Acknowledgments. The author is indebted to Tom Lindstrom, Matti Rubin, and Keith Stroyan, who all provided encouragement and helpful suggestions at various stages of this work.

References

[1] R. M. Anderson, *Star-finite representations of measure spaces*, Trans. Amer. Math. Soc. 271 (1982), 667–687.

[2] A. Bernstein and F. Wattenberg, *Nonstandard measure theory*, in *Applications of model theory to algebra, analysis, and probability* (Holt, Rinehart, and Winston, New York 1969), 171–185.

[3] G. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. 17 (1931), 656–660.

[4] J. R. Choksi, *Measurable transformations on compact groups*, Trans. Amer. Math. Soc. 184 (1973), 101–124.

[5] N. Cutland, *Nonstandard measure theory and its applications*, Bull. London Math. Soc. 15 (1983), 529–589.

[6] G. A. Edgar, *Measurable weak selections*, Illinois J. Math. 20 (1976), 630–646.

[7] D. H. Fremlin, *Measurable functions and almost continuous functions*, Manus. Math. 33 (1981), 387–405.

[8] C. W. Henson, *On the nonstandard representation of measures*, Trans. Amer. Math. Soc. 172 (1972), 437–466.

[9] D. Hoover, *Probability logic*, Annals of Math. Logic 14 (1978), 287–313.

[10] A. and C. Ionescu Tulcea, *Topics in the theory of lifting* (Springer-Verlag, New York 1969).

[11] T. Kamae, *A simple proof of the ergodic theorem using nonstandard analysis*, Israel J. Math. 42 (1982), 284–290.

[12] H. J. Keisler, *Probability quantifiers*, in *Abstract model theory and logics of mathematical concepts* (Ed., J. Barwise and S. Feferman, Springer-Verlag, to appear), ch. 14.

[13] P. A. Loeb, *An introduction to nonstandard analysis and hyperfinite probability theory*, in *Probabilistic analysis and related topics*, Vol. 2 (Ed., A. T. Bharucha-Reid, Academic Press, New York 1979), 105–142.

[14] J. von Neumann, *Einige Sätze über meßbare Abbildungen*, Ann. of Math. 33 (1932), 574–586.

[15] R. Panzone and C. Segovia, *Measurable transformations on compact spaces and o. n. systems on compact groups*, Union Math. Arg. Revista 22 (1964), 83–102.

[16] D. Ross, *Measurable transformations in saturated models of analysis* (unpublished Ph. D. thesis, 1983).

[17] K. Stroyan and J. M. Bayod, *Foundations of infinitesimal stochastic analysis* (North Holland, to appear).

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Received 3 June 1985

FUNDAMENTA MATHEMATICAE 128 (1987)

On characterizations of classes of metrizable spaces that have transfinite dimension

by

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Abstract. We are concerned with a characterization of two classes of infinite-dimensional spaces. First, we characterize the class of metrizable spaces which have large transfinite dimension, in terms of partitions, a special base and a dimension-raising mapping. Second, we give a characterization of the class of metrizable spaces which have strong large transfinite dimension, in terms of a dimension-raising mapping and a special refinement.

1. Introduction. In this paper we are concerned with a characterization of two classes of metrizable spaces of transfinite dimension. We say that a metrizable space is countable-dimensional if it can be expressed as the union of countably many zero-dimensional subsets (in the sense of dim or equivalently of Ind). We have been inspired by the following interesting theorem, obtained by J. Nagata [11] and K. Nagami and J. H. Roberts [10], which characterizes the class of countable-dimensional metrizable spaces.

