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More on distributive ideals

by

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Abstract. In this paper we present further results concerning ideals on uncountable cardinals whose quotient algebra is distributive. We show that such ideals are related to completely ineffable and weakly compact cardinals, flipping properties, V-ultrafilters, ideal theoretic partition relations and a closure property of the generic ultrapower.

In [14] we commenced our study of ideals on uncountable cardinals whose quotient algebra is distributive, and in particular we showed that distributivity is related to some ideal theoretic partition properties. In this paper we present some further results concerning such "distributive ideals", and whilst for the most part not strictly necessary, a familiarity with [14] would be useful.

In § 1 we show that if $\varkappa$ is completely ineffable then $\varkappa$ carries a natural normal $(\varkappa, \varkappa)$-distributive ideal, the completely ineffable ideal. It is a well-known question whether natural normal ideals (especially the non-stationary ideal) can ever be saturated (or precipitous). We answer this question for the completely ineffable ideal by showing it to be non-precipitous.

§ 2 contains some brief remarks connecting distributive ideals to V-ultrafilters and flipping properties. Using distributivity, we also give a simple proof of a theorem of Kleinberg [18] characterizing completely ineffable cardinals in terms of the existence of certain V-ultrafilters.

In § 3 we make some further remarks concerning normal WC ideals. It follows easily from results of Baumgartner [2] and of [14] that the existence of a normal WC ideal on $\varkappa$ is equivalent to the weak compactness of $\varkappa$. Indeed, the existence of such an ideal may be regarded as a (normal) ideal theoretic analogue of the "strong inaccessibility and tree property" equivalent of weak compactness. This ideal theoretic analogue is shown to have considerable power easily yielding Π^1_1-indescribability and a combinatorial equivalent of weak compactness due to Shelah [21].

In § 4 various forms of weak distributivity are considered, and are shown to be related to ideal theoretic versions of partition relations akin to those defining Rowbottom cardinals. Also, using a forcing argument similar to that of [14, Theorem 9], we give a new proof of a partition theorem for saturated ideals originally due to Solovay.

In § 5 we briefly mention a connection between distributivity and a closure property of the generic ultrapower.

§ 0. Notation and terminology. Our set-theoretical terminology is reasonably standard (see [11]), and background results, notation and terminology not defined here concerning ideals may be found in [5], [12] or [14].

Lower case greek letters will denote ordinals and when a set of ordinals is written as $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ it is assumed that $\zeta_1 < \zeta_2 < \ldots < \zeta_n$ and if $m \leqslant n$ then $\{\zeta_1, \zeta_2, \ldots, \zeta_n\} | m = \{\zeta_1, \zeta_2, \ldots, \zeta_m\}$. Throughout the paper $\varkappa$ will denote a regular uncountable cardinal, and I a proper non-principal $\varkappa$-complete ideal on $\varkappa$ (see

[5. p. 7]). $\mathrm{NS}_\varkappa$ will denote the ideal of non-stationary subsets of $\varkappa$, $I_\varkappa = \{X \subseteq \varkappa \mid |X| < \varkappa\}$, and $R_\varkappa = \{\xi < \varkappa \mid \xi$ is regular$\}$.

If $A \in I^+$ $(= P(\varkappa) - I)$, then an I-partition of A is a maximal collection $W \subseteq P(A) \cap I^+$ such that $X \cap Y \in I$ whenever $X, Y \in W$, $X \neq Y$. The I-partial W is said to be *disjoint* if distinct members of W are disjoint, and in this case for $\xi \in \bigcup W$, $W(\xi)$ denotes the unique member of W containing ξ. If W and T are I-partitions of A, we say W refines T (and write $W \leqslant T$) if for each $X \in W$ there is a $Y \in T$ such that $X \subseteq Y$. If $B \in P(A) \cap I^+$, then $W \restriction B = \{X \in W \mid B \cap X \in I^+\}$.

Definition 0.1. For $\lambda \leqslant \varkappa$ we say I is $(\mu, <\eta, \lambda)$-*distributive* if whenever $A \in I^+$ and $\langle W_\alpha \mid \alpha < \mu\rangle$ is a sequence of disjoint I-partitions of A such that for each $\alpha < \mu$, $|W_\alpha| \leqslant \lambda$, there is a $B \in P(A) \cap I^+$ such that $|W_\alpha \restriction B| < \eta$ for each $\alpha < \mu$. I is said to be (μ, λ)-*distributive* if I is $(\mu, <2, \lambda)$-distributive.

It is easy to check that for $\mu \leqslant \varkappa$, $\lambda = \varkappa$ this definition is equivalent to that given in [14, Definition 1]. Also as mentioned in [14], for $\lambda \leqslant \varkappa$, $(\mu, <\eta, \lambda)$-distributivity is equivalent to the property obtained by omitting the condition that each W_α is disjoint (since, by $\varkappa$-completeness, any I-partition of cardinality $\leqslant \lambda$ has a disjoint refinement of cardinality $\leqslant \lambda$), and hence I is $(\mu, <\eta, \lambda)$-distributive iff the Boolean algebra $P(\varkappa)/I$ is $(\mu, <\eta, \lambda)$-distributive in the usual sense (although our notation here may be slightly non standard).

Definition 0.2. (a) A function $h\colon A \to \varkappa$ is said to be an *I-small I-function* if $A \in I^+$ and for each $\varrho < \varkappa$, $h^{-1}(\{\varrho\}) \in I$.

(b) An I-small I-function h is said to be a *minimal unbounded I-function* if there is no I-small I-function f such that $\mathrm{dom}(f) \subseteq \mathrm{dom}(h)$ and for each $\xi \in \mathrm{dom}(f)$, $f(\xi) < h(\xi)$. (Hence I is normal if the identity function $\mathrm{id}\colon \varkappa \to \varkappa$ is minimal unbounded.)

(c) I is *weakly selective* if every I-small I-function is injective on a set in I^+.

A collection $U \subseteq P(\varkappa) \cap V$ is said to be a *V-$\varkappa$-complete ultrafilter on $\varkappa$* (where V is our ground model) if U is a proper non-principal ultrafilter on $P(\varkappa) \cap V$ such that whenever $\beta < \varkappa$, $f\colon \beta \to U$ and $f \in V$ then $\bigcap \{f(\alpha) \mid \alpha < \beta\} \in U$. U is said to be *normal* if $\Delta f = \{\xi < \varkappa \mid \forall \alpha < \xi,\ \xi \in f(\alpha)\} \in U$ whenever $f\colon \varkappa \to U$, $f \in V$.

Given U, we may form the (not necessarily well-founded) ultrapower $V^\varkappa/U$ (see [12]). The ordinals of $V^\varkappa/U$ will contain an initial segment in order type $\varkappa$ (or $\varkappa+1$ if, say, U is normal) which we identify with $\varkappa$ (or $\varkappa+1$), and if $j\colon V \to V^\varkappa/U$ is the natural embedding then $j\xi = \xi$ for each $\xi < \varkappa$.

$R(I)$ is the notion of forcing whose conditions are sets $A \in I^+$ with $A \leqslant A'$ iff $A \subseteq A'$. If D is $R(I)$-V-generic then D is a V-$\varkappa$-complete ultrafilter on $\varkappa$ extending I^*, the filter dual to I. If I is normal then D will be normal, and, in general, if I is weakly normal (in the sense of [12]), then $V^\varkappa/D$ will contain a $\varkappa$th ordinal. I is said to by *precipitous* iff $R(I) \Vdash$ "$V^\varkappa/D$ is well-founded", and in this case we identify $V^\varkappa/D$ with its transitive collapse. For more on $V^\varkappa/D$ see [12].

For $P, Q \subseteq P(\varkappa)$, $n < \omega$ and $\eta \leqslant \varkappa$, $P \to [Q]^n_{r,<\eta}$ denotes the assertion: whenever $A \in P$ and $f\colon [A]^n \to \varkappa$ is regressive (i.e. $f(x) < \min(x)$), there is a $B \in P(A) \cap Q$

such that $|f([B]^n)| < \eta$. (Similarly with "$<\omega$" replacing "n".) In the case where $P = \{A\}$ we write $A \to [Q]^n_{r,<\eta}$ and when $\eta = 2$ we write $P \to [Q]^n_r$.

§ 1. The completely ineffable ideal. For α an ordinal define $\mathrm{In}^\varkappa_\alpha$ by induction on α as follows: $\mathrm{In}^\varkappa_0 = \mathrm{NS}^+_\varkappa$, $\mathrm{In}^\varkappa_{\alpha+1} = \{X \subseteq \varkappa \mid X \to [\mathrm{In}^\varkappa_\alpha]^2_r\}$ and for $\lim(\alpha)$, $\mathrm{In}^\varkappa_\alpha = \bigcap \{\mathrm{In}^\varkappa_\beta \mid \beta < \alpha\}$. Note that $\beta < \alpha$ implies $\mathrm{In}^\varkappa_\alpha \subseteq \mathrm{In}^\varkappa_\beta$, and hence there exists an α for which $\mathrm{In}^\varkappa_\alpha = \mathrm{In}^\varkappa_{\alpha+1}$.

$\mathrm{In}^\varkappa_1$ is the set of ineffable subsets of $\varkappa$, and so the sets $\mathrm{In}^\varkappa_\alpha$ may be regarded as a means of iterating the operation which produces ineffable sets from stationary sets. Baumgartner [3] defined $\varkappa$ to be *completely ineffable* iff ($\varkappa$ is regular) and there exists an ordinal α such that $\mathrm{In}^\varkappa_\alpha = \mathrm{In}^\varkappa_{\alpha+1} \neq \emptyset$. In [14, Corollary 3] we proved that $\varkappa$ is completely ineffable iff $\varkappa$ carries a normal $(\varkappa, \varkappa)$-distributive ideal, and in particular we mentioned that if $\mathrm{In}^\varkappa_\alpha = \mathrm{In}^\varkappa_{\alpha+1} \neq \emptyset$ then $I = P(\varkappa) - \mathrm{In}^\varkappa_\alpha$ is a normal $(\varkappa, \varkappa)$-distributive ideal on $\varkappa$, and is indeed the minimal such ideal on $\varkappa$. If $\varkappa$ is completely ineffable we refer to this ideal as the completely ineffable ideal on $\varkappa$.

We first make the observation that the completely ineffable ideal may be characterized in terms of an infinite game. For $A \subseteq \varkappa$ define a game G_A between 2 players, One and Two as follows: One plays first and chooses a set $A_0 \in P(A) \cap \mathrm{NS}^+_\varkappa$. Two then picks a regressive function $f_0\colon [A_0]^2 \to \varkappa$. One then chooses a set

$$A_1 \in P(A_0) \cap \mathrm{NS}^+_\varkappa$$

homogeneous for f_0. Two then chooses a regressive function $f_1\colon [A_1]^2 \to \varkappa$ and then One picks a set $A_2 \in P(A_1) \cap \mathrm{NS}^+_\varkappa$ homogeneous for f_1. They continue in this fashion to produce two sequences $A \supseteq A_0 \supseteq A_1 \supseteq \ldots$ and $f_0, f_1, f_2, \ldots$ If after a finite number of moves Player One cannot find the required set the game terminates there and Player Two wins. Otherwise the game continues for ω moves and in this case Player One wins.

