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## Generic properties of proper foliations

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**Abstract.** The basic result in this note is the existence of an open dense saturated subset of  $\mathbb{R}^3$  without holonomy in a foliation of a paracompact manifold for which every leaf is embedded. Under the additional hypothesis of codimension one, there is an open dense saturated subset consisting of leaves having open saturated neighborhoods which are foliated as products.

Let  $M$  denote a paracompact  $m$ -dimensional manifold and  $\mathcal{F}$  a codimension  $k$  foliation. A leaf of  $\mathcal{F}$  is said to be *proper* if its leaf topology is the same as its subspace topology in  $M$ . If every leaf of a foliation is proper the foliation is said to be proper. Examples of proper foliations include large classes of foliations: compact foliations (in which all leaves are compact), closed foliations (in which all leaves are closed in the subspace topology), Reeb type foliations of  $T^2$  or  $S^3$ , as well as two further examples worthy of special notice.

The first example is a closed foliation of  $\mathbb{R}^3$  by planar surfaces each of which is diffeomorphic to the complement in  $\mathbb{R}^2$  of the natural numbers on the  $x$ -axis. A construction of this type was suggested by Palmeira. List the rational numbers in  $\mathbb{R}$  by  $\{q_i\}_{i=1}^\infty$  and consider the foliation induced on the complement of  $X = \{(x, 0, z) \mid x = j \text{ and } z \geq q_j \text{ for some } 1 \leq j \leq \infty\}$  in  $\mathbb{R}^3$  by planes of constant  $z$ -coordinate. There is a diffeomorphism of  $\mathbb{R}^3 \setminus X$  onto  $\mathbb{R}^3$ , such that every leaf is closed, giving the desired foliation of  $\mathbb{R}^3$ .

The second example is a foliation of  $T^2 \times [0, 1]$ , which could be “doubled” to provide a foliation of  $T^3$  if desired, and which arises as a smooth deformation of the longitudinal foliation of  $T^2$  by circles. This deformation is illustrated, in the universal cover of  $T^2$ , in Fig. 1, where (i) denotes the initial longitudinal foliation of  $T^2 \times \{0\}$ , (ii) denotes an intermediate foliation of  $T^2 \times \{t\}$ ,  $0 \leq t \leq 1$ , and (iii) denotes the limiting “Reeb type” foliation of  $T^2 \times \{1\}$ .

Although foliations in general can exhibit a startling complexity of structure one might hope for some sort of regularity in the context of, for example, compact foliations. Indeed, this is the case for codimensions 1 and 2. The situation for codimension 2 was first described by D.B.A. Epstein [2] via a suprisingly delicate argument for 3-dimensional compact manifolds. This was later extended to higher dimensions independently by Edwards, Millett, and Sullivan [1] and by Vogt [12].

Examples of Reeb [9] and Epstein [2] showed that such structure was not possible for noncompact manifolds. Later an example of Sullivan [10, 11], followed by examples of Epstein and Vogt [5] and Vogt [13], showed that in general one must expect very complicated structure for compact foliations of compact manifolds.

