega^p_n(s^{p-1})^{-1}\tilde{i}(x)$. If we take $(-1)^n y \in H_{n+p-1}(C^{p-1}_*)$, then from Lemma 3 it follows that $\omega^p_n t^{p-1}(-1)^n y = \omega^p_n(s^{p-1})^{-1}\tilde{i}(x)$. Let us define the element $z = (i)^{-1}[(s^{p-1})^{-1}\tilde{i}(x) - (-1)^n t^{p-1} y] \in K^p_n$. Since $\tilde{i}$ is a monomorphism and equality (9) holds, we have $\tilde{s}z = x$. The lemma is proved.

LEMMA 5. *We have the exact sequence*

$$E^{p-1}_{n+p-1} \to K^p_n \to K^{p-1}_n.$$

Proof. Let us consider the diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E^{p-1}_{n+p-1} \arrow[r, "\bar{f}"] \arrow[d, "\bar{\tau}"] & K^p_n \arrow[d, "i"] & & \\
 & H_{n+p-1}(C^{p-1}_*)/\operatorname{Im}\delta^{p-2}_{n+p-1} \arrow[r, "\bar{f}"] \arrow[d, "\tau"] & H_n(R^{p-1}_*) \arrow[r, "s^{p-1}"] \arrow[d, "\omega^p_n"] & K^{p-1}_n \arrow[r] \arrow[d, "\omega"] & 0 \\
0 \arrow[r] & H_{n+p-1}(C^{p-1}_*)/\operatorname{Ker}\delta^{p-1}_{n+p-1} \arrow[r, "\bar{\delta}"] \arrow[d] & H_{n+p-1}(C^p_*) \arrow[r, "\bar{\delta}"] & H_{n+p-1}(C^p_*)/\operatorname{Im}\delta^{p-1}_{n+p-1} \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

The exactness of the left column follows from the definition of $E^{p-1}_{n+p-1} = \operatorname{Ker}\delta^{p-1}_{n+p-1}/\operatorname{Im}\delta^{p-2}_{n+p-1}$; the exactness of the last row from the isomorphism $H_{n+p-1}(C^p_*)/\operatorname{Ker}\delta^{p-1}_{n+p-1} \approx \operatorname{Im}\delta^{p-1}_{n+p-1}$. Let us prove the exactness of the middle row. From Lemma 2 it follows that the homomorphism $t^{p-1}\colon H_{n+p-1}(C_*^{p-1}) \to H_n(R_*^{p-1})$ induces the homomorphism $\bar{t}$. The exactness of the middle row follows from the sequence (5_{p-1}) and the commutative diagram

$$
(10)\qquad
\begin{tikzcd}
H_{n+p-1}(C_*^{p-1}) \arrow[rr, "t^{p-1}"] \arrow[dr, "\bar\delta"'] & & \operatorname{Ker} s^{p-1} \arrow[r] & 0 \\
& H_{n+p-1}(C_*^{p-1})/\operatorname{Im}\delta^{p-2}_{n+p-1} \arrow[ur, "\bar t"'] & &
\end{tikzcd}
$$

The homomorphisms $\tilde{t}$ and ω are induced by the homomorphisms $\bar{t}$ and ω_n^p, respectively.

Let us prove that the composition

$$E^{p-1}_{n+p-1} \xrightarrow{\tilde t} K^p_n \xrightarrow{\tilde s} K^{p-1}_n,$$

where $\tilde{s} = s^{p-1}\cdot i$, is exact. Let $x\in \operatorname{Ker}\tilde{s}$. The existence of $y \in H_{n+p-1}(C_*^{p-1})/\operatorname{Im}\delta^{p-2}_{n+p-1}$ such that $\bar{t}(y) = i(x)$ follows from the exactness of the middle row. Since $\omega_n^p \bar{t}(y) = 0$, using Lemma 3 and the monomorphism $\bar{\delta}$, we have $\tau(y) = 0$. The exactness of the left column ensures the existence of $z\in E^{p-1}_{n+p-1}$ such that $i\cdot\tilde{t}(z) = \bar{t}\bar{\tau}(z) = \bar{t}(y) = i(x)$. Since i is a monomorphism, $\tilde{t}(z) = x$. The lemma is proved.

LEMMA 6. *We have the exact sequence*

$$K^{p-2}_{n+1} \to E^{p-1}_{n+p-1} \to K^p_n$$

Proof. We consider the commutative diagram

$$
(11)\qquad
\begin{tikzcd}
& H_{n+1}(R_*^{p-2}) \arrow[r, "s^{p-2}"] \arrow[d, "\omega^{p-1}"] & K^{p-2}_{n+1} \arrow[d, "\tilde\omega"] & & \\
& \operatorname{Ker}\delta^{p-1}_{n+p-1} \arrow[r, "r"] \arrow[d, "j"] & E^{p-1}_{n+p-1} \arrow[r, "\tilde t"] \arrow[d, "\bar\tau"] & K^p_n \arrow[d, "i"] & \\
& H_{n+p-1}(C_*^{p-1}) \arrow[r, "\bar\delta"] & H_{n+p-1}(C_*^{p-1})/\operatorname{Im}\delta^{p-2}_{n+p-1} \arrow[r, "\bar t"] & H_n(R_*^{p-1}) \arrow[r, "s^{p-1}"] & K_n^{p-1} \arrow[r] & 0
\end{tikzcd}
$$

Assume $x\in\operatorname{Ker}\tilde{t}$. Since r is an epimorphism, $y\in\operatorname{Ker}\delta^{p-1}_{n+p-1}$ such that $r(y) = x$ can be found. From diagram (10) it follows that $t^{p-1}\cdot j(y) = \bar{t}\bar{\delta}j(y) = \bar{t}\bar{\tau}r(y) = \bar{t}\bar{\tau}(x) = i\tilde{t}(x) = 0$. Since $j\omega^{p-1} = \omega^{p-1}_{n+1}$ and $\operatorname{Ker} t^{p-1} = \operatorname{Im}\omega^{p-1}_{n+1}$, by virtue of the exactness of the sequence (5_{p-1}), there exists $z\in H_{n+1}(R_*^{p-2})$

such that $\omega^{p-1}(z)=y$. Since $r\omega^{p-1}=\tilde{\omega}s^{p-2}$, we have $\tilde{\omega}(s^{p-2}(z))=x$. The lemma is proved.

