Continuous and discrete flows and the property of $ch0^{\pm}$

by

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Abstract. Consider a continuous or a discrete flow (X, T) where X is a locally compact connected metric space. Let B denote the set of points $x \in X$ such that x is not a point either of $ch0^+$ or $ch0^-$. Assume that $\bar{B}$ does not disconnect X and has only compact components. This paper answers the question: How many components can $\bar{B}$ have. For continuous flows the question is completely answered, for discrete flows it is answered completely if we assume furthermore that $A = X - \bar{B}$ is a semi-continuum.

Introduction. Let (X, T) be a continuous or a discrete flow. Let B denote the set of points $x \in X$ where T fails to be either of $ch0^+$ or $ch0^-$ (for definition see [1]), and let $A = X - \bar{B}$. Assume that X is a locally compact, connected metric space, that $\bar{B}$ does not disconnect X, that is, A is connected, and that every component of $\bar{B}$ is compact

We wish to study the cardinality of the components of $\bar{B}$ and the behaviour of the trajectories of points in A relative to the components of $\bar{B}$. The major steps in the study are as follows: First we look at the case where $B = \emptyset$. It turns out that if such a flow contains one recurrent point then it is pointwise equicontinuous and each point of X has compact orbit closure or equivalently the flow is pointwise almost periodic. Next we look at the case where $B \neq \emptyset$ and $\bar{B}$ is totally disconnected. In this case the results for discrete flows require that A be a semi-continuum. It is then shown that B consists of at most two points, both left fixed by each element of T, and any point of B acts either as an attractive point or a repulsive point for all elements of A. Lastly in the general case $B \neq \emptyset$ and not zero dimensional, it is shown that if the flow is such that some point of A is not a recurrent point then again $\bar{B}$ has at most two components and they behave like the points of the second case above. An example is given to show that this additional condition is also necessary for such a result.

We give below precise statements of these results after preliminary definitions.

Throughout this paper a *transformation group* (X, T, π) or simply (X, T) [2] is a discrete or a continuous flow, that is, T is either the additive group of integers with discrete topology or the additive group of reals with the usual topology and X is a locally compact, connected metric space. T^+ (T^-) denote the nonnegative (non-

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positive) set of elements of T. A sequence $\{t_i\}$ in T is said to *diverge* if it has no convergent subsequence. We use $t_i \to \infty$ ($t_i \to -\infty$) to indicate that the sequence diverges to ∞ (or $-\infty$).

We recall some definitions and elementary results. For any $x \in X$, the positive limit set of x, $L^+(x) = \{y \in X: xt_i \to y$ for some $\{t_i\}$ diverging to $\infty\}$. It is known that $L^+(x)$ is a closed subset of X and is invariant under T. The positive prolongation limit set of x, $J^+(x) = \{y \in X: x_i t_i \to y$, for some sequence $\{x_i\}$ converging to x and $\{t_i\}$ diverging to $\infty\}$. The positive prolongation set of x, $D^+(x) = \{y \in X: x_i t_i \to y$, for some $\{x_i\}$ converging to x and some sequence $\{t_i\}$ in $T^+\}$. The positive orbit closure of x is denoted by $K^+(x)$. Similar sets are defined for T^- and denoted by $L^-(x)$, $J^-(x)$, $D^-(x)$ and $K^-(x)$. $K(x)$ denotes the orbit closure of x. It is easy to see that if $y \in J^+(x)$ then $x \in J^-(y)$.

A point $x \in X$ is said to be of $ch0^+$ ($ch0^-$) if $K^+(x) = D^+(x)$ ($K^-(x) = D^-(x)$). Let B denote the set of points of X which fail to be either of $ch0^+$ or of $ch0^-$ and set $A = X - \bar{B}$. Thus each point of A is both of $ch0^+$ and $ch0^-$. For terms not defined here see [3].

Definition (0.1). A flow is said to be of $ch0^\pm$ if $B = \emptyset$. A flow is said to be of $ch0^\pm$ *almost everywhere* (CAE for short) if $B \neq \emptyset$, $\bar{B}$ is zero dimensional and A is connected.