Theorem 1.1. *If $\mathrm{In}^\varkappa_\alpha = \mathrm{In}^\varkappa_{\alpha+1}$, then $A \in \mathrm{In}^\varkappa_\alpha$ iff Player One has a winning strategy in G_A.*

Proof. $(\to)$ Player One's strategy is clear; since $\mathrm{In}^\varkappa_\alpha \to [\mathrm{In}^\varkappa_\alpha]^2_r$, he may choose each of the sets $A_n \in \mathrm{In}^\varkappa_\alpha$.

$(\to)$ Suppose $A \notin \mathrm{In}^\varkappa_\alpha$. Firstly, if $A \in \mathrm{NS}_\varkappa$ then trivially Player Two has a winning strategy in G_A since there is no set $A_0 \in P(A) \cap \mathrm{NS}^+_\varkappa$, hence suppose $A \in \mathrm{NS}^+_\varkappa$. For $B \in P(A) \cap \mathrm{NS}^+_\varkappa$, let $f(B)$ be the least ordinal β such that $B \notin \mathrm{In}^\varkappa_\beta$, then $0 < f(B) \leqslant \alpha$, $f(B)$ is a successor ordinal, say $f(B) = \gamma + 1$, and hence there is a regressive function $g(B)\colon [B]^2 \to \varkappa$ having no homogeneous set $C \in P(B) \cap \mathrm{In}^\varkappa_\gamma$. In particular, if $C \in P(B) \cap \mathrm{NS}^+_\varkappa$ is homogeneous for $g(B)$ then $f(C) < f(B)$. g is clearly a winning strategy for Player Two in G_A for if $A_0, g(A_0), A_1, g(A_1), \ldots$ is a play of G_A in which Two plays according to the strategy g, then for each n, $f(A_n) > f(A_{n+1})$, and hence the play must terminate in a finite number of moves. ■

Corollary 1.2. *For each $A \subseteq \varkappa$ the game G_A is determined.*

Corollary 1.3. *$\varkappa$ is completely ineffable iff Player One has a winning strategy in $G_\varkappa$.*

COROLLARY 1.4. *If $\varkappa$ is completely ineffable and I is the completely ineffable ideal on $\varkappa$, then for each $A \subseteq \varkappa$, $A \in I$ iff Player Two has a winning strategy in G_A.*

We now turn to the main theorem of this section, the non-precipitousness of the completely ineffable ideal. As mentioned by Baumgartner [3], if $\varkappa$ is measurable then $\varkappa$ is completely ineffable and indeed $\varkappa$ must be much greater than the least completely ineffable; for instance, if U is a normal measure on $\varkappa$ then $\{\xi < \varkappa \mid \xi$ is completely ineffable$\} \in U$. Our next theorem strengthens this remark and will be of use later. It also gives additional information about the size of completely ineffable cardinals.

THEOREM 1.5. *If I is normal, precipitous and $(\varkappa, \varkappa)$-distributive then $\{\xi < \varkappa \mid \xi$ is completely ineffable$\} \in I^*$.*

Proof. By Łoś's theorem it suffices to show that $R(I) \models$ "$V^\varkappa/D \Vdash \varkappa$ is completely ineffable", hence let D be $R(I)$-V-generic, M denote (the transitive collapse of) the ultrapower $V^\varkappa/D$ and let $j\colon V \to M$ be the natural embedding.

By the remark following [14, Theorem 1], if $f\colon \varkappa \to \varkappa$ with $f \in M$ then $f \in V$, and conversely (since $j \restriction \varkappa$ is the identity) if $g\colon \varkappa \to \varkappa$ with $g \in V$ then $g = jg \restriction \varkappa \in M$. Hence it is clear that $\varkappa$ is regular in M, $P^V(\varkappa) = P^M(\varkappa)$ and it is trivial to check (by induction) that for each ordinal β, $(\mathrm{In}^\varkappa_\beta)^V = (\mathrm{In}^\varkappa_\beta)^M$. Since $\varkappa$ is completely ineffable in V, there is an ordinal α such that $(\mathrm{In}^\varkappa_\alpha)^V = (\mathrm{In}^\varkappa_{\alpha+1})^V \neq \emptyset$, and hence $\varkappa$ is completely ineffable in M. ■

In fact, a version of Theorem 1.5 also holds in the case where the hypothesis of normality is removed: if I is precipitous and $(\varkappa, \varkappa)$-distributive then I is weakly normal (see [12, p. 13]) and (using [14, Theorem 9]) it is easy to show that if h is a minimal unbounded I-function then $h_*(I) = \{X \subseteq \varkappa \mid h^{-1}(X) \in I\}$ is a normal $(\varkappa, \varkappa)$-distributive ideal on $\varkappa$. Hence $\varkappa$ is completely ineffable and as in Theorem 1.5, $R(I) \Vdash$ "$V^\varkappa/D \models \varkappa$ is completely ineffable".

THEOREM 1.6. *If $\varkappa$ is completely ineffable then the completely ineffable ideal on $\varkappa$ is not precipitous.*

Proof. Suppose not; then by Theorem 1.5, if I is the completely ineffable ideal on $\varkappa$, $A = \{\xi \in R_\varkappa \mid \xi$ is completely ineffable$\} \in I^*$. For $\xi \in A$, let J_ξ be the completely ineffable ideal on ξ; then as in the proof of Theorem 1.5, if D is $R(I)$-V-generic and M denotes (the transitive collapse of) the ultrapower $V^\varkappa/D$,

$$I = P^V(\varkappa) - \bigcap_\beta (\mathrm{In}^\varkappa_\beta)^V = P^M(\varkappa) - \bigcap_\beta (\mathrm{In}^\varkappa_\beta)^M = [\langle J_\xi \mid \xi \in A\rangle].$$

Hence by Łoś's theorem, if $X \in I$ then $\{\xi \in A \mid X \cap \xi \in J_\xi\} \in I^*$ and if $X \in I^+$ then

$$\{\xi \in A \mid X \cap \xi \in J_\xi^+\} \in I^*. \tag{1.1}$$

Now suppose $\varkappa$ is the least regular cardinal carrying a normal $(\varkappa, \varkappa)$-distributive ideal I for which there is an $A \in I^*$ and a sequence of ideals $\langle J_\xi \mid \xi \in A\rangle$ satisfying (1.1). If D is $R(I)$-V-generic then, as in Theorem 1.5, $P^V(\varkappa) = P(\varkappa) \cap V^\varkappa/D$, and hence by (1.1), $V^\varkappa/D \models$ "$I = [\langle J_\xi \mid \xi \in A\rangle]$ is a normal $(\varkappa, \varkappa)$-distributive ideal on $\varkappa$".

Also if $j\colon V \to V^\varkappa/D$ is the natural embedding then by strong inaccessibility of $\varkappa$, $V^\varkappa/D \models$ "$\xi = j\xi$ is regular and $J_\xi = jJ_\xi$ is an ideal on ξ", for each $\xi \in A$. Hence $\langle J_\xi \mid \xi \in A\rangle = j\langle J_\xi \mid \xi \in A\rangle \restriction A \in V^\varkappa/D$ and it is now clear that I and $\langle J_\xi \mid \xi \in A\rangle$ satisfy (1.1) in $V^\varkappa/D$. By Łoś's theorem this contradicts the minimality of $\varkappa$. ■

Similarly we may prove that the completely ineffable ideal is nowhere precipitous, i.e. for each $A \in I^+$, $I|A$ is not precipitous.

As mentioned in the introduction, the question of whether natural normal ideals can be saturated (or precipitous) is a common one. Certain (negative) results are known (see [12] and [16, § 11]) but as far as we known Theorem 1.6 is the only result in which a natural normal ideal is shown outright to be non-precipitous.

Of course, precipitousness is a well-known consequence of saturation. Another such consequence is the completeness of the quotient algebra (see [16, § 11]).

COROLLARY 1.7. *If $\varkappa$ is completely ineffable and I is the completely ineffable ideal on $\varkappa$, then $P(\varkappa)/I$ is not complete.*

Proof. Suppose $P(\varkappa)/I$ is complete; then since $P(\varkappa)/I$ is $(\varkappa, \varkappa)$-distributive and $(2^\varkappa)^+$-saturated, it is clearly $(\varkappa, \infty)$-distributive (see [6, p. 57]), and hence I is precipitous contradicting Theorem 1.6. ■

Given Theorem 1.6, one might wonder whether every atomless normal $(\varkappa, \varkappa)$-distributive ideal on $\varkappa$ must be non-precipitous. In fact, this is not the case, and indeed in [15, Theorem 4.22] we showed (relative to the measurability of $\varkappa$) that $\varkappa$ may carry an atomless normal ideal whose quotient algebra is $(\varkappa, \infty)$-distributive. We do not, however, know how to construct an atomless $\varkappa^+$-saturated $(\varkappa, \varkappa)$-distributive ideal on $\varkappa$. One remark which may be of use here is that if I is such an ideal and $2^\varkappa = \varkappa^+$, then $P(\varkappa)/I$ has a dense subset isomorphic to a $\varkappa^+$-Suslin tree.

§ 2. Flipping properties and V-ultrafilters.

In this section we show that distributive ideals are related to flipping properties and V-ultrafilters.

Recall [1] that if $f\colon \mu \to P(\varkappa)$ then a flip of f is a function $g\colon \mu \to P(\varkappa)$ such that for each $\alpha < \mu$, $g(\alpha)$ is either $f(\alpha)$ or $\varkappa - f(\alpha)$. If $f\colon \varkappa \to P(\varkappa)$ then

$$\Delta f = \{\xi < \varkappa \mid \forall \alpha < \xi,\ \xi \in f(\alpha)\}.$$

THEOREM 2.1. *I is normal and $(\varkappa, \varkappa)$-distributive iff whenever $f\colon \varkappa \to P(\varkappa)$ and $A \in I^+$ there is a flip of f, g such that $A \cap \Delta g \in I^+$.*

Proof. ($\to$) Suppose $A \in I^+$ and $f\colon \varkappa \to P(\varkappa)$; then by $(\varkappa, 2)$-distributivity there is a $B \in P(A) \cap I^+$ such that for each $\alpha < \varkappa$, $B - f(\alpha) \in I$ or $B \cap f(\alpha) \in I$. Let $g\colon \varkappa \to P(\varkappa)$ be the flip of f given by $g(\alpha) = f(\alpha)$ iff $B - f(\alpha) \in I$; then by normality $C = \{\xi \in B \mid \exists \alpha < \xi,\ \xi \notin g(\alpha)\} \in I$, and hence $B - C \subseteq A \cap \Delta g \in I^+$.