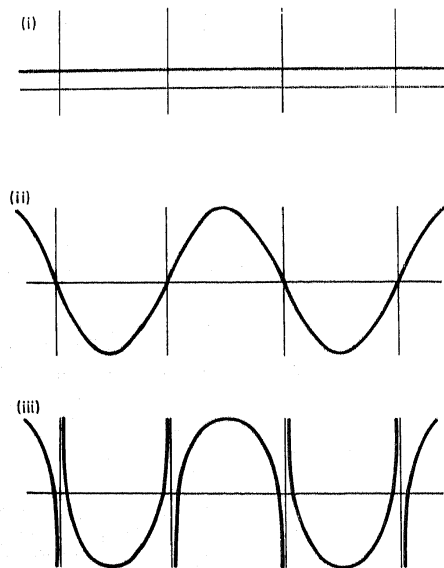


Fig. 1

There is a fundamental concept developed in Epstein [2] which has proved to be central to succeeding studies of compact foliations. This is the Epstein filtration of the subset of leaves having holonomy. This arises by considering a fixed riemannian metric on a paracompact manifold  $M$ , the induced riemannian metric on the tangent bundle of the foliation and the subsequently induced volume function on each leaf of the foliation. Although the essential facts for this discussion will be presented below the reader is referred to Epstein [2, 3] or Edwards, Millett, and Sullivan [1], especially sections 4 and 6, for a detailed discussion. This volume function is lower-semicontinuous. As a consequence, it is continuous on an open dense saturated subset  $M_1$  of  $M$ . Because of the continuity the leaves of  $M_1$  have trivial holonomy and, therefore, by the Reeb stability theorem [9], are *stable*, i.e. have open saturated neighborhoods in  $M$  which are foliated as a product. As a consequence one says that the generic leaf of a compact foliation has trivial holonomy and, furthermore,

is stable. The goal of this paper is the study of the extent to which this is also true for proper foliations.

Returning to the definition of the filtration, let  $X_1 = M \setminus M_1$  denote the closed saturated nowhere dense subset of leaves for which this volume function is discontinuous. Since  $X_1$  is locally compact, the restricted volume function is continuous on an open dense saturated subset of  $X_1$  and, as a consequence, there is an open dense saturated subset of  $X_1$  and, as a consequence, there is an open dense saturated subset of  $M$ ,  $M_2$ , containing  $M_1$  such that  $M_2 \setminus M_1$  is an open dense saturated subset of  $X_1$  of leaves with relatively trivial holonomy and therefore, by a generalization of the Reeb stability theorem, having relatively open saturated neighborhoods foliated as products, i.e. relatively stable leaves.

This process may be continued to define an ascending family of open dense saturated subsets  $\{M_\alpha\}_{\alpha \leq \gamma}$  indexed by a countable ordinal  $\gamma$ , such that (i)  $M_0 = \emptyset$ , (ii)  $M_\gamma = M$ , (iii) if  $\alpha$  is a limit ordinal then  $M = \bigcup_{\beta < \alpha} M_\beta$ , and (iv) if  $\alpha$  is not a limit ordinal and is not 0 then  $M_\alpha \setminus M_{\alpha-1}$  is an open dense saturated subset of leaves of  $M \setminus M_{\alpha-1}$  having relatively trivial holonomy. This family is related to the "fine" Epstein filtration [1] of the set of leaves having holonomy by taking complements, i.e.  $X_\alpha = M \setminus M_\alpha$ . Such a family of dense open saturated subsets need not exist in general, even for smooth codimension 1 foliations of compact manifolds, as exhibited by the examples in Epstein, Millett, and Tischler [4] where the following theorem was proved:

**THEOREM.** *Let  $M$  be a paracompact manifold with a foliation of codimension  $k$ . Let  $T$  be the union of all leaves with trivial holonomy. The  $T$  is a dense  $G_\delta$  in  $M$ .*

In an example in [4] the set  $T$  has no interior. This occurs, essentially, because the leaves without holonomy are not proper. Even in the case where the leaf is proper and the holonomy group is trivial there are still problems as shown by a recent example of Inaba [8]. In an earlier paper [7], Inaba shows that there is, however, a natural condition which implies openness, at least for codimension one foliations. Employing the definition of the holonomy pseudogroup provided in [4], we shall say that a leaf  $L$  of a foliation  $(M, \mathcal{F})$  has *locally trivial holonomy pseudogroup* if there exists an open transversal,  $h(\{p\} \times Q)$  which intersects  $L$  and such that each leaf of  $(M, \mathcal{F})$  meets the transversal in at most one point. It is easy to see that if a leaf has locally trivial holonomy pseudogroup, then its holonomy group is trivial. The converse is not true, even for proper leaves.