In § 1 we study flows of $ch0^\pm$. The main result is:

Theorem A. *Let (X, T) be a flow of $ch0^\pm$. Then the following are equivalent.*

(i) *Some point $x_0 \in X$ is T-recurrent.*

(ii) *(X, T) is pointwise almost periodic.*

(iii) *(X, T) is pointwise equicontinuous and for each $x \in X$, $K(x)$ is compact.*

The notion of "indivisibility" was first introduced in [4]. We give a definition of indivisible applicable to our case of CAE flows in § 2 and obtain a crucial result, Lemma (2.1). The following structure theorems for CAE flows are proved in § 3.

Theorem B. *Let (X, R) be a continuous CAE flow. Then it has at most one point each not of $ch0^+$ and $ch0^-$. Furthermore, if there exists a point of $ch0^+$ ($ch0^-$) then R^- (R^+) is equicontinuous on A.*

Theorem C. *Let (X, Z) be a discrete CAE flow. Then the conclusions of Theorem* B *hold if A is a semi-continuum.*

Remark. Theorem B is proved by Lam [5, Th. 4, p. 145] with the additional assumption that A contain a point which is not almost periodic. Theorem A helps us to drop this assumption. Theorem B yields Theorem (3.2) for discrete flows similar to Lam's Theorem 7 [5, p. 146] which is for continuous flows only.

§ 1. In this section we shall prove Theorem A starting with some preliminary results.

Lemma (1.1). *Let (X, T, π) be a continuous or a discrete flow, $x \in X$ be of $ch0^+$ and $K^+(x)$ be compact. If U is an open set in X containing $K^+(x)$ ($L^+(x)$) then there exists an open set V containing x such that for each $y \in V$, $K^+(y) \subset U$ [$L^+(y) \subset U$].*

Proof. We first prove the lemma for $T = Z$. Let W be an open set in X such that $K^+(x) \subset W \subset \bar{W} \subset U$, and $\bar{W}$ is compact. Suppose no such open set V as claimed in the lemma exists. Then using the continuity of π we can find a sequence $\{t_n\}$ in Z^+ such that $x_n t_n \in W$ but $x_n(t_n+1) \notin W$ for some sequence $\{x_n\}$ converging to x. Consequently, $x_n(t_n+1) \in \bar{W}(1)$ gives a point $y \in \bar{W}(1) - W$ which lies in $D^+(x)$. Since x is of $ch0^+$, this is a contradiction. This proves the lemma for $T = Z$.

Now suppose that $T = R$ and the lemma is not true. Then using the continuity of π again, and W as above, the fact that $K^+(x) \subset W$ gives a sequence $\{x_n\}$ converging to x, and sequences $\{s_n\}$ and $\{t_n\}$ in R^+ such that $x_n t_n \notin W$ but $x_n s_n \in W$, so that there is a γ_n between s_n and t_n for each n, such that $x_n \gamma_n \in \partial W$. This implies that $D^+(x) \cap \partial W \neq \emptyset$ and since x is of $ch0^+$, $K^+(x) = D^+(x)$, contradicting the choice of W. This proves the lemma for $T = R$.

It is easy to see that the same proof applies to $L^+(x)$.

Lemma (1.2). *Let (X, T) be a continuous or a discrete flow. Then the following are true.*

(a) *If x is of $ch0^+$ ($ch0^-$) then $L^+(x) = J^+(x)$ ($L^-(x) = J^-(x)$).*

(b) *If x is of $ch0^+$ and $y \in L^+(x)$ is of $ch0^-$, then $x \in L^+(x)$.*

Proof. For proof of (a) see Lemma (1.1) [4, p. 808]. (b) Since $L^+(x)$ is T-invariant and closed, $y \in L^+(x)$ implies that $K(y) \subset L^+(x) = J^+(x)$. Therefore, $x \in J^-(y) = L^-(y) \subset K(y) \subset L^+(x)$. This completes the proof.