($\leftarrow$) Suppose $A \in I^+$ and $h\colon A \to \varkappa$ is regressive. For $\alpha < \varkappa$ let $f(\alpha) = h^{-1}(\{\alpha\})$ and g be a flip of f such that $A \cap \Delta g \in I^+$. It is easy to check that $h \restriction A \cap \Delta g$ is constant.

Clearly, if I is $(\varkappa, 2)$-distributive then I is $(\varkappa, \varkappa)$-distributive; hence suppose $A \in I^+$ and $\langle W_\alpha \mid \alpha < \varkappa\rangle$ is a sequence of I-partitions of A with each $|W_\alpha| = 2$, say $W_\alpha = \{X^0_\alpha, X^1_\alpha\}$. Let $f\colon \varkappa \to P(\varkappa)$ be given by $f(\alpha) = X^0_\alpha$ and g be a flip of f such that $A \cap \Delta g \in I^+$. It is easy to check that for each $\alpha < \varkappa$, $|W_\alpha \restriction A \cap \Delta g| = 1$. ■

Similarly for $\mu < \varkappa$, I is $(\mu, 2)$-distributive iff whenever $f\colon \mu \to P(\varkappa)$ and $A \in I^+$ there is a flip of f, g such that $A \cap \bigcap \{g(\alpha)|\ \alpha < \mu\} \in I^+$.

Now for the connection between V-ultrafilters and distributivity. For the remainder of this section U will denote a V-$\varkappa$-complete ultrafilter on $\varkappa$. Since U itself may not belong to V, properties which capture more information about U inside V are of interest. Property (a) of the following theorem was used by Kunen [19] in his work on iterated ultrapowers.

THEOREM 2.2. *For $\eta < \varkappa$ the following are equivalent*

(a) *for each $g \in V$, if $g\colon \eta \to P(\varkappa)$ then $\{\alpha < \eta|\ g(\alpha) \in U\} \in V$;*

(b) *for each $f \in V^\varkappa/U$, if $f\colon \eta \to 2$ then $f \in V$.*

Proof (a) → (b). Suppose $h \in V$ is such that $h\colon \varkappa \to V$ and $V^\varkappa/U \models$ "$[h]\colon \eta \to 2$"; then, by Łoś's theorem, $A = \{\xi < \varkappa|\ h(\xi)\colon \eta \to 2\} \in U$. For $\alpha < \eta$ let

$$g(\alpha) = \{\xi \in A|\ h(\xi)(\alpha) = 0\}\ ;$$

then $\{\alpha < \eta|\ g(\alpha) \in U\} \in V$, and hence if $f\colon \eta \to 2$ is given by $f(\alpha) = 0$ iff $g(\alpha) \in U$ then $f \in V$ and $f = [h]$.

(b) → (a). Suppose $g\colon \eta \to P(\varkappa)$, $g \in V$. For $\xi < \varkappa$ define $h(\xi)\colon \eta \to 2$ by $h(\xi)(\alpha) = 0$ iff $\xi \in g(\alpha)$, then $V^\varkappa/U \models$ "$[h]\colon \eta \to 2$", and hence there is an $f \in V$ such that $f\colon \eta \to 2$ and $f = [h]$. Clearly, for each $\alpha < \eta$, $g(\alpha) \in U$ iff $f(\alpha) = 0$. ■

As in [14, Theorem 1], replacing η by $\varkappa$ may render condition (b) meaningless (as $V^\varkappa/U$ may not contain a $\varkappa$th ordinal). However, as in Theorem 2.2, we may show that $\forall f \in V^\varkappa/U$ (if $f\colon j\varkappa \to 2$ then $f \restriction \varkappa \in V$) iff $\forall g \in V$ (if $g\colon \varkappa \to P(\varkappa)$ then $\{\alpha < \varkappa|\ g(\alpha) \in U\} \in V$).

As in [14, Theorem 2] we may also prove the following

THEOREM 2.3. *Suppose η is a cardinal, $\eta < \varkappa$; then the following are equivalent*

(a) *for each $f \in V^\varkappa/U$, if $f\colon \eta \to \varkappa$ then $f \in V$;*

(b) *$V \models$ "$\forall \lambda < \varkappa,\ \lambda^\eta < \varkappa$" and if $\varkappa \in V^\varkappa/U$ then $V^\varkappa/U \models$ "$\mathrm{cf}\,\varkappa > \eta$".*

We leave the proof to the reader.

Kleinberg [18] proved that $V \models$ "$\varkappa$ is completely ineffable" iff there is, in some generic extension, a normal V-$\varkappa$-complete ultrafilter on $\varkappa$, U such that whenever $g\colon \varkappa \to P(\varkappa)$ with $g \in V$ then $\{\alpha < \varkappa|\ g(\alpha) \in U\} \in V$. Using distributivity, we may give a simple proof of this result. First recall ([14, Corollary 3]) that $\varkappa$ is completely ineffable iff $\varkappa$ carries a normal $(\varkappa, \varkappa)$-distributive ideal.

THEOREM 2.4. *$\varkappa$ carries a normal $(\varkappa, \varkappa)$-distributive ideal iff there is, in some generic extension, a normal V-$\varkappa$-complete ultrafilter on $\varkappa$, U such that whenever $g\colon \varkappa \to P(\varkappa)$ with $g \in V$ then $\{\alpha < \varkappa|\ g(\alpha) \in U\} \in V$.*

Proof (→). Suppose I is a normal $(\varkappa, \varkappa)$-distributive ideal on $\varkappa$; then by the remarks following Theorem 2.2 and [14, Theorem 1], if D is $R(I)$-V-generic then D is the required ultrafilter.

(←). Suppose P is a forcing notion and $U \in V^P$ is such that $P \Vdash$ "U is a normal V-$\varkappa$-complete ultrafilter on $\varkappa$ such that whenever $g\colon \varkappa \to P(\varkappa)$, $g \in V$ then $\{\alpha < \varkappa|\ g(\alpha) \in U\} \in V$".

Let $I = \{X \in P(\varkappa) \cap V|\ P \Vdash X \notin U\}$; then clearly I is an ideal on $\varkappa$. Using Theorem 2.1, we show that I is normal and $(\varkappa, \varkappa)$-distributive. Suppose $f\colon \varkappa \to P(\varkappa)$, $(f \in V)$ and $A \in I^+$; then $P \Vdash$ "$\{\alpha < \varkappa|\ f(\alpha) \in U\} \in V$", and hence we may find a condition $p \in P$ and a flip of f, g, with $g \in V$ such that $p \Vdash$ "$A \in U$ and for each $\alpha < \varkappa$, $g(\alpha) \in U$". By normality of U, $p \Vdash$ "$A \cap \Delta g \in U$", and hence $A \cap \Delta g \in I^+$. ■

Note that in [18] Kleinberg proves a slightly stronger version of Theorem 2.4 (←), i.e. if in any extension of V there is such an ultrafilter then $\varkappa$ is completely ineffable in V. However, it is not clear that this hypothesis gives rise to any particular normal $(\varkappa, \varkappa)$-distributive ideal other than the completely ineffable ideal (see the proof of [18, Theorem 2]).

§ 3. **Normal WC ideals.** In [14, Definition 1] we defined I to be WC if

(3.1) I is $(\mu, \varkappa)$-distributive for each $\mu < \varkappa$,

(3.2) whenever $A \in I^+$ and $\langle W_\alpha|\ \alpha < \varkappa\rangle$ is a sequence of disjoint I-partitions of A such that for each $\alpha < \beta < \varkappa$, $W_\beta \leqslant W_\alpha$, there is a sequence $\langle X_\alpha|\ \alpha < \varkappa\rangle$ such that $X_\alpha \in W_\alpha$ and $X_\beta \subseteq X_\alpha$ whenever $\alpha < \beta < \varkappa$.

(Such a sequence of I-partitions is said to be *decreasing* and the sequence $\langle X_\alpha|\ \alpha < \varkappa\rangle$ is said to be a *branch through* $\langle W_\alpha|\ \alpha < \varkappa\rangle$.)

Baumgartner [2] showed that every weakly compact cardinal carries a natural normal ideal, the Π^1_1-indescribable ideal ($= \{X \subseteq \varkappa|\ X$ is not ordinal Π^1_1-indescribable$\}$). It follows easily from (the proof of) [2, Theorem 3.2] that if $\varkappa$ is weakly compact and I is normal and extends the Π^1_1-indescribable ideal on $\varkappa$, then $I^+ \to (I^+, \varkappa)^2$. This partition property is related to WC in that ([14, Theorem 8]) $I^+ \to (I^+, \varkappa)^2$ iff I is weakly selective and WC, and hence from indescribability considerations we have that $\varkappa$ is weakly compact iff $\varkappa$ carries a normal WC ideal.

However, the existence of a normal WC ideal on $\varkappa$ may also be regarded as a (normal) ideal theoretic analogue of the "strong inaccessibility and tree property" equivalent of weak compactness. Firstly, note that in (3.2) we have a tree of height $\varkappa$ whose levels are of cardinality $\leqslant \varkappa$ (rather that $< \varkappa$, as in the tree property). However, since there are no $I_\varkappa$-partitions of cardinality exactly $\varkappa$, $\varkappa$ has the tree property iff $I_\varkappa$ satisfies (3.2) iff $\varkappa$ carries an ideal satisfying (3.2). Also by [14, Theorem 2] (and the remark following), $\varkappa$ is strongly inaccessible iff $I_\varkappa$ satisfies (3.1) iff $\varkappa$ carries an ideal satisfying (3.1).

In this section we show that this normal ideal theoretic analogue is rather powerful, easily yielding Π^1_1-indescribability and a combinatorial equivalent of weak compactness due to Shelah [21].

Baumgartner [3, Theorem 2.1] showed (via two equivalent characterizations) that if $X \subseteq \varkappa$ and $X \to (\mathrm{NS}^+_\varkappa, \varkappa)^2$ then X is (strongly) Π^1_1-indescribable, and hence it follows that the Π^1_1-indescribable ideal is the minimal normal WC ideal. We first give a direct proof of this result.

THEOREM 3.1. *If I is normal and* WC *then $\varkappa$ is Π^1_1-indescribable and I extends the Π^1_1-indescribable ideal on $\varkappa$.*

Proof. Suppose $S \subseteq \varkappa$, $\forall X \Phi(X, Y)$ is Π^1_1, $(\varkappa, \varepsilon, S) \models \forall X \Phi(X, S)$ and $A = \{\alpha < \varkappa \mid (\alpha, \varepsilon, S \cap \alpha) \models \neg \forall X \Phi(X, S \cap \alpha)\} \in I^+$. By [14, Theorem 2], $R_\varkappa \in I^*$ and for each $\alpha \in R_\varkappa \cap A$ let $T_\alpha \subseteq \alpha$ be such that $(\alpha, \varepsilon, S \cap \alpha, T_\alpha) \models \neg \Phi(T_\alpha, S \cap \alpha)$, and $f_\alpha \colon \alpha \to \alpha$ be normal and such that for each $\delta < \alpha$, $(f_\alpha(\delta), \varepsilon, S \cap f_\alpha(\delta), T_\alpha \cap f_\alpha(\delta)) \models \neg \Phi(T_\alpha \cap f_\alpha(\delta), S \cap f_\alpha(\delta))$ (see [16, § 1]).