From Lemmas 4, 5 and 6 follows

THEOREM 1. *We have the exact sequence*

$$\ldots \to K_{n+1}^{q-n-1} \to E_q^{q-n} \to K_n^{q-n+1} \to K_n^{q-n} \to E_q^{q-n+1} \to \ldots \tag{12}$$

LEMMA 7. *We have the isomorphism*

$$R_*^\infty \approx \varprojlim_p R_*^p$$

LEMMA 8. *We have the exact sequence*

$$0 \to \varprojlim_p{}^{(1)} H_{n+1}(R_*^p) \to H_n(R_*^\infty) \to \varprojlim_p H_n(R_*^p) \to 0.$$

COROLLARY 1. *We have the exact sequence*

$$0 \to \varprojlim_p{}^{(1)} K_{n+1}^p \to H_n(R_*^\infty) \to \varprojlim_p K_n^p \to 0.$$

Proof. From sequence (5*p*) we have $\operatorname{Im} s_n^p = K_n^p$ and from Proposition 2.2.IX [4] we have the isomorphism

$$\varprojlim_p{}^{(i)} \operatorname{Im} s^p \approx \varprojlim_p{}^{(i)} H_n(R_*^p), \qquad i=0,1.$$

Using Lemma 8, we obtain the required statement.

COROLLARY 2. (a) *Let* n *and* p *be fixed integers. If* $E_{n+t}^t = E_{n+t}^{t+1} = 0$ *for* $t \geqslant p$, *then*

$$H_{n-1}(R_*^\infty) \approx \varprojlim_t K_{n-1}^t$$

and the sequence

$$0 \to \varprojlim{}^{(1)} K_{n+1}^t \to H_n(R_*^\infty) \to K_n^p \to 0$$

is exact;

(b) *If condition* (a) *is fulfilled and* $E_{n+t+1}^t = 0$ *for* $t \geqslant p$, *then*

$$H_n(R_*^\infty) \approx K_n^t$$

for all $t \geqslant p$.

The complex C_*^* (2) will be called restricted from the left (right) if $C_*^i = 0$ when $i < p$ ($i > p$), for some fixed p.

The complex C_*^* (2) will be called restricted from below (above), if $C_j^* = 0$ when $j < q$ ($j > q$), for some fixed q.

LEMMA 9. (a) *If the complex C^*_* is restricted from the left, then we have the isomorphism*

$$E^p_{n+p} \xrightarrow{\sim} K^{p+1}_n$$

for all $n \in \mathbf{Z}$;

(b) *If the complex C^*_* is restricted from below, then we have the isomorphism*

$$E^{q-n}_q \xrightarrow{\sim} K^{q-n+1}_n$$

for all $n \in \mathbf{Z}$.

COROLLARY 3. (a) *If the complex C^*_* is restricted from the left and $E^t_q = 0$ for all $t > p' > p$ and $q \in \mathbf{Z}$, then for all $n \in \mathbf{Z}$ we have the finite exact sequence*

$$0 \to E^{p+1}_{n+p+1} \to K^{p+2}_n \to E^p_{n+p} \to E^{p+2}_{n+p+1} \to K^{p+3}_{n-1} \to K^{p+2}_{n-1}$$
$$\to E^{p+3}_{n+p+1} \to \ldots \to K^{p'-1}_{n+s+1} \to E^{p'}_{n+p+1} \to H^\infty_{n+s} \to K^{p'}_{n+s} \to 0,$$

where $s = p+1-p'$.

In particular, if $p' = p+1$, we have the exact sequence

$$0 \to E^{p+1}_{n+p+1} \to H^\infty_n \to E^p_{n+p} \to 0.$$

(b) *If the complex C^*_* is restricted from below and $E^p_t = 0$ for all $t > q' > q$ and $p \in \mathbf{Z}$, then for all $n \in \mathbf{Z}$, we have $q'-q$ exact sequences*

$$\ldots \to E^{q-n-1}_q \to E^{q-n+1}_{q+1} \to K^{q-n+2}_n \to E^{q-n}_q \to E^{q-n+2}_{q+1} \to K^{q-n+3}_{n-1} \to \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots \to K^{q'-n-1}_{n+1} \to E^{q'-n}_q \to H^\infty_n \to K^{q'-n}_n \to E^{q'-n+1}_q \to H^\infty_{n-1} \to K^{q'-n+1}_{n-1} \to \ldots$$

In particular, if $q' = q+1$, then we have the exact sequence

$$\ldots \to E^{q-n-1}_q \to E^{q-n+1}_{q+1} \to H^\infty_n \to E^{q-n}_q \to E^{q-n+2}_{q+1} \to H^\infty_{n-1} \to E^{q-n+1}_q \to \ldots$$