Lemma (1.3). *Let (X, T) be a continuous or a discrete flow of $ch0^\pm$. Then the set $Y = \{x \in X: T$ is recurrent at $x\}$ is both open and closed in X.*

Proof. Recall that x is T recurrent if and only if $x \in L^+(x)$ and $x \in L^-(x)$ [2, ch. 7], and that Y is invariant under T [2, Th. (7. 03), p. 64]. The proof is divided into three parts:

(a) Y is closed in X: Let $y \in \bar{Y}$ and $\{y_n\}$ be a sequence in Y converging to y. Then $y_n \in L^+(y_n) = J^+(y_n)$ from Lemma (1.2). Hence $y \in J^+(y) = L^+(y)$. Similarly $y \in L^-(y)$ and Y is closed. Consequently Y is locally compact.

(b) For any $y \in Y$, the orbit closure of y is a minimal set: Since $L^+(y)$ is closed and invariant, $y \in L^+(y) = K(y)$ [2, Remark (7. 02), p. 64]. Let $x \in K(y)$; then $x \in L^+(y) = J^+(y)$, from Lemma (1.2), hence $y \in J^-(x) = L^-(x) = K(x)$ and $K(y)$ is minimal.

(c) Now consider the flow (Y, T). Then by Theorem (7.05) [2, p. 64], (Y, T) is pointwise almost periodic and hence for any $y \in Y$, $K(y)$ is compact. Therefore, from Lemma (1.1), for any $y \in Y$ there is an open set V containing y such that for any $x \in V$, $K(x)$ is compact hence $L^+(x) \neq \emptyset$ and $L^-(x) \neq \emptyset$. Hence $V \subset Y$ and Y is open. This completes the proof of the Lemma.

Proof of Theorem A. (i)$\Rightarrow$(ii): By assumption (i) the set Y defined in Lemma (1.3) is nonempty. Since X is connected $Y = X$ and, by (c) in the proof of Lemma (1.3) (X, T) is pointwise almost periodic.

(ii)$\Rightarrow$(iii): That $K(x)$ is compact for each $x \in X$ follows from Theorem (4.09) [2, p. 32]. From Lemma (1.1) it is easy to see then that (X, T) is pointwise equicontinuous.

(iii)$\Rightarrow$(i): Trivial.

We give an example to show that if in Theorem A the assumption that a recurrent point exists is dropped then the theorem is not true.

EXAMPLE (1.4). Consider the differential equation $\dot{\gamma} = -\gamma^2 \sin\theta$ in polar coordinates (γ, θ) in the plane. The trajectories given by the polar equation $\gamma = c - \cos\theta$ for arbitrary $c \geqslant 0$ describe a continuous flow (X, R) where X is the plane $\boldsymbol{R}^2$.

It is easy to see that no point on the parabola $c = 1$ is a point of $ch0^+$ or $ch0^-$ as every such point is a limiting point of periodic points on the ellipses given by $c > 1$. But now consider the subspace Y of the plane obtained by deleting all ellipses from the plane X. Then (Y, R) is again a continuous flow on a connected locally compact metric space Y. It has no recurrent points. Every point of Y is a point of $ch0^+$ but no point on the line $c = 0$ $(x = -1)$ is a point of equicontinuity.

§ 2. For a general definition of indivisibility see [4]. We give below a definition more suitable for our purposes.

DEFINITION (2.0). A CAE flow (X, T) is said to be *indivisible* if, whenever for some $x_0 \in A$ there exists a sequence $\{t_n\}$ in T such that $x_0 t_n \to p \in B$, then $xt_n \to p$ for all $x \in A$.

Our first result is a lemma crucial to the study of CAE flows. For any subset W of X we denote the component of any point $x \in W$ in W by $c(x, W)$.

LEMMA (2.1). *Let (X, T) be a* CAE *flow. If p is not of $ch0^+$ $(ch0^-)$, then there exists a point $y \in A$ such that $p \in L^-(y)$ $(p \in L^+(y))$.*

Proof. Since p is not of $ch0^+$, there is a point $q \in D^+(p) - K^+(p)$ and consequently sequences $\{p_n\}$ in X and $\{t_n\}$ in T^+ such that $p_n \to p$ and $p_n t_n \to q$. Let U, V be open sets in X such that $q \in U \subset V$, $\overline{V}$ is compact, $\partial U \cup \partial V \subset A$, and $\overline{V} \cap K^+(p) = \emptyset$. Then the following two cases are exhaustive. Either

(a) there exists an open set W containing p such that $c(p_n, \overline{W}) t_n \subset U$ for all $n \geqslant n_0$ for some positive integer n_0, or

(b) given any open W containing p there are infinitely many values of n for which $c(p_n, \overline{W}) t_n \not\subset U$.