**THEOREM (Inaba [7]).** *Let  $M$  be a smooth manifold and let  $\mathcal{F}$  be a continuous codimension 1 foliation. If  $L$  is a proper relatively compact leaf with locally trivial holonomy pseudogroup then  $L$  is contained in an open saturated neighborhood foliated as a product, i.e.  $L$  is stable.*

The example of the smooth codimension 1 closed foliation of  $\mathbb{R}^3$ , described earlier, has the property that all leaves are closed and have locally trivial holonomy pseudogroup and yet no leaf is stable. In the second example, giving a proper foliation of  $T^2 \times [0, 1]$ , the non compact leaves are relatively compact and have locally trivial

holonomy pseudogroup and yet not stable showing the necessity of the codimension 1 assumption.

A foliation will be called *relatively compact* if each leaf of the foliation is relatively compact, i.e. is contained in a compact subset of the manifold. The following theorem is an extension of the holonomy theorem of Epstein, Millett, and Tischler stated above.

**THEOREM 1.** *Let  $M$  be a smooth paracompact manifold and let  $\mathcal{F}$  be a proper relatively compact continuous codimension 1 foliation of  $M$ . There is an ascending family of open dense saturated subsets  $\{M_\alpha\}_{\alpha \leq \gamma}$  indexed by a countable ordinal,  $\gamma$ , such that (i)  $M_0 = \emptyset$ , (ii)  $M_\gamma = M$ , (iii) if  $\alpha$  is a limit ordinal then  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ , and (iv) if  $\alpha$  is not a limit ordinal and is not 0, then  $M_\alpha \setminus M_{\alpha-1}$  is an open dense saturated subset of relatively stable leaves of  $M \setminus M_{\alpha-1}$ .*

**COROLLARY 2.** *The generic leaf of a proper relatively compact  $C^0$  codimension 1 foliation of a smooth paracompact manifold is stable.*

The corollary is simply the existence of  $M_1$  in the theorem. For higher codimensions and for not necessarily relatively compact foliations we are not able to employ Inaba's stability theorem. There is, nevertheless, the following generalization of the holonomy theorem whose statement and conceptual approach is modeled on the corresponding result of Glimm [6] for locally compact transformation groups.

**THEOREM 3.** *Let  $M$  be a paracompact manifold and let  $\mathcal{F}$  be a continuous codimension  $k$  foliation. Then the following are equivalent:*

- (1)  $\mathcal{F}$  is a proper foliation,
- (2) for each leaf of the foliation there is an open transversal meeting the leaf in exactly one point,
- (3) each leaf is relatively open its closure,
- (4) the leaf space  $M/\mathcal{F}$  is  $T_0$ ,
- (5) there is an ascending family of open dense saturated subsets  $\{M_\alpha\}_{\alpha \leq \gamma}$  indexed by a countable ordinal,  $\gamma$ , such that (i)  $M_0 = \emptyset$ , (ii)  $M_\gamma = M$ , (iii) if  $\alpha$  is a limit ordinal then  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ , and (iv) if  $\alpha$  is not a limit ordinal and is not 0, then  $M_\alpha \setminus M_{\alpha-1}$  is an open dense saturated subset of leaves having locally trivial relative holonomy pseudogroup in  $M \setminus M_{\alpha-1}$ .

As above we have the following:

**COROLLARY 4.** *The generic leaf of a proper foliation of a paracompact manifold has locally trivial holonomy pseudogroup.*

With these theorems and current examples of proper foliations, it appears reasonable to conjecture that Theorem 1 can be extended to higher codimensions. This seems, at least for the present, an attractive conjecture since the present proof applied to various test examples gives precisely the property.

Since Theorem 1 follows directly from Theorem 3 and Inaba's Theorem, the remainder of this note will be devoted to the proof of Theorem 3.