Since I is normal and $(\mu, \varkappa)$-distributive for each $\mu < \varkappa$, we may find a decreasing sequence of disjoint I-partitions of $R_\varkappa \cap A$, $\langle W_\nu \mid \nu < \varkappa \rangle$, such that, for each $\nu < \varkappa$, if $X \in W_\nu$ then $\xi \geqslant \nu$, $T_\xi \cap \nu = T_\chi \cap \nu$ and $f_\xi \restriction \nu = f_\chi \restriction \nu$ whenever $\xi, \chi \in X$. By WC there is a branch $\langle X_\nu \mid \nu < \varkappa \rangle$ through $\langle W_\nu \mid \nu < \varkappa \rangle$ and let $T \subseteq \varkappa$, $f \colon \varkappa \to \varkappa$ be such that whenever $\nu < \varkappa$, $\xi \in X_\nu$; then $T \cap \nu = T_\xi \cap \nu$ and $f \restriction \nu = f_\xi \restriction \nu$. Now $(\varkappa, \varepsilon, S, T) \models \Phi(T, S)$; hence $C = \{\beta < \varkappa \mid (\beta, \varepsilon, S \cap \beta, T \cap \beta) \models \Phi(T \cap \beta, S \cap \beta)\} \in \mathrm{NS}^*_\varkappa$ and by Łoś's theorem,

$$R_\varkappa \cap A \Vdash \text{“} V^\varkappa/D \models (\beta, \varepsilon, S \cap \beta, T \cap \beta) \models \Phi(T \cap \beta, S \cap \beta)$$

and

$$([f_\alpha](\delta), \varepsilon, S \cap [f_\alpha](\delta), [T_\alpha] \cap [f_\alpha](\delta)) \models \neg \Phi([T_\alpha] \cap [f_\alpha](\delta), S \cap [f_\alpha](\delta))\text{”}$$

for each $\delta < \varkappa$ and $\beta \in C$. ... (3.3)

f is clearly normal, and hence pick $\gamma < \varkappa$ such that $f(\gamma) \in C$ and ϱ such that $\gamma, f(\gamma) < \varrho < \varkappa$. But then if D is $R(I)$-V-generic with $X_\varrho \in D$ then in $V^\varkappa/D$, $[T_\alpha] \cap \varrho = T \cap \varrho$, $[f_\alpha] \restriction \varrho = f \restriction \varrho$, and hence by (3.3), $(f(\gamma), \varepsilon, S \cap f(\gamma), T \cap f(\gamma)) \models \neg \Phi(T \cap f(\gamma), S \cap f(\gamma))$, contradicting $f(\gamma) \in C$.

Hence $\varkappa$ is ordinal Π^1_1-indescribable. By [14, Theorem 2], $\varkappa$ is strongly inaccessible and hence Π^1_1-indescribable. ■

In [21] Shelah gave a combinatorial proof that $\varkappa$ is strongly inaccessible and has the tree property iff

(3.4) for every family of functions $f_\alpha \colon \alpha \to \alpha$ $(\alpha < \varkappa)$, there is a function $f \colon \varkappa \to \varkappa$ such that $(\forall \alpha < \varkappa)(\exists \beta)(\alpha \leqslant \beta < \varkappa$ and $f_\beta \restriction \alpha = f \restriction \alpha)$.

We will show that the property (3.4) is related to normal WC ideals and in particular the Π^1_1-indescribable ideal. Firstly (3.4) follows very easily from the existence of a normal WC ideal.

THEOREM 3.2. *If I is normal and* WC, *then for each $A \in I^+$ and family of functions $f_\alpha \colon \alpha \to \alpha$ $(\alpha < \varkappa)$ there is a function $f \colon \varkappa \to \varkappa$ such that $(\forall \alpha < \varkappa)(\exists \beta \in A)(\alpha \leqslant \beta < \varkappa$ and $f_\beta \restriction \alpha = f \restriction \alpha)$.*

Proof. Suppose $A \in I^+$ and for each $\alpha < \varkappa$, $f_\alpha \colon \alpha \to \alpha$. As in Theorem 3.1 we may find a decreasing sequence of disjoint I-partitions of A, $\langle W_\nu \mid \nu < \varkappa \rangle$ such that, for each $\nu < \varkappa$, if $X \in W_\nu$ then $\xi \geqslant \nu$ and $f_\xi \restriction \nu = f_\chi \restriction \nu$ whenever $\xi, \chi \in X$. By WC

there is a branch $\langle X_\nu \mid \nu < \varkappa \rangle$ through $\langle W_\nu \mid \nu < \varkappa \rangle$ and let $f \colon \varkappa \to \varkappa$ be such that whenever $\nu < \varkappa$ and $\xi \in X_\nu$ then $f_\xi \restriction \nu = f \restriction \nu$. Clearly, f is our required function. ■

Conversely, (3.4) naturally gives rise to a normal WC ideal.

THEOREM 3.3. *If* (3.4) *holds then $\varkappa$ carries a normal* WC *ideal.*

Proof. Let I be the set of subsets S of $\varkappa$ such that (3.4) is not satisfied if we replace “$(\forall \alpha < \varkappa)(\exists \beta)$...” by “$(\forall \alpha < \varkappa)(\exists \beta \in S)$...”. Shelah [21] showed that I is a normal ideal on $\varkappa$. We show that I is WC.

Firstly, $\varkappa$ is strongly inaccessible for if $\mu < \varkappa$ and $g \colon \varkappa \to 2^\mu$ is injective define a family of functions $f_\alpha \colon \alpha \to \alpha$ $(\alpha \in \varkappa - \mu)$ by $f_\alpha(\delta) = g(\alpha)(\delta)$ if $\delta < \mu$; $f_\alpha(\delta) = 0$ otherwise. Clearly, $\{f_\alpha \mid \alpha \in \varkappa - \mu\}$ then contradicts (3.4).

Suppose now that $\eta < \varkappa$; then for $\xi \in A = \{\chi < \varkappa \mid \operatorname{cf} \chi = \eta\}$ let f_ξ be any function $f_\xi \colon \xi \to \xi$ such that $f_\xi \restriction \eta$ is cofinal in ξ. The family $\{f_\xi \mid \xi \in A\}$ is easily seen to witness that $A \in I$, and hence, by [14, Theorem 2] and normality, I is $(\mu, \varkappa)$-distributive for each $\mu < \varkappa$.

Finally, suppose that $A \in I^+$ and $\langle W_\delta \mid \delta < \varkappa \rangle$ is a decreasing sequence of disjoint I-partitions of A. Let $h \colon \bigcup \{W_\delta \mid \delta < \varkappa\} \to \varkappa$ be injective; then, by normality,

$$B = \{\xi \in A \mid \text{for each } \delta < \xi,\ \xi \in \bigcup W_\delta \text{ and } h(W_\delta(\xi)) < \xi\} \in (I|A)^*.$$

Hence by the definition of I there is a function $f \colon \varkappa \to \varkappa$ such that $(\forall \alpha < \varkappa)$ $(\exists \xi \in B - \alpha)$ $(\forall \delta < \alpha)$ $(f(\delta) = h(W_\delta(\xi)))$. For each $\delta < \varkappa$ let $X_\delta \in W_\delta$ be such that $f(\delta) = h(X_\delta)$. If $\delta < \gamma < \varkappa$ then we may find $\xi \in B - (\gamma + 1)$ such that $f(\gamma) = h(W_\gamma(\xi))$ and $f(\delta) = h(W_\delta(\xi))$, and hence since $W_\gamma \leqslant W_\delta$ and W_δ is disjoint, we must have $X_\gamma = W_\gamma(\xi) \subseteq W_\delta(\xi) = X_\delta$. Therefore $\langle X_\delta \mid \delta < \varkappa \rangle$ is a branch through $\langle W_\delta \mid \delta < \varkappa \rangle$.

Note that by Theorem 3.2 the ideal I given in the proof of Theorem 3.3 is the minimal normal WC ideal on $\varkappa$, and hence is the Π^1_1-indescribable ideal.

§ 4. Weak forms of distributivity. In this section we present some results concerning the weak forms of distributivity given in Definition 0.1, and in particular we we will show that such forms of distributivity are related to partition relations of the form $I^+ \to [I^+]^n_{r, <\varkappa}$.

Firstly we mention the corresponding versions of [14, Theorems 1, 2 and 4] which hold for “weakly distributive ideals”. As the proofs are similar to those in [14] we leave the details to the reader.

THEOREM 4.1. *Suppose $\mu < \varkappa$, $\eta, \lambda \leqslant \varkappa$; then I is $(\mu, <\eta, \lambda)$-distributive iff whenever D is $R(I)$-V-generic and $f \in V^\varkappa/D$ with $f \colon \mu \to \lambda$, there exists $g \in V$ such that $g \colon \mu \to P(\lambda)$ and for each $\alpha < \mu$, $|g(\alpha)|^V < \eta$ and $f(\alpha) \in g(\alpha)$.*

A similar result may also be stated for the case $\mu = \varkappa$ (see [14] or § 2).