Suppose (a) holds. Without loss of generality we may suppose that $\overline{W}$ is compact, $X - \overline{W} \neq \emptyset$ and $\partial \overline{W} \subset A$. Then X being connected $\partial W \neq \emptyset$ and by (10.1) [6, p. 16], $c(x, \overline{W}) \cap \partial W \neq \emptyset$ for any $x \in W$. Using the terminology of [6] and Theorem (9.1) [6, p. 14], clearly $p \in \liminf_{n\to\infty} c(p_n, \overline{W})$, hence $\limsup_{n\to\infty} c(p_n, \overline{W}) = E$ is connected. Also $E \cap \partial W \neq \emptyset$ since $c(p_n, \overline{W}) \cap \partial W \neq \emptyset$ and ∂w is compact. Let $z \in E \cap \partial W \subset E \cap A$. We may assume without loss of generality (or else we work with a subsequence) that there is a sequence $z_n \in c(p_n, \overline{W})$ $n = 1, 2, \ldots$ converging to z. Then, by assumption (a), $z_n t_n \in U$, and $\overline{U}$ being compact, $J^+(z) \cap \overline{U} \neq \emptyset$. Since z is of $ch0^+$, by Lemma (1.2) (a), $L^+(z) \cap \overline{U} \neq \emptyset$.

Now let $\{W_n\}$, where $W_1 = W$, be a decreasing sequence of neighbourhood base at p, and choose inductively for each integer n an element $z_n \in c(p, \overline{W}_n)$ as above so that $L^+(z_n) \cap \overline{U} \neq \emptyset$. Thus, for each n, there exists an $s_n \in T^+$ such that $z_n s_n \in V$. Since $K^+(p) \cap \overline{V} = \emptyset$, for sufficiently large values of n, $ps_n \notin \overline{V}$. Hence $c(p, \overline{W}_n) s_n \cap \partial V \neq \emptyset$. Consequently, as $z_n \to p$, $J^+(p) \cap \partial V \neq \emptyset$. Let $y \in J^+(p) \cap \partial V$, then $p \in J^{-1}(y) = L^-(y)$ by Lemma (1.2) (a) because $\partial V \subset A$. This completes the proof of the Lemma in case (a).

Case (b): Given a neighbourhood base $\{W_n\}$ at p as above, there exist points $p_{m_n} = q_n \in W_{m_n} = 0_n$ and $t_{m_n} = s_n$, such that $c(q_n, 0_n) s_n \not\subset \overline{U}$. Since $q_n s_n \to q$, $c(q_n, 0_n) s_n \cap \partial U \neq \emptyset$. Thus $J^+(p) \cap \partial U \neq \emptyset$, and as in the last paragraph above there is a point $y \in \partial U$ such that $p \in J^{-1}(y)$. This completes the proof of the lemma.

THEOREM (2.2). *If (X, T) is a* CAE *flow, then T is not recurrent at any point of A.*

Proof. Note that A is invariant under T, and (A, T) satisfies all the conditions of Theorem A. Thus, if it contains a recurrent point then for all points $x \in A$, $K'(x)$, the orbit closure of x in (A, T), is compact. Hence $K'(x) = K(x)$, where $K(x)$ is the orbit closure of x in (X, T). Consequently, for each $x \in A$, $K(x) \cap \overline{B} = \emptyset$. Since $B \neq \emptyset$ this contradicts Lemma (2.1) and completes the proof.