(1)  $\rightarrow$  (2). Suppose that  $\mathcal{F}$  is a proper foliation, i.e. that each leaf of  $\mathcal{F}$  has the subspace topology of  $M$ . Thus, given  $x \in L \in \mathcal{F}$  and a euclidean neighborhood of  $x$  in  $L$ , there is an open subset of  $M$  whose intersection with  $L$  is precisely the neighborhood. Within this open set there is an open transversal meeting the leaf in exactly the point  $x$ . ■

(2)  $\rightarrow$  (3). Let  $L$  be a leaf of  $\mathcal{F}$  and  $x \in L$  having an open transversal  $Q$  such that  $Q \cap L = \{x\}$ . The saturation of  $Q$ ,  $U$ , is an open saturated subset of  $M$  containing  $L$ . Suppose that  $y \in U \cap (\bar{L} \setminus L) \neq \emptyset$ . Then  $L_y$ , the leaf containing  $y$ , is contained in  $U \cap (\bar{L} \setminus L)$  since it is a saturated set. Therefore,  $y_0 \in L_y \cap Q \neq \emptyset$ . Let  $\bar{Q}$  be an open transversal at  $y$ . Since  $y \in \bar{L} \setminus L$ , there is a sequence of points  $\{x_i\}_{i=1}^\infty$  in  $\bar{Q} \cap L$  converging to  $y$ . The holonomy pseudogroup map taking a neighborhood of  $y$  in  $\bar{Q}$  to a neighborhood of  $y_0$  in  $Q$  takes a final segment of the sequence  $\{x_i\}$  to a sequence  $\{x'_i\}$  converging to  $y_0$  in  $Q$ . Since  $\{x'_i\} \subset L \cap Q = \{x\}$ , this is impossible because  $L_y \neq L$ . Thus  $U \cap (\bar{L} \setminus L) = \emptyset$ , and hence  $L$  is relatively open its closure. ■

(3)  $\rightarrow$  (4). Suppose that  $L_x$  and  $L_y$  are two distinct leaves of  $\mathcal{F}$ . If  $L_y \not\subset \bar{L}_x$  there is a saturated open set  $U = M \setminus \bar{L}_x$  such that  $L_y \subset U$  and  $L_x \cap U = \emptyset$ . This provides an open subset of  $M/\mathcal{F}$  containing  $[L_y]$  and disjoint from  $[L_x]$ . If, however,  $L_y \subset \bar{L}_x$  there is, by (3), an open set  $U$  of  $M$  such that  $U \cap \bar{L}_x = L_x$ . Its saturation,  $\bar{U}$ , gives a saturated open set containing  $L_x$  and missing  $L_y$  and, therefore, a neighborhood of  $[L_x]$  missing  $[L_y]$  in  $M/\mathcal{F}$ . Therefore,  $M/\mathcal{F}$  is  $T_0$ . ■

(4)  $\rightarrow$  (5). Suppose that  $M/\mathcal{F}$  is  $T_0$ . Let  $\{Q_j\}_{j \in J}$  be a collection of transversals arising from a locally finite family of foliation charts such as constructed in [4] for a locally compact Hausdorff paracompact foliated space,  $h_j: Q_j \rightarrow \mathbb{R}^k$ . Let  $Q$  denote the disjoint union of the  $Q_i$  and let  $\mathcal{R}$  denote the equivalence relation induced on  $Q$  by the foliation, i.e.  $(x, y) \in \mathcal{R} \subset Q \times Q$  if and only if  $x$  and  $y$  lie in the same leaf of  $\mathcal{F}$ . Then  $Q/\mathcal{R}$  is naturally homeomorphic to  $M/\mathcal{F}$ .

Let  $\|u - v\|$  denote the standard norm on  $\mathbb{R}^k$ . Define a family of functions

$$\varrho_i: Q_i \times Q_i \rightarrow [0, \infty) \quad \text{by}$$

$$\varrho_i(x, y) = \max\{d(x, y), d(y, x)\}$$

where

$$d(x, y) = \inf\{\|h_i(x) - h_i(y')\| \mid (y, y') \in \mathcal{R} \cap Q_i \times Q_i\}.$$