THEOREM 4.2. *Suppose μ is a cardinal, $\mu < \varkappa$; then I is $(\mu, <\varkappa, \varkappa)$-distributive iff for every minimal unbounded I-function h, $\{\xi \in \operatorname{dom}(h) \mid \operatorname{cf}(h(\xi)) \leqslant \mu\} \in I$.*

COROLLARY 4.3. *For $\mu < \varkappa$, I is $(\mu, \varkappa)$-distributive iff $\forall \lambda < \varkappa (\lambda^\mu < \varkappa)$ and I is $(\mu, <\varkappa, \varkappa)$-distributive.*

COROLLARY 4.4. *If I is normal and $(\mu, <\varkappa, \varkappa)$-distributive for each $\mu < \varkappa$, then $R_\varkappa \in I^*$.*

Recall that G_I is the ideal game studied by Galvin, Jech and Magidor [10]. In [15, Theorem 2.10] we showed that if Nonempty has a winning strategy in G_I then for every minimal unbounded I-function h, $\{\xi \in \operatorname{dom}(h) |\ \operatorname{cf}(h(\xi)) = \omega\} \in I$; hence we have

COROLLARY 4.5. *If Nonempty has a winning strategy in G_I then I is $(\omega, <\varkappa, \varkappa)$-distributive.*

We now move on to partition relations. We wish to show that weak forms of distributivity are equivalent to simultaneous partition relations. Firstly we need the following

LEMMA 4.6. *Suppose η is regular, $\eta \leqslant \lambda < \varkappa$ and I is weakly selective. Then the following are equivalent:*

(a) *I is $(\varkappa, <\eta, \lambda)$-distributive;*

(b) *if $A \in I^+$ then for any family of functions $f_\alpha\colon A \to \lambda$ $(\alpha < \varkappa)$, there is a $B \in P(A) \cap I^+$ and a function $t\colon \varkappa \to \varkappa$ such that for each $\alpha < \varkappa$, $|f_\alpha(B - t(\alpha))| < \eta$;*

(c) *if $A \in I^+$ then for any family of functions $s_\xi\colon \xi \to \lambda$ $(\xi < \varkappa)$, there is a $B \in P(A) \cap I^+$ and a function $s\colon \varkappa \to [\lambda]^{<\eta}$ such that for each $\{\xi, \chi\} \in [B]^2$, if $\alpha \leqslant \xi$ then $s_\chi(\alpha) \in s(\alpha)$.*

Proof (a) $\to$ (b). Suppose $A \in I^+$ and $f_\alpha\colon A \to \lambda$ for each $\alpha < \varkappa$. By $\varkappa$-completeness, for $\alpha < \varkappa$, $W_\alpha = \{f_\alpha^{-1}(\{\varrho\}) |\ \varrho < \lambda$ and $f_\alpha^{-1}(\{\varrho\}) \in I^+\}$ is a disjoint I-partition of A of cardinality $\leqslant \lambda$, and hence by $(\varkappa, <\eta, \lambda)$-distributivity there is a $C \in P(A) \cap I^+$ such that for each $\alpha < \varkappa$, $|W_\alpha \restriction C| < \eta$. For each $\xi \in C$ let $g(\xi)$ be the least α such that $f_\alpha^{-1}(\{f_\alpha(\xi)\}) \notin W_\alpha \restriction C$ if such an α exists; $g(\xi) = \varkappa$ otherwise. Clearly each $|f_\alpha(g^{-1}(\{\varkappa\}))| < \eta$, and hence if $g^{-1}(\{\varkappa\}) \in I^+$ we are finished. Otherwise, $g^{-1}(\{\varkappa\}) \in I$, $E = C - g^{-1}(\{\varkappa\}) \in I^+$ and $g \restriction E$ must be I-small. Hence by weak selectivity there is a $B \in P(E) \cap I^+$ such that $g \restriction B$ is injective and so we may find a function $t\colon \varkappa \to \varkappa$ such that for each $\alpha < \varkappa$ and $\xi \in B - t(\alpha)$, $g(\xi) > \alpha$ and hence $|f_\alpha(B - t(\alpha))| < \eta$.

(b) $\to$ (c). Suppose $A \in I^+$ and $s_\xi\colon \xi \to \lambda$ for each $\xi < \varkappa$. For $\alpha < \varkappa$ and $\xi \in A$ let $f_\alpha(\xi) = s_\xi(\alpha)$ if $\alpha < \xi$; $f_\alpha(\xi) = 0$ otherwise. Let $E \in P(A) \cap I^+$ and $t\colon \varkappa \to \varkappa$ be such that for each $\alpha < \varkappa$, $s(\alpha) = f_\alpha(E - t(\alpha))$ has cardinality $<\eta$. Let

$$C = \{\delta < \varkappa |\ \forall \alpha < \delta,\ t(\alpha) < \delta\}\ ;$$

then C is closed unbounded in $\varkappa$, and so by weak selectivity there is a $B \in P(E) \cap I^+$ such that for each $\{\xi, \chi\} \in [B]^2$ there is a $\delta \in C$ with $\xi < \delta < \chi$ (see [5, p. 60, Lemma]). Hence if $\{\xi, \chi\} \in [B]^2$ and $\alpha \leqslant \xi$ then $t(\alpha) < \chi$ and therefore

$$s_\chi(\alpha) = f_\alpha(\chi) \in f_\alpha(E - t(\alpha)) = s(\alpha)\,.$$

(c) $\to$ (a). Suppose $A \in I^+$ and $\langle W_\alpha |\ \alpha < \varkappa\rangle$ is a sequence of disjoint I-partitions of A, each of cardinality $\leqslant \lambda$. For $\alpha < \varkappa$ let $h_\alpha\colon W_\alpha \to \lambda$ be injective, and for $\xi \in A$ let $s_\xi\colon \xi \to \lambda$ be given by $s_\xi(\beta) = h_\beta(W_\beta(\xi))$ if $\xi \in \bigcup W_\beta$; $s_\xi(\beta) = 0$ otherwise. If $B \in P(A) \cap I^+$ and $s\colon \varkappa \to [\lambda]^{<\eta}$ are such that for each $\{\xi, \chi\} \in [B]^2$ and $\beta \leqslant \xi$, $s_\chi(\beta) \in s(\beta)$, then clearly for each $\alpha < \varkappa$, $|W_\alpha \restriction B| < \eta$. ∎

In fact we will not make further use of property (c). It is included here as it clearly relates to Theorem 4.1 and also to a combinatorial equivalent of weak compactness due to Baumgartner [2, Theorem 5.4].

THEOREM 4.7. *Suppose η is regular, $\eta \leqslant \lambda < \varkappa$, and I is weakly selective. Then I is $(\varkappa, <\eta, \lambda)$-distributive iff whenever $A \in I^+$, $n < \omega$ and $f_\alpha\colon [A]^n \to \lambda$ $(\alpha < \varkappa)$ is a family of functions, there is a $B \in P(A) \cap I^+$ and a function $t\colon \varkappa \to \varkappa$ such that for each $\alpha < \varkappa$, $|f_\alpha([B - t(\alpha)]^n)| < \eta$.*

Proof ($\to$). By induction on n; the case $n = 1$ is proved in Lemma 4.6; hence suppose $A \in I^+$ and $f_\alpha\colon [A]^{n+1} \to \lambda$ for each $\alpha < \varkappa$. For each $(\alpha, \xi) \in \varkappa \times A$ define $g_\xi^\alpha\colon [A]^n \to \lambda$ by

$$g_\xi^\alpha(a) = \begin{cases} f_\alpha(\{\xi\} \cup a) & \text{if } \min(a) > \xi\,, \\ 0 & \text{otherwise} \end{cases}$$

for each $a \in [A]^n$. By inductive hypothesis there is a $B \in P(A) \cap I^+$ and $\{h_\alpha |\ \alpha < \varkappa\}$ such that for each $(\alpha, \xi) \in \varkappa \times A$, $h_\alpha\colon A \to \varkappa$ and $|g_\xi^\alpha([B - h_\alpha(\xi)]^n)| = \tau_\xi^\alpha < \eta$. By Lemma 4.6 there is a $C \in P(B) \cap I^+$ and a function $t\colon \varkappa \to \varkappa$ such that for each $\alpha < \varkappa$, $\{\tau_\xi^\alpha |\ \xi \in C - t(\alpha)\} \in [\eta]^{<\eta}$, and hence since η is regular,

$$\varrho_\alpha = \bigcup \{\tau_\xi^\alpha |\ \xi \in C - t(\alpha)\} < \eta\,.$$

For $(\alpha, \xi) \in \varkappa \times (C - t(\alpha))$ let $t_\xi^\alpha\colon \varrho_\alpha \to g_\xi^\alpha([B - h_\alpha(\xi)]^n)$ be onto and for $\alpha < \varkappa$, $\beta < \varrho_\alpha$ and $\xi \in C$ let

$$s_\beta^\alpha(\xi) = \begin{cases} t_\xi^\alpha(\beta) & \text{if } \xi \geqslant t(\alpha)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

By Lemma 4.6 we may find an $E \in P(C) \cap I^+$ and $\langle \sigma_\beta^\alpha |\ \alpha < \varkappa, \beta < \varrho_\alpha\rangle$ such that $\sigma_\beta^\alpha < \varkappa$ and $|s_\beta^\alpha(E - \sigma_\beta^\alpha)| < \eta$ whenever $\alpha < \varkappa$ and $\beta < \varrho_\alpha$. Let $\sigma_\alpha = \bigcup \{\sigma_\beta^\alpha |\ \beta < \varrho_\alpha\}$, then $\sigma_\alpha < \varkappa$ and $|\bigcup \{s_\beta^\alpha(E - \sigma_\alpha) |\ \beta < \varrho_\alpha\}| < \eta$.

For $\alpha < \varkappa$ let $C_\alpha = \{\delta < \varkappa |\ \forall \xi \in \delta \cap A,\ h_\alpha(\xi) < \delta\}$ and

$$C_\varkappa = \{\delta < \varkappa |\ \forall \alpha < \delta,\ \delta \in C_\alpha\}\,,$$

then $C_\varkappa$ is closed unbounded in $\varkappa$ and as in Lemma 4.6 we may find an $F \in P(E) \cap I^+$ such that whenever $\{\xi, \chi\} \in [F]^2$ there is a $\delta \in C_\varkappa$ with $\xi < \delta < \chi$.

Suppose $\alpha < \varkappa$ and $\{\xi\} \cup a \in [F - (\alpha \cup t(\alpha) \cup \sigma_\alpha)]^{n+1}$ with $\xi < \min(a)$. Pick $\delta \in C_\varkappa$ such that $\xi < \delta < \min(a)$; then $\delta > \alpha$; hence $\delta \in C_\alpha$ and $h_\alpha(\xi) < \delta$. So $g_\xi^\alpha(a) \in g_\xi^\alpha([B - h_\alpha(\xi)]^n)$ and since $\xi \geqslant t(\alpha)$, there is a $\beta < \varrho_\alpha$ such that $g_\xi^\alpha(a) = t_\xi^\alpha(\beta) = s_\beta^\alpha(\xi) \in \bigcup \{s_\beta^\alpha(E - \sigma_\alpha) |\ \beta < \varrho_\alpha\}$. Also $f_\alpha(\{\xi\} \cup a) = g_\xi^\alpha(a)$, hence

$$|f_\alpha([F - (\alpha \cup t(\alpha) \cup \sigma_\alpha)]^{n+1})| < \eta\,.$$

($\leftarrow$) is immediate from Lemma 4.6. ∎

Note that to obtain an equivalent of $(\varkappa, <\eta, \lambda)$-distributivity we have had to use $\varkappa$ many partitions. Whilst it is easy to show that just one such partition implies

$(\omega, <\eta, \lambda)$-distributivity (using the fact that if $\{W_n|\ n<\omega\}$ are I-partitions of cardinality $\leqslant \lambda$ we may assume, by refining if necessary, that $W_0 \geqslant W_1 \geqslant W_2 \geqslant \ldots$) we do not know that, say, $(\omega_1, <\eta, \lambda)$-distributivity is implied.