(c) *If the complex C^*_* is restricted from the right, then for all $n \in \mathbf{Z}$ we have the exact sequence*

$$\ldots \to K^{p-1}_{n+1} \to E^p_{n+p} \to H^\infty_n \to K^p_n \to 0.$$

Let $f^* = \{f^i\}$: $C^*_* \to \bar{C}^*_*$ be a mapping between two cochain complexes of the type (2), where f^i: $C^i_* \to \bar{C}^i_*$ is a chain mapping. It is clear that f^* induces the commutative diagrams

$$(13q) \qquad \begin{tikzcd}
\cdots \arrow[r] & K^{p-2}_{n+1} \arrow[r] \arrow[d, "\bar{f}^{p-2}_{n+1}"] & E^{p-1}_q \arrow[r] \arrow[d, "\bar{f}^{p-1}_q"] & K^p_n \arrow[r] \arrow[d, "\bar{f}^p_n"] & K^{p-1}_n \arrow[r] \arrow[d, "\bar{f}^{p-1}_n"] & E^p_q \arrow[r] \arrow[d, "\bar{f}^p_q"] & \cdots \\
\cdots \arrow[r] & \bar{K}^{p-2}_{n+1} \arrow[r] & \bar{E}^{p-1}_q \arrow[r] & \bar{K}^p_n \arrow[r] & \bar{K}^{p-1}_n \arrow[r] & \bar{E}^p_q \arrow[r] & \cdots
\end{tikzcd}$$

where $p = q-n+1$, and

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_r^{(1)} K^r_{n+1} \arrow[r] \arrow[d, "\varprojlim^{(1)} \bar f"] & H^\infty_n \arrow[r] \arrow[d, "f^\infty_n"] & \varprojlim_r K^r_n \arrow[r] \arrow[d, "\varprojlim \bar f"] & 0 \\
0 \arrow[r] & \varprojlim_r^{(1)} \bar K^r_{n+1} \arrow[r] & \bar H^\infty_n \arrow[r] & \varprojlim_r \bar K^r_n \arrow[r] & 0
\end{tikzcd}$$

COROLLARY 4. *If* $\bar f^r_n$ *is an isomorphism for* n *and* $n+1$ *and any* $r \geqslant p$, *then* f^∞_n *is an isomorphism.*

LEMMA 10. (a) *If the complexes* C^*_* *and* $\bar C^*_*$ *are restricted from the left and* $\bar f^t_q$: $E^t_q \to \bar E^t_q$ *are isomorphisms for all* $t \geqslant p$ *and* $q \in \boldsymbol{Z}$, *then* f^∞_n *is an isomorphism for all* $n \in \boldsymbol{Z}$.

(b) *If the complexes* C^*_* *and* $\bar C^*_*$ *are restricted from below and* $\bar f^p_t$: $E^p_t \to \bar E^p_t$ *are isomorphisms for all* $t \geqslant q$ *and* $p \in \boldsymbol{Z}$, *then* f^∞_n *is an isomorphism,* $n \in \boldsymbol{Z}$.

Proof. (a) From statement (a) of Lemma 9, diagram (13q) for $q = n+p+1$ and condition (a) of the lemma it follows that in the diagram

$$\begin{tikzcd}
0 \arrow[r] & E^{p+1}_{n+p+1} \arrow[r] \arrow[d] & K^{p+2}_n \arrow[r] \arrow[d, "\bar f^{p+2}_n"] & E^p_{n+p} \arrow[r] \arrow[d] & E^{p+2}_{n+p+1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \bar E^{p+1}_{n+p+1} \arrow[r] & \bar K^{p+2}_n \arrow[r] & \bar E^p_{n+p} \arrow[r] & \bar E^{p+2}_{n+p+1} \arrow[r] & \cdots
\end{tikzcd}$$

the homomorphism $\bar f^{p+2}_n$ is an isomorphism for all $n \in \boldsymbol{Z}$.

For $q = n+p+2$ diagram (13q) has the form

$$\begin{tikzcd}
\cdots \arrow[r] & E^p_{n+p+1} \arrow[r] \arrow[d] & E^{p+2}_{n+p+2} \arrow[r] \arrow[d] & K^{p+3}_n \arrow[r] \arrow[d, "\bar f^{p+3}_n"] & K^{p+2}_n \arrow[r] \arrow[d, "\bar f^{p+2}_n"] & E^{p+3}_{n+p+2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bar E^p_{n+p+1} \arrow[r] & \bar E^{p+2}_{n+p+2} \arrow[r] & \bar K^{p+3}_n \arrow[r] & \bar K^{p+2}_n \arrow[r] & \bar E^{p+3}_{n+p+2} \arrow[r] & \cdots
\end{tikzcd}$$

since $E^p_{n+p+1} \xrightarrow{\sim} K^{p+1}_{n+1}$ (Lemma 9 (a)).

Using condition (a) of the lemma, the isomorphism $\bar f^{p+2}_n$ and the five lemma, we conclude that $\bar f^{p+3}_n$ is an isomorphism for all $n \in \boldsymbol{Z}$. Continuing by induction, for all $q = n+t+1$ and $t \geqslant p+2$ from diagram (13q), and condition (a) of the lemma it can be deduced that $\bar f^{t+2}_n$ is an isomorphism, $t \geqslant p+2$. Applying now Corollary 4, we obtain the required statement.