THEOREM A. *For a metrizable space X, the following conditions are equivalent:*

(a) *X is countable-dimensional.*

(b) *For every sequence $\{(A_i, B_i): i \in N\}$ of pairs of disjoint closed sets of X, there is a sequence $\{L_i: i \in N\}$ of closed sets such that each L_i is a partition between A_i and B_i in X and the family $\{L_i: i \in N\}$ is point finite.*

(c) *X has a σ-discrete base $\mathscr{B}$ such that the family $\{\mathrm{Bd}\, B: B \in \mathscr{B}\}$ is point finite.*

(d) *There are a metrizable space Z and a closed continuous mapping f of Z onto X such that $\dim Z \leqslant 0$ and $f^{-1}(x)$ consists of at most finitely many points, for each point $x \in X$.*

In [5], R. Engelking and R. Pol characterized the class of metrizable spaces of large transfinite dimension by use of a strongly point finite family (see § 2 for the definition) of partitions. But the concept of strong point finiteness cannot characterize this class in terms of a σ-discrete base. In Section 2 we characterize this class in terms of partitions and of a σ-discrete base simultaneously by use of a new concept of "point finiteness". A characterization of the same class in terms of a dimension-raising closed continuous mapping from a zero-dimensional metrizable space is also obtained. In Section 3 we give a characterization of a narrower class than that considered in Section 2, namely, the class of metrizable spaces which have strong large transfinite dimension, in terms of a dimension-raising closed continuous mapping and of a special refinement of an arbitrary open cover.

Throughout the paper all spaces are assumed to be Hausdorff. Unless otherwise stated the term dimension means the large inductive dimension Ind (and equivalently, the covering dimension dim for metrizable spaces). Let X be a space and A a subset of X. Then we denote by $\bar{A}$ (respectively $\mathrm{Bd}\,A$) the closure (the boundary) of A in X. Let N denote the set of all natural numbers. For a subset Y and a family $\mathscr{A}$ of subsets of a space X, we write $\mathscr{A}^{\#} = \bigcup\{A\colon A \in \mathscr{A}\}$ and $\mathscr{A}|Y = \{A \cap Y\colon A \in \mathscr{A}\}$. For a point x of X, the order of $\mathscr{A}$ at x, denoted by $\mathrm{ord}_x\mathscr{A}$, is the cardinality of the subfamily $\{A \in \mathscr{A}\colon x \in A\}$ of $\mathscr{A}$ and we write $\mathrm{ord}\,\mathscr{A} = \sup\{\mathrm{ord}_x\mathscr{A}\colon x \in X\}$. We refer the reader to [3], [9], [12] for the terminology and basic results on dimension theory, and especially to [4] for transfinite dimension.

The author wishes to express his thanks to Professor J. Nagata for his valuable advices.

2. Characterization of metrizable spaces which have large transfinite dimension.

A family of sets is called *strongly point finite* if it has no infinite subfamily having the finite intersection property. This notion was introduced by R. Engelking and R. Pol in [5] and, by making use of this notion, they characterized the class of metrizable spaces that have large transfinite dimension as follows:

THEOREM B ([5; Theorem 3]). *A metrizable space X has large transfinite dimension if and only if, for every sequence $\{(A_i, B_i)\colon i \in N\}$ of pairs of disjoint closed sets of X, there is a sequence $\{L_i\colon i \in N\}$ of closed sets such that each L_i is a partition between A_i and B_i in X and the family $\{L_i\colon i \in N\}$ is strongly point finite.*

In view of Theorems A and B, we may expect that a metrizable space X has large transfinite dimension if and only if X has a σ-discrete base $\mathscr{B}$ such that the family $\{\mathrm{Bd}\,B\colon B \in \mathscr{B}\}$ is strongly point finite. But the following theorem makes this conjecture false, because there is a complete separable metric space that has small transfinite dimension but does not have large transfinite dimension (see [4; Example 2.1]).

THEOREM C ([5; Theorems 1 and 2]). *A metrizable space X is a subspace of a completely metrizable countable-dimensional space if and only if X has a σ-discrete base $\mathscr{B}$ such that the family $\{\mathrm{Bd}\,B\colon B \in \mathscr{B}\}$ is strongly point finite. Furthermore, if X is separable, then X has such a base if and only if X has small transfinite dimension.*

In this section, we obtain a characterization of the class of metrizable spaces which have large transfinite dimension in terms of partitions and a σ-discrete base having a special property. For this purpose we need the following definition.

2.1. DEFINITION. A family $\mathscr{A}$ of subsets of a space X is said to be *completely point finite* if, for every closed discrete subset F of X, $\sup\{\mathrm{ord}_x\mathscr{A}\colon x \in F\} < \infty$.