Similarly we may prove the following.

THEOREM 4.8. *Suppose η is regular, $\eta<\varkappa$, then I is normal and $(\varkappa, <\eta, \varkappa)$-distributive iff whenever $A \in I^+$, $n<\omega$ and f_α: $[A]^n \to \varkappa$ $(\alpha<\varkappa)$ is a family of regressive functions, there is a $B \in P(A) \cap I^+$ such that for each $\alpha<\varkappa$, $|f_\alpha([B-(\alpha+1)]^n)|<\eta$.*

Let us say that I is *strongly $(\varkappa, <\varkappa, \varkappa)$-distributive* iff whenever $A \in I^+$ and $\langle W_\alpha|\ \alpha<\varkappa\rangle$ is a sequence of disjoint I-partitions of A, there is a $B \in P(A) \cap I^+$ and a $\lambda<\varkappa$ such that for each $\alpha<\varkappa$, $|W_\alpha \restriction B|<\lambda$.

THEOREM 4.9. *I is normal and strongly $(\varkappa, <\varkappa, \varkappa)$-distributive iff whenever $A \in I^+$, $n<\omega$ and f_α: $[A]^n \to \varkappa$ $(\alpha<\varkappa)$ is a family of regressive functions, there is a $B \in P(A) \cap I^+$ and a $\lambda<\varkappa$ such that for each $\alpha<\varkappa$, $|f_\alpha([B-(\alpha+1)]^n)|<\lambda$.*

We leave to the reader the proofs of Theorems 4.8 and 4.9 and also the derivation of the analogues of Lemma 4.6 (c).

Of course, if I is η-saturated it is clearly $(\varkappa, <\eta, \varkappa)$-distributive, and in this case it is known that a stronger partition property holds.

THEOREM 4.10 (Solovay). *Suppose η is regular, $\omega<\eta<\varkappa$; then I is normal and η-saturated iff $\varkappa \to [I^*]^{<\omega}_{r,<\eta}$.*

Using a forcing argument similar to that of [14, Theorem 9] we give a new proof of Solovay's theorem.

Proof ($\to$). By $\varkappa$-completeness it suffices to show that $\varkappa \to [I^*]^n_{r,<\eta}$ holds for each $n<\omega$. We proceed by induction on n; the case $n=1$ is immediate from normality and η-saturation.

Suppose f: $[\varkappa]^{n+1} \to \varkappa$ is regressive. Then $R(I) \Vdash$ "jf: $[j\varkappa]^{n+1} \to j\varkappa$ is regressive". Let D be $R(I)$-V-generic then by a well-known lemma of Prikry (see [16, § 11]) and [13, Theorem 2.1], in $V(D)$, I generates a normal η-saturated ideal J, and hence by inductive hypothesis applied in $V(D)$ there is an $X \in J^*$ and an $S \subseteq \varkappa$ such that $|S|^{V(D)}<\eta$ and for each $\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \in [X]^n$, $jf(\{\zeta_1, \zeta_2, \ldots, \zeta_n, \varkappa\}) \in S$. By η-saturation and the definition of J there is a $Y \in P(X) \cap V \cap I^*$ and a $T \in P(\varkappa) \cap V$ such that $|T|^V<\eta$ and $S \subseteq T$.

Hence $R(I)$ forces these facts, and so in V now we may find an I-partition of $\varkappa$, $\{B_\alpha|\ \alpha<\beta<\eta\}$ and sets $\{Y_\alpha|\ \alpha<\beta\} \subseteq I^*$ and $\{T_\alpha|\ \alpha<\beta\} \subseteq [\varkappa]^{<\eta}$ such that for each $\alpha<\beta$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \in [Y_\alpha]^n$, $B_\alpha \Vdash$ "$jf(\{\zeta_1, \zeta_2, \ldots, \zeta_n, \varkappa\}) \in T_\alpha$".

Let $Y = \bigcap\{Y_\alpha|\ \alpha<\beta\}$ and $T = \bigcup\{T_\alpha|\ \alpha<\beta\}$. Then $Y \in I^*$, $|T|^V<\eta$, and for each $\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \in [Y]^n$, $R(I) \Vdash$ "$jf(\{\zeta_1, \zeta_2, \ldots, \zeta_n, \varkappa\}) \in T$".

Suppose

$$C = \{\xi \in Y|\ \exists\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \in [Y \cap \xi]^n, f(\{\zeta_1, \zeta_2, \ldots, \zeta_n, \xi\}) \notin T\} \in I^+;$$

then by normality there is a $\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \in [Y]^n$ such that

$$E = \{\xi \in Y|\ f(\{\zeta_1, \zeta_2, \ldots, \zeta_n, \xi\}) \notin T\} \in I^+,$$

and hence $E \Vdash$ "$jf(\{\zeta_1, \zeta_2, \ldots, \zeta_n, \varkappa\}) \notin jT = T$", a contradiction. Therefore $C \in I$, $Y-C \in I^*$ and $f([Y-C]^{n+1}) \subseteq T$.

($\leftarrow$) is trivial. ∎

Trivially, Theorem 4.10 could be strengthened to accomodate $\varkappa$ many such partitions since, by normality, if g: $\varkappa \to I^*$ then $\Delta g \in I^*$. Also Theorem 4.10 suggests the problem of obtaining an equivalent of $I^+ \to [I^+]^{<\omega}_{r,<\eta}$ (or of $\varkappa$ many such partitions) in terms of distributivity. However, the natural candidate for the analogue of $(\omega, \varkappa)^{<\omega}$-distributivity (see [14, Theorem 12]) seems rather artificial and so we omit the details.

The next theorem gives us additional information about the partition property $I^+ \to [I^+]^2_{r,<\varkappa}$, and also brings us to $(\varkappa, <\varkappa, \varkappa)$-distributivity.

THEOREM 4.11. *If $I^+ \to [I^+]^2_{r,<\varkappa}$ then*

(a) *$R_\varkappa \in I^*$;*

(b) *whenever $A \in I^+$ and $\langle S_\xi|\ \xi \in A\rangle$ is a sequence such that for each $\xi \in A$, $|S_\xi| = |\xi|$ and $S_\xi \subseteq \xi$, there is a $B \in P(A) \cap I^+$ such that $|S_\xi \cap S_\chi| = |\xi|$ whenever $\{\xi, \chi\} \in [B]^2$;*

(c) *$\varkappa$ has the tree property;*

(d) *I is $(\varkappa, <\varkappa, \varkappa)$-distributive.*

Proof (a). Clearly, I is normal; hence if $R_\varkappa \notin I^*$ there is a $\mu<\varkappa$ such that $A = \{\xi<\varkappa|\ \mathrm{cf}\,\xi = \mu\} \in I^+$. For each $\xi \in A$ let $S_\xi \subseteq \xi$ be cofinal in ξ in order type μ, and for $\{\xi, \chi\} \in [A]^2$ let $f(\{\xi, \chi\}) = \bigcup S_\xi \cap S_\chi$. f is regressive and so we may find a $B \in P(A) \cap I^+$ and $\varrho<\varkappa$ such that $f([B]^2) \subseteq \varrho$. Hence if $g(\xi) \in S_\xi-(\varrho+1)$ for each $\xi \in B-(\varrho+1)$, then g is regressive and injective, contradicting the normality of I.

(b) This is similar to (a); if $\langle S_\xi|\ \xi \in A\rangle$ is as given then for $\{\xi, \chi\} \in [A \cap R_\varkappa]^2$ define $f(\{\xi, \chi\}) = 0$ if $|S_\xi \cap S_\chi| = \xi$; $f(\{\xi, \chi\}) = \bigcup S_\xi \cap S_\chi$ otherwise.

(c) Suppose $T = \langle\varkappa, \leqslant_T\rangle$ is a tree of height $\varkappa$ such that for each $\alpha<\varkappa$, T_α (the αth level of T) has cardinality $<\varkappa$. For $\xi<\varkappa$ let $S_\xi = \{\zeta<\varkappa|\ \zeta<_T\xi\}$. Clearly, we may assume that for each $\xi<\varkappa$, $\xi \in \bigcup\{T_\alpha|\ \alpha \leqslant \xi\}$, and if $\alpha<\varkappa$, $\lim(\alpha)$ and $\xi, \chi \in T_\alpha$ with $S_\xi = S_\chi$ then $\xi = \chi$. By normality,

$$A = \{\xi \in R_\varkappa|\ \xi \in T_\xi \text{ and } S_\xi \subseteq \xi_\xi\} \in I^*,$$

and so by (b) there is a $B \in P(A) \cap I^+$ such that $|S_\xi \cap S_\chi| = \xi$ whenever $\{\xi, \chi\} \in [B]^2$. Clearly B forms a branch through T.

(d) Suppose $A \in I^+$ and $\langle W_\alpha|\ \alpha<\varkappa\rangle$ is a sequence of disjoint I-partitions of A. We first construct a sequence of disjoint I-partitions of A, $\langle T_\alpha|\ \alpha<\varkappa\rangle$ such that for each $\alpha<\beta<\varkappa$, $T_\alpha \leqslant W_\alpha$ and $|T_\alpha \restriction X|<\varkappa$ whenever $X \in T_\beta$. Let $T_0 = W_0$; hence suppose $\beta<\varkappa$ and we have constructed $\langle T_\alpha|\ \alpha<\beta\rangle$ satisfying the given properties. If $\beta = \gamma+1$ let $T_\beta = \{X \cap Y|\ X \in T_\gamma, Y \in W_\beta \text{ and } X \cap Y \in I^+\}$. If $\lim(\beta)$ let h: $\bigcup\{T_\alpha|\ \alpha<\beta\} \to \varkappa$ be injective; then by $\varkappa$-completeness $E = \{\xi \in A \cap R_\varkappa|\ \forall\alpha<\beta, \xi \in \bigcup T_\alpha \text{ and } h(T_\alpha(\xi))<\xi\} \in (I|A)^*$. For $\xi \in E$ let $k(\xi) = \bigcup\{h(T_\alpha(\xi))\ \alpha<\beta\}$; then by normality $T = \{k^{-1}(\{\varrho\})|\ \varrho<\varkappa$ and $k^{-1}(\{\varrho\})\in I^+\}$ is a disjoint I-partition of A such that for each $\alpha<\beta$ and $X\in T$, $|T_\alpha \restriction X|<\varkappa$. Let

$$T_\beta = \{X\cap Y|\ X\in T,\ Y\in W_\beta \text{ and } X\cap Y\in I^+\}.$$

In either case it is easy to check that $\langle T_\alpha|\ \alpha\leqslant\beta\rangle$ has the required properties, hence we have our sequence $\langle T_\alpha|\ \alpha<\varkappa\rangle$. For each $\alpha<\varkappa$ let $q_\alpha\colon T_\alpha\to\varkappa$ be injective and such that $q_\alpha(T_\alpha)\cap q_\beta(T_\beta)=\emptyset$ whenever $\alpha<\beta<\varkappa$, then by normality

$$B=\{\xi\in A\cap R_\varkappa|\ \forall\alpha<\xi,\ \xi\in\bigcup T_\alpha \text{ and } q_\alpha(T_\alpha(\xi))<\xi\}\in(I|A)^*.$$

For $\xi\in B$ let $S_\xi=\{q_\alpha(T_\alpha(\xi))|\ \alpha<\xi\}$; then by (b) we may find a $C\in P(B)\cap I^+$ such that for each $\{\xi,\chi\}\in[C]^2$, $|S_\xi\cap S_\chi|=\xi$.