(b) From statement (b) of Lemma 9, diagram (13($q+1$)) and condition (b) of the lemma it follows that in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E_q^{q-n-1} \arrow[r] \arrow[d] & E_{q+1}^{q-n+1} \arrow[r] \arrow[d] & K_n^{q-n+2} \arrow[r] \arrow[d, "\bar f_n^{q-n+2}"] & E_q^{q-n} \arrow[r] \arrow[d] & E_{q+1}^{q-n+2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bar E_q^{q-n-1} \arrow[r] & \bar E_{q+1}^{q-n+1} \arrow[r] & \bar K_n^{q-n+2} \arrow[r] & \bar E_q^{q-n} \arrow[r] & \bar E_{q+1}^{q-n+2} \arrow[r] & \cdots
\end{tikzcd}$$

the homomorphism $\bar f_n^{q-n+2}$ is an isomorphism for all $n \in \boldsymbol{Z}$.

Using condition (b) of the lemma, the isomorphism $\bar f_n^{q-n+2}$ and $\bar f_{n+1}^{q-n+1}$ and the five lemma, from diagram $(13(q+2))$ we deduce that $\bar f^{q-n+3}$ is an isomorphism for all $n \in \boldsymbol{Z}$. Continuing by induction, it can be proved that $\bar f_n^{q-n+t}$ is an isomorphism for all $t \geqslant 4$ and $n \in \boldsymbol{Z}$. Using now Corollary 4, we obtain the required statement.

Let $f^* = \{f^i\}$, $g^* = \{g^i\}$: $C_*^* \to \bar C_*^*$. The q-th components of the mappings f^i, g^i will be denoted by f_q^i, g_q^i: $C_q^i \to \bar C_q^i$.

By a homotopy (s_1, s_2): $f^* \sim g^*$ connecting the mappings f^*, g^*: $C_*^* \to \bar C_*^*$, we mean a pair of homomorphisms $s_1 = \{s_1^i\}$, $s_2 = \{s_2^i\}$: $C_*^* \to \bar C_*^*$, where s_1^i: $C_*^i \to \bar C_*^i$ is a homomorphism of degree $(+1)$ and s_2^i: $C_*^i \to \bar C_*^{i-1}$ is a homomorphism of the 0-th degree; their q-th components are $s_{1,q}^i$: $C_q^i \to C_{q+1}^i$, $s_{2,q}^i$: $C_q^i \to \bar C_q^{i-1}$, respectively such that for all $i, q \in \boldsymbol{Z}$ we have

$$\bar\partial_{q+1}^i s_{1,q}^i + s_{1,q-1}^i \partial_q^i + (-1)^{q-i+1} \bar\delta_q^{i-1} s_{2,q}^i + (-1)^{q-i} s_{2,q}^{i+1} \delta_q^i = g_q^i - f_q^i,$$
$$s_1 \delta = \bar\delta s_1, \qquad \bar\partial s_2 + s_2 \partial = 0.$$

For the cones R_*^∞ and $\bar R_*^\infty$ of the corresponding cochain complexes C_*^* and $\bar C_*^*$ the homotopy (s_1, s_2) defines the homomorphism $s = \{s^n\}$: $R_*^\infty \to \bar R_*^\infty$ of degree $(+1)$, where s^n: $R_n^\infty \to \bar R_{n+1}^\infty$ is given by the equality $s^n = \{s_{1,n+i}^i\} + \{s_{2,n+i}^i\}$.

Lemma 11. *The homomorphism s is a homotopy connecting the induced mappings f^*, g^*: $R_*^\infty \to \bar R_*^\infty$.*

Lemma 12. *If (s_1, s_2) is a homotopy connecting the mappings f^*, g^*: $C_*^* \to \bar C_*^*$, then*

$$\bar f_n^p = \bar g_n^p \colon K_n^p \to \bar K_n^p.$$

Lemma 13. *If (s_1, s_2) is a homotopy connecting the mappings f^*, g^*: $C_*^* \to \bar C_*^*$, then*

$$\tilde f_q^p = \tilde g_q^p \colon E_q^p \to \bar E_q^p.$$

3. Applications

Let $\mathscr{K}$ be an abelian category with a sufficient supply of injective objects, Ab the category of abelian groups, T: $\mathscr{K} \to$ Ab a left-exact additive covariant functor and $T^{(i)}$: $\mathscr{K} \to$ Ab the right derivatives of the functor T, $i \geqslant 0$. It is clear that $T^{(0)} = T$.

Consider a chain complex in the category $\mathcal{K}$:

$$K_* = \ldots \leftarrow K_{n-1} \xleftarrow{\partial_n} K_n \xleftarrow{\partial_{n+1}} K_{n+1} \leftarrow \ldots$$

The conditions imposed on the category $\mathcal{K}$ ensure the existence of the resolvent $I(K_*)$ of the complex K_*:

$$0 \to K_* \to I_*^0 \to I_*^1 \to \ldots \to I_*^{p-1} \to I_*^p \to \ldots,$$

where I_*^p are chain complexes, $p \geq 0$.

Denote $\operatorname{Ker} \partial_n$, $\operatorname{Im} \partial_{n+1}$, Z_n/B_n by Z_n, B_n, H_n, respectively, and their resolvents by $I(Z_n)$, $I(B_n)$, $I(H_n)$, respectively.