Clearly, every completely point finite family of subsets of a space is point finite and these concepts coincide in the case of families of subsets of countably compact spaces. The following simple examples show that there are no relations between the concepts of complete point finiteness and of strong point finiteness, in general.

2.2. EXAMPLES. (1) Let $X = \bigoplus\{I_n\colon n \in N\}$ be the discrete sum of countably many copies I_n, $n \in N$, of the unit interval I. For each $n \in N$ and each $i \leqslant n$, let $A_n^i = [0, 1/i] \subset I_n$, $\mathscr{A}_n = \{A_n^i\colon i \leqslant n\}$ and $\mathscr{A} = \bigcup\{\mathscr{A}_n\colon n \in N\}$. Then $\mathscr{A}$ is strongly point finite but is not completely point finite.

(2) Let $X = I$ and $A_n = (0, 1/n]$ for each $n \in N$. Then $\mathscr{A} = \{A_n\colon n \in N\}$ is completely point finite but is not strongly point finite.

Let us notice that the family described in Example 2.2 (2) consists of nonclosed sets. For families consisting of closed sets of spaces, we have the following

2.3. PROPOSITION. *A completely point finite family of closed subsets of a space is strongly point finite.*

Proof. Let $\mathscr{F}$ be a completely point finite family of closed sets of a space X. Assume that $\mathscr{F}$ is not strongly point finite. There is a countable subfamily $\{F_n\colon n \in N\}$ of $\mathscr{F}$ which has the finite intersection property such that $F_m \neq F_n$ if $m \neq n$. By induction on n, we can find sequences $\{x_n\colon n \in N\}$ of points of X and $\{i_n\colon n \in N\}$ of natural numbers such that $x_n \in \bigcap\{F_i\colon i = 1, \ldots, i_n\}$, $x_n \notin F_{i_n+1}$ and $i_m < i_n$ if $m < n$. Put $F = \{x_n\colon n \in N\}$. It follows that F is a closed discrete subset of X such that $\sup\{\mathrm{ord}_x\mathscr{F}\colon x \in F\} = \infty$. This contradicts the complete point finiteness of $\mathscr{F}$.

Now, we are ready to give a characterization of the class of metrizable spaces that have transfinite dimension. Recall from [13] that a metrizable space X satisfies the condition (K) if there is a compact subset K of X such that $\mathrm{Ind}\,H < \infty$ for every closed subset H of X which does not meet K. The following lemma is already known (cf. [5], [7] and [13]).

2.4. LEMMA. *For a metrizable space X, the following conditions are equivalent:*

(a) X has large transfinite dimension.

(b) X is a countable-dimensional space satisfying the condition (K).

(c) X is a countable-dimensional space which contains no discrete family $\{U_n\colon n \in N\}$ of open sets of X such that $\mathrm{Ind}\,U_n \geqslant n$ for each $n \in N$.

2.5. THEOREM. *For a metrizable space X, the following conditions are equivalent:*

(a) X has large transfinite dimension.

(b) For every sequence $\{(A_i, B_i)\colon i \in N\}$ of pairs of disjoint closed sets of X, there is a sequence $\{L_i\colon i \in N\}$ of closed sets of X such that each L_i is a partition between A_i and B_i in X and the family $\{L_i\colon i \in N\}$ is completely point finite.

(c) X has a σ-discrete base $\mathscr{B}$ such that the family $\{\mathrm{Bd}\,B\colon B \in \mathscr{B}\}$ is completely point finite.

Proof. Let X be a metrizable space of large transfinite dimension. By Lemma 2.4, there is a compact subset K of X such that $\mathrm{Ind}\,H < \infty$ for every closed set H of X which does not meet K. For each $i \in N$, we put $H_i = X - S(K, 1/i)$, where