Given $\alpha<\varkappa$, pick $\xi\in C-(\alpha+1)$. Then for each $\chi\in C-(\xi+1)$ there is a β such that $\alpha<\beta<\xi$ and $T_\beta(\xi)=T_\beta(\chi)$. Hence $C-(\xi+1)\subseteq\bigcup\{T_\beta(\xi)|\ \alpha<\beta<\xi\}$ and since $|T_\alpha\restriction T_\beta(\xi)|<\varkappa$ whenever $\alpha<\beta<\xi$, we must have $|T_\alpha\restriction C|<\varkappa$. Finally since $T_\alpha\leqslant W_\alpha$, $|W_\alpha\restriction C|<\varkappa$. ■

We will see shortly that the converse of Theorem 4.11 (d) does not hold.

THEOREM 4.12. *The following are equivalent*

(a) *I is normal and $(\varkappa,<\varkappa,\varkappa)$-distributive*;

(b) *whenever $A\in I^+$, $m<\omega$ and $f\colon[A]^{m+1}\to\varkappa$ is such that $f(a)<\max(a)$ for each $a\in[A]^{m+1}$, there is a $B\in P(A)\cap I^+$ and a function $g\colon[B]^m\to\varkappa$ such that for each $a\in[B]^{m+1}$, $f(a)<g(a|m)$.*

Proof (a) → (b). Suppose $A\in I^+$ and $f\colon[A]^{m+1}\to\varkappa$ is as given. For $a\in[A]^m$ and $\varrho<\varkappa$ let $X_a^\varrho=\{\xi\in A|\ \xi>\max(a)$ and $f(a\cup\{\xi\})=\varrho\}$ and

$$W_a=\{X_a^\varrho|\ \varrho<\varkappa \text{ and } X_a^\varrho\in I^+\}.$$

By normality W_a is a disjoint I-partition of A; hence there is a $B\in P(A)\cap I^+$ such that for each $a\in[A]^m$, $|W_a\restriction B|<\varkappa$. Let $g\colon[A]^m\to\varkappa$ be given by $g(a)=\bigcup\{\varrho<\varkappa|\ X_a^\varrho\in W_a\restriction B\}$. Then by normality we must have that

$$C=\{\xi\in B|\ \exists b\in[B\cap\xi]^m,\quad f(b\cup\{\xi\})>g(b)\}\in I,$$

and hence $B-C$ is our required set.

(b) → (a). Normality is clear from the case $m=0$. We first show that $R_\varkappa\in I^*$. Suppose not; then by normality there is a $\nu<\varkappa$ such that $A=\{\xi<\varkappa|\ \mathrm{cf}\,\xi=\nu\}\in I^+$. For $\xi\in A$ let $\langle\xi_\alpha|\ \alpha<\nu\rangle$ be a strictly increasing sequence cofinal in ξ, and for $\{\xi,\chi\}\in[A]^2$ let $f(\{\xi,\chi\})=\chi_\alpha$ where α is the least ordinal less than ν such that $\chi_\alpha>\xi$. By hypothesis there is a $B\in P(A)\cap I^+$ such that for each $\xi\in B$, $g(\xi)=\bigcup\{f(\{\xi,\chi\})|\ \chi\in B-(\xi+1)\}<\varkappa$. Pick $\{\xi^\delta|\ \delta\leqslant\nu\}\subseteq B$ such that for each $\delta\leqslant\nu$, $\xi^\delta>\bigcup\{g(\xi^\beta)|\ \beta<\delta\}$; then for each $\beta<\gamma<\nu$,

$$\xi^\beta<f(\{\xi^\beta,\xi^\nu\})\leqslant g(\xi^\beta)<\xi^\gamma<f(\{\xi^\gamma,\xi^\nu\}).$$

Hence, for each $\alpha<\nu$, $\xi_\alpha^\nu<\bigcup\{g(\xi^\beta)|\ \beta<\nu\}<\xi^\nu$, contradicting the choice of $\langle\xi_\alpha^\nu|\ \alpha<\nu\rangle$.

Suppose now that $A\in I^+$ and $\langle W_\alpha|\ \alpha<\varkappa\rangle$ is a sequence of disjoint I-partitions of A. As in the proof of Theorem 4.11 we may assume that for each $\alpha<\beta<\varkappa$, $|W_\alpha\restriction X|<\varkappa$ whenever $X\in W_\beta$, and if $q\colon\bigcup\{W_\alpha|\ \alpha<\varkappa\}\to\varkappa$ is injective then by normality, $C=\{\chi\in A|\ \forall\alpha<\chi,\ \chi\in\bigcup W_\alpha$ and $q(W_\alpha(\chi))<\chi\}\in(I|A)^*$.

For $\{\xi,\chi\}\in[C]^2$, let $f(\{\xi,\chi\})=q(W_\xi(\chi))$; then by our hypothesis we may find an $E\in P(C)\cap I^+$ such that for each $\xi\in E$, $|\{f(\{\xi,\chi\})|\ \chi\in E-(\xi+1)\}|<\varkappa$, and hence $|W_\xi\restriction E|<\varkappa$. So, given $\alpha<\varkappa$, pick $\xi\in E-(\alpha+1)$; then $W_\alpha\restriction E\subseteq\bigcup\{W_\alpha\restriction X|\ X\in W_\xi\restriction E\}$ and since $|W_\alpha\restriction X|<\varkappa$ for each $X\in W_\xi$, $|W_\alpha\restriction E|<\varkappa$. ■

It is immediate from Theorems 4.8 and 4.9 that a version of Theorem 4.12 (a) → (b) also holds for $(\varkappa,<\eta,\varkappa)$ and strong $(\varkappa,<\varkappa,\varkappa)$-distributivity. For example,

THEOREM 4.13. *If η is regular, $\eta<\varkappa$ and I is normal and $(\varkappa,<\eta,\varkappa)$-distributive then whenever $A\in I^+$, $m,n<\omega$ and $f\colon[A]^{m+n}\to\varkappa$ is such that for each $a\in[A]^{m+n}$, $f(a)<\min(a-a|m)$, there is a $B\in P(A)\cap I^+$ such that for each $b\in[B]^m$,*

$$|\{f(b\cup a)|\ a\in[B]^n,\ \min(a)>\max(b)\}|<\eta.$$

Again just one such partition does not seem to imply the appropriate form of distributivity. Also the corresponding version of Theorem 4.13 for normal $(\varkappa,<\varkappa,\varkappa)$-distributive ideals does not hold in the case when $n\geqslant 2$: Consider the case where $\varkappa$ is strongly inaccessible and I is normal, atomless and $\varkappa$-saturated; then trivially I is $(\varkappa,<\varkappa,\varkappa)$-distributive but by Theorem 4.11 (c), $I^+\not\to[I^+]^2_{r,<\varkappa}$ (since I naturally gives rise to a $\varkappa$-Suslin tree, see [20, p. 50]).

However, in the case where $\varkappa$ is weakly compact, $(\varkappa,<\varkappa,\varkappa)$-distributive ideals on $\varkappa$ do satisfy a partition property.

THEOREM 4.14. *If $\varkappa$ is weakly compact and I is weakly selective and $(\varkappa,<\varkappa,\varkappa)$-distributive then $I^+\to(I^+,\varkappa)^2$.*

Proof. By [14, Theorem 8] we need to show that I is WC. By Corollary 4.3 I is $(\mu,\varkappa)$-distributive for each $\mu<\varkappa$, so suppose $A\in I^+$ and $\langle W_\alpha|\ \alpha<\varkappa\rangle$ is a decreasing sequence of disjoint I-partitions of A. By $(\varkappa,<\varkappa,\varkappa)$-distributivity there is a $B\in P(A)\cap I^+$ such that for each $\alpha<\varkappa$, $|W_\alpha\restriction B|<\varkappa$, but then $\bigcup\{W_\alpha\restriction B|\ \alpha<\varkappa\}$ naturally forms a tree (whose αth level is $W_\alpha\restriction B$) which, by weak compactness, has a branch $\langle X_\alpha|\ \alpha<\varkappa\rangle$. Clearly, $\langle X_\alpha|\ \alpha<\varkappa\rangle$ is a branch through $\langle W_\alpha|\ \alpha<\varkappa\rangle$. ■

Hence, by Theorem 3.1, if $\varkappa$ is weakly compact and I is normal and $(\varkappa,<\varkappa,\varkappa)$-distributive then I extends the Π_1^1-indescribable ideal on $\varkappa$. Indeed, even in the case where $\varkappa$ is not weakly compact, normal $(\varkappa,<\varkappa,\varkappa)$-distributive ideals have properties reminiscent of indescribability.

THEOREM 4.15. *If I is normal and $(\varkappa,<\varkappa,\varkappa)$-distributive then for each $S\in\mathrm{NS}_\varkappa^+$, $\{\xi\in R_\varkappa|\ S\cap\xi\in\mathrm{NS}_\xi^+\}\in I^*$.*

Proof. Suppose not; then by Corollary 4.4 there is an $S\in\mathrm{NS}_\varkappa^+$ such that $A=\{\xi\in R_\varkappa|\ S\cap\xi\in\mathrm{NS}_\xi\}\in I^+$. For $\xi\in A$ let $C_\xi\subseteq\xi$ be closed unbounded in ξ such that $C_\xi \cap S = \emptyset$, and let $f_\xi\colon \xi \to \xi$ be the normal enumeration of C_ξ. Then $A \Vdash$ "$[f_\xi] \in V^\varkappa/D$ and $[f_\xi]\colon \varkappa \to \varkappa$", and so by Theorem 4.1 there is a $B \in P(A) \cap I^+$ and a function $g\colon \varkappa \to \varkappa$ $(g \in V)$ such that for each $\alpha < \varkappa$, $B \Vdash$ "$[f_\xi](\alpha) < g(\alpha)$". Hence by normality, $E = \{\xi \in B|\ \forall \alpha < \xi,\ f_\xi(\alpha) < g(\alpha)\} \in (I|B)^*$.