**LEMMA 1.**  $\varrho_i(x, y) = 0$  if and only if  $(x, y) \in \mathcal{R} \cap Q_i \times Q_i$ .

**Proof.** If  $(x, y) \in \mathcal{R} \cap Q_i \times Q_i$ , then  $d(x, y) = 0$ , and hence  $\varrho_i(x, y) = 0$ . If  $(x, y) \notin \mathcal{R} \cap Q_i \times Q_i$  and  $\varrho_i(x, y) = 0$  then  $L_x \neq L_y$ ,  $L_x \subset L_y$ , and  $L_y \subset \bar{L}_x$ . Thus  $[L_x]$  and  $[L_y]$  are two distinct points in  $M/\mathcal{F}$  and any saturated open subset of  $M$  containing  $L_x$  contains  $L_y$  and *vice versa*. Thus there is neither an open subset of  $M/\mathcal{F}$  containing  $[L_x]$  and not  $[L_y]$  nor *vice versa*. This contradicts the assumption that  $M/\mathcal{F}$  is  $T_0$ . ■

**LEMMA 2.**  $\varrho_i$  is upper semicontinuous.

Proof. First note that  $d(x, y)$  is continuous in  $x$ . To show that  $d(x, y)$  is upper semicontinuous it, therefore, is sufficient to show that it is upper semicontinuous in  $y$ , i.e. we must show that for any  $s \in \mathbb{R}$  the set  $\{y | d(x, y) < s\}$  is an open subset of  $Q_i$ . Thus suppose we are given  $(x_0, y_0) \in Q_i$  such that  $d(x_0, y_0) = t < s$ . By the definition of  $d$  there is a  $y' \in Q_i$  such that  $(y_0, y') \in \mathcal{H} \cap Q_i \times Q_i$  and  $\|h_i(x_0) - h_i(y')\| < t + (s - t)/4$ . There is an open neighborhood of  $y_0 \in Q_i$ ,  $U$ , and a holonomy pseudo-group map of this neighborhood to a neighborhood of  $y'$  such that the diameters of the images of these neighborhoods in  $\mathbb{R}^k$  under  $h_i$  are less than  $(s - t)/4$ . Then, if  $w \in U$ ,

$$d(x_0, w) < \|h_i(x_0) - h_i(y')\| + (s - t)/4 < t + (s - t)/2 < s,$$

so that  $(x_0, y_0) \in U \subset \{y | d(x_0, y) < s\}$ , and hence  $d$  is upper semicontinuous. Since the maximum of two upper semicontinuous functions is upper semicontinuous,  $q_i$  is upper semicontinuous. ■

Since  $Q_i \times Q_i$  is a locally compact space, the set of points of continuity of  $p_i$  is a dense  $G_\delta$  subspace,  $W'_i$ , of  $Q_i \times Q_i$ . Since  $M/\mathfrak{F}$  is  $T_0$ ,  $\mathcal{H}$  is nowhere dense in  $Q_i \times Q_i$ . Therefore,  $W_i = W'_i \setminus (\mathcal{H} \cap Q_i \times Q_i)$  is a dense subset of  $Q_i \times Q_i$  and  $q_i$  is positive on  $W_i$ .

LEMMA 3. For each  $i$  there is a dense subset of points,  $Y_i \subset Q_i$ , such that each  $y \in Y_i$  has a neighborhood  $U_y$  such that  $U_y \times U_y \cap \mathcal{H} = \Delta$ .

Proof. Suppose that  $(x, y) \in W_i$ , i.e.  $(x, y)$  is a point of continuity of  $q_i$  of nonzero value  $\varepsilon$ . There are neighborhoods  $U_x$  and  $U_y$ , of  $x$  and  $y$ , respectively, whose images under  $h_i$  have diameter less than  $\varepsilon/100$  and such that if  $(u, v) \in U_x \times U_y$  then  $|q_i(x, y) - q_i(x, y)| < \varepsilon/100$ . Suppose that

$$(u, u') \in (\mathcal{H} \setminus \Delta) \cap U_x \times U_x \quad \text{and} \quad (v, v') \in (\mathcal{H} \setminus \Delta) \cap U_y \times U_y.$$