Applying the functor T to the injective resolvent $I(K_*)$, we obtain the cochain complex of the type (2) restricted from the left.

$$TI_*^0 \to TI_*^1 \to \ldots \to TI_*^{p-1} \to TI_*^p \to TI_*^{p+1} \to \ldots \tag{14}$$

THEOREM 2. *For cochain complex* (14) *and any* $n \in \mathbf{Z}$ *we have the exact sequence*

$$0 \to T^{(1)} H_{n+1} \to K_n^2 \to TH_n \to T^{(2)} H_{n+1} \to K_{n-1}^3 \to K_{n-1}^2 \to T^{(3)} H_{n+1} \to \ldots$$

Proof. Since the functor T is left exact, from the commutative diagrams

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & TB_n \arrow[r] \arrow[d] & TZ_n \arrow[r] \arrow[d] & TH_n \arrow[d] & \\
0 \arrow[r] & TI(B_n) \arrow[r] & TI(Z_n) \arrow[r] & TI(H_n) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & TZ_{n+1} \arrow[r] \arrow[d] & TK_{n+1} \arrow[r] \arrow[d] & TB_n \arrow[d] & \\
0 \arrow[r] & TI(Z_{n+1}) \arrow[r] & TI(K_{n+1}) \arrow[r] & TI(B_n) \arrow[r] & 0
\end{tikzcd}$$

it follows that for all $n \in \mathbf{Z}$ we have the isomorphism

$$H_n(TI_*^p) \xrightarrow{\sim} TI^p(H_n).$$

Therefore for $p \geq 0$ we have the isomorphism

$$E_n^p \xrightarrow{\sim} T^{(p)} H_n.$$

Using Lemma 9 (a) and Theorem 1, we obtain the required exact sequence.

Denote by K_*^∞ the cone of the cochain complex $TI(K_*)$.

COROLLARY 5. *If for a chain complex $K_* \in \mathscr{K}$ the condition $T^{(i)} H_n = 0$ is fulfilled for all $n \in \boldsymbol{Z}$ and $i \geqslant 2$, then we have the exact sequence*

$$0 \to T^{(1)} H_{n+1} \to H_n(K_*^\infty) \to TH_n \to 0.$$

Let K_* and $\bar{K}_*$ be two chain complexes and $f_*\colon K_* \to \bar{K}_*$ a chain mapping in the category $\mathscr{K}$.

COROLLARY 6. *If the chain mapping f_* induces the isomorphism $\tilde{f}_*\colon H(K_*) \xrightarrow{\sim} H(\bar{K}_*)$, then for all $n \in \boldsymbol{Z}$ the induced homorphism*

$$f_n^\infty\colon H_n(K_*^\infty) \to H_n(\bar{K}_*^\infty)$$

is an isomorphism.

COROLLARY 7. *If chain mappings $f_*, g_*\colon K_* \to \bar{K}_*$ are chain homotopic, then for all $n \in \boldsymbol{Z}$ we have*

$$f_n^\infty = g_n^\infty\colon H_n(K_*^\infty) \to H_n(\bar{K}_*^\infty),$$

$$\bar{f}_n^p = \bar{g}_n^p\colon K_n^p \to \bar{K}_n^p,$$

$$\tilde{f}_n^p = \tilde{g}_n^p\colon E_q^p \to \bar{E}_q^p.$$

Remark 1. From Corollary 7 it follows, in particular, that for any chain complex K the groups $H_n(K_*^\infty)$ and K_n^p, $n, p \in \boldsymbol{Z}$, are uniquely defined, independent of the choice of the injective resolvent $I(K_*)$.

THEOREM 3. *If the functor T is exact, then we have the isomorphism*

$$H_n(TK_*) \xrightarrow{\sim} H_n(K_*^\infty)$$

for all $n \in \boldsymbol{Z}$.

Proof. Since the functor T is exact, it follows that $T^{(p)} H_n(K_*) = 0$, $p > 0$. Using Theorem 2, we obtain the isomorphism

$$TH_n \approx K_n^t \tag{15n}$$

for all $n \in \boldsymbol{Z}$ and $t \geqslant 2$.

Since in the inverse system $\{K_{n+1}^t\}_t$ the projections are isomorphisms, $\varprojlim^{(1)} K_{n+1}^t = 0$. By virtue of Corollary 1 and the isomorphism (15n), we have

$$TH_n \approx K_n^2 \approx \varprojlim K_n^t \approx H_n(K_*^\infty). \tag{16}$$

On the other hand, since T is exact, we have

$$\operatorname{Ker} T\partial_n = T \operatorname{Ker} \partial_n = TZ_n,$$

$$\operatorname{Im} T\partial_{n+1} = T \operatorname{Im} \partial_{n+1} = TB_n.$$

Therefore

(17) $$H_n(TK_*) = \operatorname{Ker} T\partial_n/\operatorname{Im} T\partial_{n+1} = TZ_n/TB_n = TH_n.$$

The required statement follows from the isomorphisms (16) and (17). The theorem is proved.

A category $\mathscr{B}$ will be called "good" if in the category of chain complexes generated by it total homologies of the second kind can be defined.

COROLLARY 8. *If a category $\mathscr{K}$ is "good", then we have the isomorphism*

$$H_n(K_*) \approx H_n(R_*^\infty(I(K_*))).$$

Write $Z_n^\infty = \operatorname{Ker} \Delta_n$, $B_n^\infty = \operatorname{Im} \Delta_{n+1}$, $H_n^\infty = Z_n^\infty/B_n^\infty = H_n(R_*^\infty(I(K_*)))$, where Δ_n is a differential in the chain complex $R_*^\infty(I(K_*))$.

COROLLARY 9. *If a category $\mathscr{K}$ is "good" and a functor T commutes with direct products and $T^{(i)}(p_n^\infty)$: $T^{(i)}Z_n^\infty \to T^{(i)}H_n^\infty$ is a direct sum, then we have the exact sequence*

$$\ldots \to T^{(3)}H_{n+2} \to T^{(1)}H_{n+1} \to H_n(K_*^\infty) \to TH_n \to T^{(2)}H_{n+1} \to \ldots$$

(see [20]).