Let $C = \{\delta < \varkappa|\ \forall \alpha < \delta,\ g(\alpha) < \delta\}$ then $C \in \mathrm{NS}_\varkappa^*$, and so we may find a $\delta \in C \cap S$. Pick $\xi \in E - (\delta+1)$, then for each $\alpha < \delta$, $f_\xi(\alpha) < g(\alpha) < \delta$ and hence, since f_ξ is normal, $\delta = f_\xi(\delta) \in C_\xi \cap S$, a contradiction. ■

COROLLARY 4.16. $(V = L)$ *If I is normal and $(\varkappa, <\varkappa, \varkappa)$-distributive then* $I^+ \to (I^+, \varkappa)^2$.

Proof. This follows immediately from Theorems 4.14, 4.15 and Jensen's result [9] that if $V = L$, then $\varkappa$ has the stationary reflection property iff $\varkappa$ is weakly compact. ■

Recall [8, Definition 0.1], $\varkappa$ is 0-Mahlo iff $\varkappa$ is regular and for $\alpha > 0$, $\varkappa$ is α-Mahlo iff for each $\beta < \alpha$, $\{\xi < \varkappa|\ \xi \text{ is } \beta\text{-Mahlo}\} \in \mathrm{NS}_\varkappa^+$. Theorem 4.15 easily yields the following

COROLLARY 4.17. *If I is normal and $(\varkappa, <\varkappa, \varkappa)$-distributive then*

$$\{\xi < \varkappa|\ \xi \text{ is } \xi\text{-Mahlo}\} \in I^*.$$

I is said to be an *M-ideal* if $M(X) = \{\xi < \varkappa|\ \mathrm{cf}\xi > \omega \text{ and } X \cap \xi \text{ is stationary in } \xi\} \in I^*$ whenever $X \in I^*$. It is well known (see [16, § 11]) that if $A \in \mathrm{NS}_\varkappa^+$ then $\mathrm{NS}_\varkappa|A$ is not an M-ideal, and hence we have

COROLLARY 4.18 (a). *If I is normal and $(\varkappa, <\varkappa, \varkappa)$-distributive then I is an M-ideal.*

(b). *If $A \in \mathrm{NS}_\varkappa^+$ then $\mathrm{NS}_\varkappa|A$ is not $(\varkappa, <\varkappa, \varkappa)$-distributive.*

$\varkappa$ is said to be *weakly ineffable* iff whenever $\langle S_\xi|\ \xi < \varkappa\rangle$ is a sequence such that for each $\xi < \varkappa$, $|S_\xi| = |\xi|$ and $S_\xi \subseteq \xi$, there is a $B \in \mathrm{NS}_\varkappa^+$ such that $|S_\xi \cap S_\chi| = |\xi|$ whenever $\{\xi, \chi\} \in [B]^2$. It follows immediately from Theorem 4.11 (b) that if $I^+ \to [I^+]^2_{r,<\varkappa}$ then $\varkappa$ is weakly ineffable and indeed a more detailed examination reveals that the weak ineffability of $\varkappa$ follows from $\varkappa \to [\mathrm{NS}_\varkappa^+]^2_{r,<\varkappa}$. This together with Theorems 4.11 and 4.15 easily yields the following corollary whose proof we leave to the reader.

COROLLARY 4.19. *If $I^+ \to [I^+]^3_{r,<\varkappa}$ then $\{\xi < \varkappa|\ \xi \text{ is weakly ineffable}\} \in I^*$.*

§ 5. Closure of the generic ultrapower. In this section we briefly mention a form of distributivity which characterises a closure property of the generic ultrapower. I is said to be *$(\mu, <\eta, \infty)$-distributive* iff I is $(\mu, <\eta, \lambda)$-distributive for each ordinal λ.

THEOREM 5.1. *For $\mu < \varkappa$, I is $(\mu, <\varkappa^+, \infty)$-distributive iff $R(I) \Vdash$ "for each f, if $f\colon \mu \to V^\varkappa/D$ then $f \in V^\varkappa/D$".*

Proof $(\to)$. If $A \in I^+$ and f is an $R(I)$ name such that $A \Vdash$ "$f\colon \mu \to V^\varkappa/D$", then for each $\alpha < \mu$ we may find an I-partition of A, $W_\alpha = \{X_\alpha^\delta|\ \delta < \delta_\alpha\}$ and a family of functions $\{g_\alpha^\delta|\ \delta < \delta_\alpha\}$ such that for each $\delta < \delta_\alpha$, $\mathrm{dom}(g_\alpha^\delta) = \varkappa$ and

$$X_\alpha^\delta \Vdash \text{"}f(\alpha) = [g_\alpha^\delta]\text{"}.$$

By hypothesis there is a $B \in P(A) \cap I^+$ such that for each $\alpha < \mu$, $|W_\alpha \upharpoonright B| \leqslant \varkappa$, say (relabelling if necessary) $W_\alpha \upharpoonright B = \{X_\alpha^\delta|\ \delta < \beta_\alpha \leqslant \varkappa\}$. For $\xi \in B$ and $\alpha < \mu$ let $h(\xi)(\alpha) = g_\alpha^\delta(\xi)$ where δ is the least ordinal less than β_α such that $\xi \in X_\alpha^\delta$; $h(\xi)(\alpha) = 0$ is no such δ exists. It is now easy to check that $B \Vdash$ "$\forall \alpha < \mu, f(\alpha) = [h](\alpha)$".

$(\leftarrow)$. The case $\mu < \omega$ is trivial; hence suppose $\mu \geqslant \omega$, $A \in I^+$ and $\langle W_\alpha|\ \alpha < \mu\rangle$ is a sequence of I-partitions of A. Clearly, I is precipitous and if $\mathrm{cf}\gamma < \varkappa$ then $R(I) \Vdash$ "$j\gamma = \bigcup \{j\nu|\ \nu < \gamma\}$" (see [12, Lemma 2.2.3]). Hence we may find an injective function

$$h\colon \bigcup \{W_\alpha|\ \alpha < \mu\} \to \{\gamma \in \mathrm{Ord}|\ R(I) \Vdash \text{"}j\gamma = \gamma\text{"}\}.$$

Also $A \Vdash$ "$\forall \alpha < \mu, |W_\alpha \cap D| = 1$", and so we may find a $B \in P(A) \cap I^+$ and a function $g\colon \varkappa \to V$ $(g \in V)$ such that for each $\xi < \varkappa$, $g(\xi)\colon \mu \to \mathrm{Ord}$ and $B \Vdash$ "$\forall \alpha < \mu([g](\alpha) = h(X)$ iff $X \in W_\alpha \cap D)$". Suppose $\alpha < \mu$ and $X \in W_\alpha \upharpoonright B$ then $B \cap X \Vdash$ "$X \in W_\alpha \cap D$", and so, by Łoś's theorem

$$W_\alpha \upharpoonright B \subseteq \{X \in W_\alpha|\ \exists \xi \in B, g(\xi)(\alpha) = h(X)\}.$$

Hence $|W_\alpha \upharpoonright B| \leqslant \varkappa$. ■

Theorem 5.1 has also been proved (independently) by others. The theorem may also be proved for the case $\mu = \varkappa$.

In [4] Baumgartner and Taylor defined an ideal I on ω_1 to be presaturated iff I is ω_2-preserving (i.e. $R(I) \Vdash$ "ω_2^V is a cardinal") and precipitous. It is clear that if I is an $(\omega, <\omega_2, \infty)$-distributive ideal on ω_1 then I is presaturated (since $R(I) \Vdash$ "$|\omega_1^V| = \omega$", see [5, p. 52]), and that the converse holds for ω_3-saturated ideals (see [4, Theorem 4.2]). Also, as in [4, Theorem 5.10] we may show (see [15, Theorem 1.17]) that $(\omega, <\omega_2, \infty)$-distributivity is a strictly weaker notion than ω_2-saturation, but are unable to construct a presaturated ideal which is not $(\omega, <\omega_2, \infty)$-distributive.

Baumgartner and Taylor also proved [4, Theorem 4.3] that if I is an ω_2-preserving ω_ω-saturated ideal on ω_1 such that for $1 < m < \omega$, $R(I) \Vdash$ "$\mathrm{cf}(\omega_m^V) > \omega$", then I is presaturated. The following strengthening of this result is clear and we leave the proof to the reader.

THEOREM 5.2. *Let I be an ω_ω-saturated ideal on ω_1, then I is $(\omega, <\omega_2, \infty)$-distributive iff for each natural number $m > 1$, $R(I) \Vdash$ "$\mathrm{cf}(\omega_m^V) > \omega$".*

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Generic properties of proper foliations

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Abstract. The basic result in this note is the existence of an open dense saturated subset of aves without holonomy in a foliation of a paracompact manifold for which every leaf is embedded. nder the additional hypothesis of codimension one, there is an open dense saturated subset consting of leaves having open saturated neighborhoods which are foliated as products.

Let M denote a paracompact m-dimensional manifold and $\mathfrak{F}$ a codimension k foliation. A leaf of $\mathfrak{F}$ is said to be *proper* if its leaf topology is the same as its subspace topology in M. If every leaf of a foliation is proper the foliation is said to be proper. Examples of proper foliations include large classes of foliations: compact foliations (in which all leaves are compact), closed foliations (in which all leaves are closed in the subspace topology), Reeb type foliations of T^2 or S^3, as well as two further examples worthy of special notice.

The first example is a closed foliation of $\boldsymbol{R}^3$ by planar surfaces each of which is diffeomorphic to the complement in $\boldsymbol{R}^2$ of the natural numbers on the x-axis. A construction of this type was suggested by Palmeira. List the rational numbers in $\boldsymbol{R}$ by $\{q_i\}_{i=1}^{\infty}$ and consider the foliation induced on the complement of $X = \{(x, 0, z)|\ x = j \text{ and } z \geqq q_j \text{ for some } 1 \leqq j \leqq \infty\}$ in $\boldsymbol{R}^3$ by planes of constant z-coordinate. There is a diffeomorphism of $\boldsymbol{R}^3 \setminus X$ onto $\boldsymbol{R}^3$, such that every leaf is closed, giving the desired foliation of $\boldsymbol{R}^3$.

The second example is a foliation of $T^2 \times [0, 1]$, which could be "doubled" to provide a foliation of T^3 if desired, and which arises as a smooth deformation of the longitudinal foliation of T^2 by circles. This deformation is illustrated, in the universal cover of T^2, in Fig. 1, where (i) denotes the initial longitudinal foliation of $T^2 \times \{0\}$, (ii) denotes an intermediate foliation of $T^2 \times \{t\}$, $0 \leqq t \leqq 1$, and (iii) denotes the limiting "Reeb type" foliation of $T^2 \times \{1\}$.

Although foliations in general can exhibit a startling complexity of structure one might hope for some sort of regularity in the context of, for example, compact foliations. Indeed, this is the case for codimensions 1 and 2. The situation for codimension 2 was first described by D. B. A. Epstein [2] via a suprisingly delicate argument for 3-dimensional compact manifolds. This was later extended to higher dimensions independently by Edwards, Millett, and Sullivan [1] and by Vogt [12].