$S(K, 1/i) = \{x \in X: \varrho(x, K) < 1/i\}$ and ϱ is a metric on X. Since $\operatorname{Ind} H_i < \infty$, we can put $H_i = \bigcup \{H_{(i,j)}: j = 1, \ldots, k_i\}$ for some $k_i \in N$, where $\operatorname{Ind} H_{(i,j)} \leqslant 0$ for each $j = 1, \ldots, k_i$. Define a function g: $\{(i,j): j \leqslant k_i \text{ and } i \in N\} \to N$ as follows: $g(1,j) = j$ for $j \leqslant k_1$ and $g(i,j) = \sum_{m=1}^{i-1} k_m + j$ for $j \leqslant k_i$ and $i \geqslant 2$. Renumber $\{H_{(i,j)}: j \leqslant k_i \text{ and } i \in N\}$ as follows: $\{G_n: n \in N\}$, where $n = g(i,j)$. Let $K = \bigcup \{K_i: i \in N\}$, where $\operatorname{Ind} K_i \leqslant 0$ for each $i \in N$ and put $X_{2i-1} = K_i$ and $X_{2i} = G_i$. Let $\{(A_i, B_i): i \in N\}$ be a sequence of pairs of disjoint closed sets of X. By [12; III.4 (A)], there are closed sets L_i, $i \in N$, such that each L_i is a partition between A_i and B_i in X and

$$\operatorname{ord}\{L_i \cap X_n: i \in N\} \leqslant n-1 .$$

To show that $\mathscr{L} = \{L_i: i \in N\}$ is completely point finite, let F be a closed discrete subset of X. Put $F_1 = F \cap K$ and $F_2 = F - K$. Since K is compact, it is clear that $\sup\{\operatorname{ord}_x \mathscr{L}: x \in F_1\} < \infty$. On the other hand, since F_2 is a closed set which does not meet K, there is a natural number n_0 such that $F_2 \subset \bigcup \{X_n: n = 1, \ldots, n_0\}$. Therefore $\sup\{\operatorname{ord}_x \mathscr{L}: x \in F_2\} \leqslant n_0 - 1$. Hence $\sup\{\operatorname{ord}_x \mathscr{L}: x \in F\} < \infty$. This completes the proof of the implication (a) $\to$ (b).

The implication (b) $\to$ (c) is proved in the standard way. We prove it for the sake of completeness. Let $\mathscr{U} = \bigcup \{\mathscr{U}_i: i \in N\}$ be a σ-discrete base of X, where each $\mathscr{U}_i = \{U_\lambda: \lambda \in \Lambda_i\}$ is a discrete family of open sets of X. For each $\lambda \in \bigcup \{\Lambda_i: i \in N\}$, let $U_\lambda = \bigcup \{U_{\lambda,j}: j \in N\}$, where $U_{\lambda,j}$, $j \in N$, are open sets of X such that $\overline{U_{\lambda,j}} \subset U_{\lambda,j+1}$ for each $j \in N$. Put $A_{i,j} = \bigcup\{\overline{U_{\lambda,j}}: \lambda \in \Lambda_i\}$ and $B_{i,j} = X - \bigcup \{U_{\lambda,j+1}: \lambda \in \Lambda_i\}$. Since $A_{i,j}$ and $B_{i,j}$ are disjoint closed sets of X, by (b), there are closed sets $L_{i,j}$, $i, j \in N$, of X such that each $L_{i,j}$ is a partition between $A_{i,j}$ and $B_{i,j}$ in X and the family $\mathscr{L} = \{L_{i,j}: i, j \in N\}$ is completely point finite. Let $P_{i,j}$ and $Q_{i,j}$ be disjoint open sets of X such that $X - L_{i,j} = P_{i,j} \cup Q_{i,j}$, $A_{i,j} \subset P_{i,j}$ and $B_{i,j} \subset Q_{i,j}$. For each $i, j \in N$ and each $\lambda \in \Lambda_i$, we put $V_{\lambda,i,j} = P_{i,j} \cap U_\lambda$ and $\mathscr{V} = \{V_{\lambda,i,j}: \lambda \in \Lambda_i \text{ and } i, j \in N\}$. It is clear that $\mathscr{V}$ is a σ-discrete base for X. We shall show that the family $\partial\mathscr{V} = \{\operatorname{Bd} V: V \in \mathscr{V}\}$ is completely point finite. Since for each $V \in \mathscr{V}$, there is an $L(V) \in \mathscr{L}$ such that $\operatorname{Bd} V \subset L(V)$ and $\{V_{\lambda,i,j}: \lambda \in \Lambda_i\}$ is discrete in X, it follows that $\operatorname{ord}_x \partial\mathscr{V} \leqslant \operatorname{ord}_x \mathscr{L}$ for each point $x \in X$. Hence, $\mathscr{L}$ being completely point finite,

$$\sup\{\operatorname{ord}_x \partial\mathscr{V}: x \in F\} \leqslant \sup\{\operatorname{ord}_x \mathscr{L}: x \in F\} < \infty,$$

for every closed discrete subset F of X.