4. Examples

EXAMPLE 1. Let $K = \operatorname{Inv}^\Omega$ be the category of inverse systems of abelian groups with the same directed set of indexes $\Omega = \{\alpha\}$ and let $T^{(p)} = \underleftarrow{\lim}^{(p)}$: $\operatorname{Inv}^\Omega \to \mathrm{Ab}$ be the derivatives of the functor of inverse limit $\underleftarrow{\lim}$, $p \geqslant 0$.

Consider an arbitrary chain complex

(18) $$\underline{K}_* = \ldots \leftarrow \underline{K}_{n-1} \leftarrow \underline{K}_n \leftarrow \underline{K}_{n+1} \leftarrow \ldots$$

from the category $\operatorname{Inv}^\Omega$.

Since in the chain complex (18) the chain complex

$$K_*^\alpha = \ldots \leftarrow K_{n-1}^\alpha \leftarrow K_n^\alpha \leftarrow K_{n+1}^\alpha \leftarrow \ldots$$

is given for each $\alpha \in \Omega$, its homologies form the inverse system $\underline{H}_* = \{H_*^\alpha\} \in \operatorname{Inv}^\Omega$ and $H_*(\underline{K}_*) = \underline{H}_*$.

By virtue of Theorem 2, for the chain complex (18) we have the exact sequence

(19) $$0 \to \underleftarrow{\lim}^{(1)} \underline{H}_{n+1} \to K_n^2 \to \underleftarrow{\lim}\, \underline{H}_n \to \underleftarrow{\lim}^{(2)} \underline{H}_{n+1}^{(2)} \to K_{n-1}^3$$
$$\to K_{n-1}^2 \to \underleftarrow{\lim}^{(3)} \underline{H}_{n+1} \to \ldots$$

For a given chain complex (18) we will define the homology groups $\check{H}'_*$,

$t = 0, 1, 2, \ldots$, and also $\overset{\infty}{H}_*(\underline{K}_*)$, using for this purpose the derivatives $\underset{\leftarrow}{\lim}^{(p)}$, $p \geqslant 0$ constructed, as was suggested by Roos [26], for the inverse system $\underline{A} = \{A_\alpha\} \in \mathrm{Inv}^\Omega$ as the cochain complex cohomology

$$\underline{\underline{A}}^* := \prod_{\alpha \in \Omega} A_\alpha \to \prod_{\alpha_1 < \alpha_2} A_{\alpha_1\alpha_2} \to \prod_{\alpha_1 < \alpha_2 < \alpha_3} A_{\alpha_1\alpha_2\alpha_3} \to \ldots,$$

where $A_{\alpha_1 \ldots \alpha_n} \equiv A_{\alpha_1}$, $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ [26].

Taking into account Remark 1, it can be shown that $\check{H}^t_* \approx K^t_*$, $t = 1, 2, \ldots$, $\overset{\infty}{H}_*(\underline{K}_*) \approx H_*(\overset{\infty}{\underline{K}}_*)$ and also that the sequence [19] is isomorphic to the sequence from Theorem 1 [23].

EXAMPLE 2. Let $\mathscr{K} = {}_R\mathscr{M}$ be the category of (left) modules over a given ring R and $\mathrm{Ext}^n_R(-, G)$: ${}_R\mathscr{M} \to \mathrm{Ab}$ the right derived functors of the functor $\mathrm{Hom}_R(-, G)$, $n \geqslant 0$. In what follows the symbol R will be omitted in the notation $\mathrm{Ext}^n_R(-, G)$.

Consider an arbitrary cochain complex

$$K^* = \ldots \to K^{n-1} \xrightarrow{\delta^{n-1}} K^n \xrightarrow{\delta^n} K^{n+1} \to \ldots$$

from the category ${}_R\mathscr{M}$, cohomology modules of which will be denoted by $H(K^*) = H^*$.

By virtue of Theorem 2, for the cochain complex K^* we have the exact sequence

$$(20) \qquad 0 \to \mathrm{Ext}(H^{n+1}, C) \to K^2_n \to \mathrm{Hom}(H^n, G) \to \mathrm{Ext}^2(H^{n+1}, G) \\ \to K^3_{n-1} \to K^2_{n-1} \to \mathrm{Ext}^3(H^{n+1}, G) \to \ldots$$

We denote by $H^\infty_*(K^*)$ the homologies of the cochain complex $\mathrm{Hom}(P_* K^*, G)$, where $P_* K^*$ is the projective resolvent of the complex K^*. We have the exact sequence

$$(21) \qquad 0 \to \underset{\overleftarrow{i}}{\lim}^{(1)} K^i_{n+1} \to H_n(K^*) \to \underset{\overleftarrow{i}}{\lim}\, K^i_n \to 0.$$

If $R = Z$, then the conditions of Corollary 5 are fulfilled and we have

$$0 \to \mathrm{Ext}(H^{n+1}, G) \to H^\infty_n(K^*) \to \mathrm{Hom}(H^n, G) \to 0.$$

EXAMPLE 3. Let X be a topological space, C^X the category of abelian group bundles over X and Γ_Φ: $C^X \to \mathrm{Ab}$ the functor, such that is $\Gamma_\Phi(F)$ the group of sections of the bundle F over X with carriers from the family Φ [8].

As is known [8], the cohomology $H^n(X, F)$ of the space X with values in the bundle F is defined as the n-th right derived functor of the functor Γ_Φ.

In the category C^X we consider a chain complex

$$L_* = \ldots \leftarrow L_{n-1} \leftarrow L_n \leftarrow L_{n+1} \leftarrow \ldots$$

the homology of which will be denoted by $\mathscr{H}(L_*) = \mathscr{H}_*$. Applying the functor Γ_Φ to the injective resolvent $I^*(L_*)$, we obtain the cochain complex $\Gamma_\Phi I^*(L_*)$. The homology of the cochain complex cone $\Gamma_\Phi I^*(L_*)$ will be denoted by $H_*(X, L_*)$.