Then  $(u, v)$ ,  $(u, v')$ ,  $(u', v) \in U_x \times U_y \cap \mathcal{H}$  that  $q_i(u, v) < \varepsilon/100$  since

$$q_i(u, v) < \max\{\|h_i(u) - h_i(v')\| < \varepsilon/100, \|h_i(u) - h_i(u')\| < \varepsilon/100\}$$

by the definition of  $q_i$ . But  $99\varepsilon/100 < q_i(u, v) < 101\varepsilon/100$  by the continuity. Thus either  $U_x \times U_x \cap \mathcal{H} \subset \Delta$ , in which case we place  $x$  in a set  $Y'_i$ , or  $U_y \times U_y \cap \mathcal{H} \subset \Delta$ , in which case we place  $y$  in a set  $Y''_i$ . Since  $X_i$  is dense in  $Q_i \times Q_i$  then either  $Y'_i$  or  $Y''_i$  is dense in  $Q_i$ . Let  $Y_i$  denote one of these which is dense. ■

The union of all neighborhoods associated to all points of all  $Y_i$  gives a dense open subset of  $Q$  whose saturation in  $M$  is a dense open saturated subset,  $M_1$ , of  $M = M \setminus M_0$ .

LEMMA 4. The leaves of  $M_1$  have locally trivial relative holonomy pseudogroup.

Proof. We need to show that for a leaf of  $M_1 \setminus M_0$ , say  $L$ , there is a point  $x \in L \cap Q_i$  and a relatively open subset of  $M_1 \cap Q_i$ ,  $V$ , such that each leaf of  $M_1 \setminus M_0$  meets  $V$  in at most one point. This follows directly from the fact that we may choose

$x \in U_y \subset Q_i$  for some  $y \in Y_i$  for some  $y$ . Since  $U_y \times U_y \cap \mathcal{H} \subset \Delta$  each leaf of  $M_1 \setminus M_0$  meets  $U_y$  in at most one point. ■

At this point we have  $M_0 = \emptyset$  and have defined  $M_1$ . We define  $X_1 = M \setminus M_1$ , a closed nowhere dense locally compact saturated subset of  $M$ . The process is then repeated with  $M$  replaced by  $X_1$  to define  $M_2$ . Transfinite induction is then employed to define the collection  $\{M_\alpha\}_{\alpha < \gamma}$  for some, possibly uncountable, ordinal  $\gamma$  satisfying the conditions of statement (5).

The fact that  $\gamma$  is countable result from a fundamental observation employed in Epstein, Millett, and Tischler [4] to the effect that the collection of all holonomy maps in a foliation of a paracompact manifold is countable, say  $\{f_i\}_{i \in \mathbb{N}}$ . Since each domain and range is homeomorphic to a subset of  $\mathbb{R}^k$  they have second countable topologies  $\{U_{ik}\}_{k \in \mathbb{N}}$  and  $\{V_{ik}\}_{k \in \mathbb{N}}$ . For each  $i$  there is a maximal  $\alpha_i$ , such that the holonomy map  $f_i$  is trivial on  $M_\alpha \setminus M_{\alpha-1}$ . For each  $\beta < \alpha$  there is a  $U_\beta \in \{U_{ik}\}_{k \in \mathbb{N}}$  such that  $\beta = \max\{\delta | U_\beta \subset M_\delta\}$ . The assignment of  $\beta$  to  $U_\beta$  is one-to-one so that  $\alpha$  has a most countably many predecessors. Since the correspondence  $i$  to  $\alpha_i$  is a terminal sequence, we see that  $\gamma$  has at most countably many predecessors and, hence, is countable.

(5)  $\rightarrow$  (1). Suppose that  $L$  is a leaf of the foliation  $\mathfrak{F}$ . There exists a minimal  $\beta$ , which cannot be a limit ordinal, such that  $L \subset M_\beta$ . Thus  $L \subset M_\beta \setminus M_{\beta-1}$ . By property (iv) for each  $x \in L$  there is an open transversal to  $L$  contained in  $M_\beta$  such that each leaf of  $M_\beta \setminus M_{\beta-1}$ , including  $L$ , meets the transversal in at most one point. Thus any sufficiently small neighborhood of  $x$  in  $L$  is the intersection of its product with a small neighborhood of  $x$  in the transversal. Therefore  $L$  has the subspace topology. Since  $L$  was arbitrary,  $\mathfrak{F}$  is a proper foliation. ■

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