THEOREM (2.3). *Let (X, T) be a* CAE *flow. If $x \in A$, then $L^+(x) \cap A = \emptyset$ and $L^-(x) \cap A = \emptyset$.*

Proof. If $L^+(x) \cap A \neq \emptyset$ then by Lemma (1.2) (b), $x \in L^+(x)$. Hence $x \in J^+(x)$, which implies that $x \in J^-(x) = L^-(x)$ by Lemma (1.2) (a). Thus x is recurrent contradicting Theorem (2.2). Similarly $L^-(x) \cap A = \emptyset$.

LEMMA (2.4). *Let (X, T) be a continuous flow. Let U be an open set in X such that ∂U is compact and disjoint with $\overline{B}$. Let $\{t_n\}$ be any given sequence in T. Then the set $Y = \{x \in A\colon \{xt_n\}$ lies eventually in $U\}$ is open in A.*

Proof. Suppose the lemma is not true. Then there exists a $y \in Y$ a sequence $\{y_k\}$ not in Y but convergent to y, and a subsequence $\{s_k = t_{n(k)}\}$ of $\{t_n\}$ such that $y_k s_k \notin U$ for all k. Then by the continuity of s_k for each k, there exists an integer $m(k)$ such that $y_{m(k)} s_k \in U$. Since $y_{m(k)} s_{m(k)} \notin U$, there is an r_k in the interval with end point $s_{m(k)}$ and s_k in R, such that, $y_{m(k)} r_k \in \partial U$. Since A is invariant under T, $\{t_n\}$ is a divergent sequence in T; hence so also is $\{r_n\}$. Since ∂U is compact and $y_{m(k)} \to y$, $J^+(y) \cap \partial U \neq \emptyset$. Since $y \in A$, by Lemma (1.2) (a), $L^+(y) \cap \partial U \neq \emptyset$ contradicting Theorem (2.3). This proves the Lemma.

THEOREM (2.5). *A continuous* CAE *flow (X, R) is indivisible.*

Proof. Let $x_0 \in A$ and $x_0 t_n \to p \in B$ for some sequence $\{t_n\}$ in R. Let U be an open set in X containing p such that $\overline{U}$ is compact and $\partial \overline{U} \cap \overline{B} = \emptyset$. Then by Lemma (2.4) the set $Y = \{x \in A\colon \{xt_n\}$ lies eventually in $U\}$ is a nonempty open set. We claim that this set is also closed. Suppose not and let $y_0 \in \overline{Y} \cap A$ but $y_0 \notin Y$. Then there exists a subsequence $\{s_n\}$ of $\{t_n\}$ such that $\{y_0 s_n\}$ lies eventually in $V = X - \overline{U}$

(Theorem (2.3)). Since $\partial V = \partial U$, by Lemma (2.4) the set $Q = \{x \in A: \{xt_n\}$ lies eventually in $V\}$ is open. But this is a contradiction since y_0 is a limit point of Y.

Thus Y is a nonempty open and closed subset of A. Since A is connected, $Y = A$. Since there exist arbitrarily small such open sets as U, clearly $xt_n \to p$ for all $x \in A$.

THEOREM (2.6). *Let (X, Z) be a discrete* CAE *flow. If A is a semi-continuum, then (X, Z) is indivisible.*

Proof. Let $x_0 \in A$ and $\{t_n\}$ be a sequence in Z such that $x_0 t_n \to p$. We want to show that $xt_n \to p$ for every $x \in A$. Suppose there is a $y_0 \in A$ for which this is not true. Since A is a semi-continuum there is a compact connected set C in A containing x_0 and y_0. Let U be an open set containing p such that ∂U is compact and $\partial U \subset A$. Then $Ct_n \cap \partial U \neq \emptyset$ for infinitely many values of n, and since C is compact, one can easily show that either $L^+(z_0) \cap \partial U \neq \emptyset$ or $L^-(z_0) \cap \partial U \neq \emptyset$, which contradicts Theorem (2.3). This proves the Theorem.

§ 3. In this section we shall prove Theorems B and C.