By virtue of Theorem 2, for the chain complex L_* we have the exact sequence

$$0 \to H^1(X, \mathscr{H}_{q+1}) \to K_q^2 \to H^0(X, \mathscr{H}_q) \to H^2(X, \mathscr{H}_{q+1})$$
$$\to K_{q-1}^3 \to K_{q-1}^2 \to H^3(X, \mathscr{H}_{q+1}) \to \ldots,$$

where $H^*(X, \mathscr{H}_p)$ is the cohomology of the space X with values in the bundle.

Moreover, we have the exact sequence

$$0 \to \varprojlim_i{}^{(1)} K_{q+1}^i \to H_q(X, L_*) \to \varprojlim_i K_q^i \to 0.$$

Using Examples 1 and 2, exact homology theories can be constructed for arbitrary topological spaces with coefficients in an abelian group or in a module.

Example 4. Let Top_2 be the category of pairs of arbitrary topological spaces. For each object $(X, A) \in \mathrm{Top}_2$ we will consider the inverse system $\{(X_\alpha, A_\alpha)\}$, consisting of pairs of simplicial complexes (X_α, A_α). This inverse system can consist either of nerves of coverings, whose refinement induces the unique simplicial mapping between the nerves or of Vietoris complexes [19]. Applying Example 1 to the inverse system $\{C_*(X_\alpha, A_\alpha, G)\}$ of chain complexes $C_*(X_\alpha, A_\alpha, G)$ over the coefficient group G, we can define the homology groups $H_*^\alpha(X, A, G) = \overset{\alpha}{H}_* \{C_*(X_\alpha, A_\alpha, G)\}$ of the pair (X, A) over the coefficient group G, satisfying all Eilenberg–Steenrod axioms [23].

Example 5. Let $(X, A) \in \mathrm{Top}_2$, R be a ring with unity and $\bar{C}^*(X, A, R)$ the Alexander–Kolmogorov cochain complex of the pair (X, A) [28]. Applying Example 2 to the cochain complex $\bar{C}^*(X, A, R)$, we define the homology $\bar{H}_*(X, A, G)$ of the pair (X, A) over the module G as $H_*^\alpha(\bar{C}^*(X, A, R))$. It will be shown that the homology defined in this manner satisfies all Eilenberg–Steenrod axioms.

To an exact sequence of cochain complexes

$$0 \to \bar{C}^*(X, A, R) \to \bar{C}^*(X, R) \to \bar{C}^*(A, R) \to 0$$

there corresponds an exact sequence of projective resolvents

$$0 \to P_* \bar{C}^*(X, A, R) \to P_* \bar{C}^*(X, R) \to P_* \bar{C}^*(A, R) \to 0.$$

Then the sequence of complexes

$$0 \to \operatorname{Hom}(P_* \bar{C}^*(A, R), G) \to \operatorname{Hom}(P_* C^*(X, R), G) \to \operatorname{Hom}(P_* C^*(X, A, R), G) \to 0$$

is exact and so is the corresponding sequence of cones

$$0 \to R_*^{\infty}(A) \to R_*^{\infty}(X) \to R_*^{\infty}(X, A) \to 0.$$

The exactness axiom for the homology $\bar{H}_*(X, A, G)$ follows.

The remaining axioms (homotopy, excision and dimension) will be proved, using exact sequences (20) and (21).

Since for any $n \in \boldsymbol{Z}$ and $p \geqslant 1$ we have the exact sequence

$$\operatorname{Ext}^p(\bar{H}^{n+p}(X, A, R), G) \to K_n^{p+1} \to K_n^p \to \operatorname{Ext}^{p+1}(\bar{H}^{n+p}X, A, R), G)$$

and the cohomology $\bar{H}^*(X, A, R)$ satisfies the dimension axiom, for the one-point space X we have

$$\bar{H}_n(X, G) \approx K_n^2 \approx \operatorname{Hom}(\bar{H}^n(X, R), G).$$

Let j: $(X-U, A-U) \to (X, A)$ be the excision mapping, where U is an open subset in X such that $\bar{U} \subset \operatorname{Int} A$. Since j induces the isomorphism j^*: $\bar{H}^*(X, A, R) \to \bar{H}^*(X-U, A-U, R)$, Corollary 5 gives us the excision axiom for the homology $\bar{H}_*(X, A, G)$. Let us have mappings g_0, g_1: $(X, A) \to (X, A) \times I$, where $g_0(x) = (x, 0)$, $g_1(x) = (x, 1)$ and p: $(X, A) \times I \to (X, A)$, where $p(x, t) = x$. The mapping p induces the cohomology isomorphism $\bar{H}^*(X, A, R) \to \bar{H}^*((X, A) \times I, R)$. Using Corollary 5, we obtain the isomorphism p_*: $\bar{H}_*((X, A) \times I, G) \to \bar{H}_*(X, A, G)$. On the other hand, we have $g_{0*} p_* = g_{1*} p_*$. Hence it follows that $g_{0*} = g_{1*}$.

If $R = \boldsymbol{Z}$, then the homology $\bar{H}_*(X, A, G)$ of the pair (X, A), with coefficients in the abelian group G, coincides with the homology introduced in [10].

As is known [28], there exists a canonical isomorphism

$$\bar{C}^*(X, A) \xrightarrow{\sim} \varinjlim \{C^*(V_\alpha, V_\alpha')\},$$

where $\bar{C}^*(X, A)$ are the integral Alexander Kolmogorov cochains of the pair (X, A) and $C^*(V_\alpha, V_\alpha')$ are the integral cochains of the pair of Vietoris complexes (V_α, V_α'), corresponding to an open covering (U_α, U_α') of the pair (X, A). Obviously, there exists a free resolvent $\{F^\alpha\}$ of the direct spectrum $\{C^*(V_\alpha, V_\alpha')\}$ such that $\varinjlim \{F^\alpha\}$ is a free resolvent for $\bar{C}^*(X, A)$. Since for all $n \geqslant 0$ we have

$$R_n^{\infty}(X, A) = \operatorname{Hom}(\varinjlim F_0^{\alpha, n}, G) + \operatorname{Hom}(\varinjlim F_1^{\alpha, n+1}, G) \approx \operatorname{Hom}(\varinjlim (F_0^{\alpha, n} + F_1^{\alpha, n+1}), G),$$

the conditions of Theorem B [20] are fulfilled, and for the homology $\bar{H}_*(X, A, G)$ we have the exact sequence

$$\ldots \varprojlim{}^{(3)} \bar{H}^\alpha_{n+2} \to \varprojlim{}^{(1)} \bar{H}^\alpha_{n+1} \to \bar{H}_n(X, A, G) \to \varprojlim \bar{H}^\alpha_n \to \varprojlim{}^{(2)} \bar{H}^\alpha_{n+1} \to \ldots$$

where $\bar{H}_p(V_\alpha, V'_\alpha, G) = H_p(\mathrm{Hom}(F^\alpha, G)) = \bar{H}^\alpha_p$.

References

[1] F. W. Bauer, *A shape theory with singular homology*, Pacific J. Math. 64 (1976), 25–65.

[2] —, *Homology functors on the shape categories*, Cahiers Topologie Géom. Différentielle 21 (1980), 427–439.

[3] A. Borel and I. C. Moore, *Homology theory for locally compact spaces*, Michigan Math. J. 7 (1960), 137–159.

[4] A. K. Bousfield and D. M. Kahn, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer, New York 1972.

[5] R. Deheuvels, *Homologie des ensembles ordonnés et des espaces topologiques*, Bull. Soc. Math. France 90 (1962), nr 2, 261–321.

[6] D. A. Edwards and H. M. Hastings, *Čech and Steenrod Homotopy Theories with Applications to Geometric Topology*, Lecture Notes in Math. 542, Springer, Berlin 1976.

[7] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, 1952.

[8] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. 9 (2, 3) (1957).

[9] P. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer, 1971.

[10] H. N. Inasaridze, *On the exact homology*, Trudy Tbiliss. Mat. Inst. 41 (1972), 128–142.

[11] V. I. Kuz'minov and I. A. Shvedov, *Covering spectra in the theory of cohomology and homology of topological spaces* (in Russian), Sibirsk. Mat. Zh. 15 (1974), 1083–1102.

[12] —, *Hyperhomology of limits of direct spectra of complexes and homology groups of topological spaces* (in Russian), Sibirsk. Mat. Zh. 16 (1975), 49–59.

[13] Yu. T. Lisitsa, *Strong shape theory and the Steenrod–Sitnikov homology* (in Russian), Sibirsk. Mat. Zb. 24 (1983), 81–99.

[14] —, *Duality theorems and dual shape and coshape categories* (in Russian), Dokl. Akad. Nauk SSSR 263 (1982), 532–536.

[15] Yu. T. Lisitsa and S. Mardešić, *Steenrod–Sitnikov homology for arbitrary spaces*, Bull. Amer. Math. Soc. 9 (1983), 207–210.

[16] —, —, *Coherent prohomotopy and a strong shape theory*, Glasnik Mat. 19 (1984).

[17] —, —, *Coherent prohomotopy and strong shape of metric compacta*, Glasnik Mat. 20 (1985).

[18] —, —, *Coherent prohomotopy and strong shape for pairs*, Glasnik Mat. 20 (1985).

[19] L. D. Mdzinarishvili, *On various space homology groups based on infinite coverings* (in Russian), Soobshch. Acad. Nauk Gruzin. SSR 38 (1) (1965), 23–30.

[20] —, *Universelle koeffizientenfolgen für den $\varprojlim$-funktor und anwendungen*, Manuscripta Math. 48 (1984), 255–273.

[21] Zh. R. Miminoshvili, *On the connection between shape homotopy groups and homologies for compact spaces* (in Russian), Soobshch. Akad. Nauk Gruzin. SSR 101 (1981), 305–308.

[22] —, *On a strong spectral shape theory* (in Russian), Trudy Tbiliss. Mat. Inst. 68 (1982), 79–102.

[23] —, *On the suquence of exact and half-exact homologies of arbitrary spaces* (in Russian), Soobshch. Akad. Nauk Gruzin. SSR 113 (1) (1984), 41–44.

[24] T. Porter, *Coherent prohomotopical algebra*, Cahiers Topologie Géom. Différentielle 18 (1977), 139–178.

[25] —, *Coherent prohomotopy theory*, Cahiers Topologie Géom. Différentielle 19 (1978), 3–46.

[26] J. E. Roos, *Sur les foncteurs derivés de lim*, C. R. Acad. Sci. Paris 252 (1961), 3702–3704.

[27] K. A. Sitnikov, *Combinatorial topology of non-closed sets*, I. *The first duality law; the spectral duality* (in Russian), Mat. Sb. 34 (1954), 3–54.

[28] E. H. Spanier, *Algebraic Topology*, 1966.

Presented to the Topology Semester
April 3 – June 29, 1984