THEOREM (3.1). *Let (X, T) be an indivisible* CAE *transformation group. If p is not of $ch0^-$ ($ch0^+$), then given any open set U containing p and any $x \in A$ there exists a $t_0 \in T^+$ ($t_0 \in T^-$) such that for $t \geqslant t_0$, ($t \leqslant t_0$) $xt \in U$.*

Proof. We assume that $\bar{U}$ is compact and $\partial \bar{U} \cap \bar{B} = \emptyset$ and prove the claim separately for discrete and continuous flows:

Suppose $T = Z$: Suppose the claim is not true. Then as in the proof of Lemma (2.1) we can find a sequence $\{s_n\}$ in Z^+ such that $xs_n \in U$ but $x(s_n+1) \notin U$. Now $\bar{U}$ being compact $\limsup_{n\to\infty} xs_n \neq \emptyset$ and by Theorem (2.3), $\limsup_{n\to\infty} xs_n \subset B \cap U$. Hence $xr_n \to q \in B \cap U$ for some subsequence $\{r_n\}$ of $\{s_n\}$. But then by indivisibility, $x(r_n+1) = (x1)r_n \to q$ which is a contradiction. Hence the claim.

Case $T = R$. Suppose the claim is not true. Then there is a sequence $\{r_n\}$ in R^+, such that, $xr_n \notin U$. Since p is not of $ch0^-$, by Lemma (2.1) there is a point $x_0 \in A$ and a sequence $\{t_n\}$ in T^+ such that $x_0 t_n \to p$ and by indivisibility $xt_n \to p$ for all $x \in A$. Hence, as argued earlier $L^+(x) \cap \partial U \neq \emptyset$ contradicting Theorem (2.3). This completes the proof.

Proof of Theorem B. Let $p \in B$ be of $ch0^-$. Then by Lemma (2.1) there is an x_0 such that $x_0 t_n \to p$ for some sequence $\{t_n\}$ in R^+. Then by Theorem (2.5) and Theorem (3.1) it follows that p is the only point not of $ch0^-$ in B. Similarly, we can show that B contains at most one point not of $ch0^+$.

It is easy to see using Lemma (1.1) that if B has a point of $ch0^-$ ($ch0^+$) then T^+ (T^-) is pointwise equicontinuous on T. This proves Theorem B.

Proof of Theorem C. In the proof of Theorem B use Theorem (2.6) instead of Theorem (2.5).

Using Theorem C the following Theorem can be proved in the same way as Theorem 7 [5, p. 146].

THEOREM (3.2). *Let (X, Z) be a discrete flow on a locally compact connected metric space X. Let $\bar{B}$ not disconnect X and have only compact components. If A contains a point which is not almost periodic, then $\bar{B}$ is the union of at most two components K and L (K may equal L) and $\omega(x) = K$; $\alpha(x) = L$ for all $x \in A$.*

EXAMPLE (3.3). The following example shows that for a flow (X, T) in which B is compact and not zero dimensional then, without the assumption that A contains a point which is not almost periodic, B may have uncountably many components. Thus, the assumption that A contains a point which is not almost periodic in Theorem (3.2) above and Theorem 7 [5, p. 146] is also necessary.

EXAMPLE (3.3). In Example (1.4) above consider the subflow (E, R), where E is the set of points on the parabola P and the ellipses. Then no point of P is a point of $ch0^+$ or $ch0^-$. Let $X = E \cup \{\infty\}$ be the one point compactification of E and (X, R) be the extended flow, where $\infty t = \infty$ for every $t \in R$. Let $B = P \cup \{\infty\}$.

Let S be the Cantor set and define $(X \times S, R, \pi)$ by $((x, s), t)\pi = (xt, s)$. Then $B \times S$ is the set of all those points which fail to be of $ch0^+$ and $ch0^-$. Let Y be the quotient space obtained from $X \times S$ by identifying all points $\{(p_0, s), s \in S\}$ where p_0 is the degenerate ellipse in (E, R). Then in the quotient transformation group (Y, R), Y is a compact connected metrizable space and $\tilde{B} = B \times S$, the set of points that fail to be of $ch0^+$ or $ch0^-$, is compact and does not disconnect the space, and $\tilde{A} = Y - \tilde{B}$ is arcwise connected but the number of components of $\tilde{B}$ is uncountable. Of course, every point of $\tilde{A}$ is a point of almost periodicity.

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