To show the implication (c) $\to$ (a), let $\mathscr{B}$ be a σ-discrete base for X such that the family $\partial\mathscr{B} = \{\operatorname{Bd} B: B \in \mathscr{B}\}$ is completely point finite. By Theorem A, it follows that X is countable-dimensional. Suppose that X contains a discrete family $\{U_n: n \in N\}$ of open sets of X such that $\operatorname{Ind} U_n \geqslant n$ for each $n \in N$. For each $n \in N$, we put $\mathscr{B}_n = \{(B \cap U_n): B \in \mathscr{B}\}$. Since $\mathscr{B}_n$ is a σ-discrete base for U_n, by Morita's theorem (see [12; Theorem II. 9]), there is a point x_n of U_n such that $\operatorname{ord}_{x_n}\{\operatorname{Bd}_{U_n}(B \cap U_n: B \in \mathscr{B}\} > n$, where $\operatorname{Bd}_{U_n}(B \cap U_n)$ denotes the boundary of $B \cap U_n$ in U_n. Since $\operatorname{Bd}_{U_n}(B \cap U_n) \subset \operatorname{Bd}_X B$ for each $B \in \mathscr{B}$, it follows that $\operatorname{ord}_{x_n} \partial\mathscr{B} > n$ for each $n \in N$. Put $F = \{x_n: n \in N\}$. Then F is a closed discrete subset of X satisfying $\sup\{\operatorname{ord}_x \partial\mathscr{B}: x \in F\} = \infty$. This contradicts the complete point finiteness of the family $\partial\mathscr{B}$. Hence, by Lemma 2.4, X has large transfinite dimension. This completes the proof.

We also have the following characterization of the same class. For a mapping f of a space Z to a space X and a point x of X, we denote by $\operatorname{ord}_x f$ the cardinality of $f^{-1}(x)$ and we write $\operatorname{ord} f = \sup\{\operatorname{ord}_x f: x \in X\}$.

2.6. Theorem. *A metrizable space X has large transfinite dimension if and only if there are a zero-dimensional metrizable space Z and a closed continuous mapping f of Z onto X such that for every closed discrete subset F of X $\sup\{\operatorname{ord}_x f: x \in F\} < \infty$.*

Proof. Let X be a metrizable space of large transfinite dimension. We use the notation introduced in the proof of the implication (a) $\to$ (b) of Theorem 2.5. By [12; III 7 (B)], there is a sequence $\{\mathscr{F}_i: i \in N\}$ of locally finite closed covers of X which satisfies the following conditions:

(2.1) For each $x \in X$ and each neighborhood U of x, there is a natural number i such that $\operatorname{St}(x, \mathscr{F}_i) \subset U$.

(2.2) $\mathscr{F}_i = \{F(\alpha_1, \ldots, \alpha_i): \alpha_k \in \Omega \text{ and } k = 1, \ldots, i\}$, where $F(\alpha_1, \ldots, \alpha_i)$ may be empty.

(2.3) $F(\alpha_1, \ldots, \alpha_i) = \bigcup \{F(\alpha_1, \ldots, \alpha_i, \beta): \beta \in \Omega\}$ for each $i \in N$.

(2.4) $\operatorname{ord}_x \mathscr{F}_i \leqslant n$ for each point $x \in X_n$ and each $i \in N$.

Let $N(\Omega)$ be the Baire's zero-dimensional space and put

$$Z = \{(\alpha_1, \alpha_2, \ldots) \in N(\Omega): \bigcap \{F(\alpha_1, \ldots, \alpha_i): i \in N\} \neq \varnothing\} \subset N(\Omega) .$$

Define a mapping f of Z to X as follows: $f(\alpha_1, \alpha_2, \ldots) = \bigcap \{F(\alpha_1, \ldots, \alpha_i): i \in N\}$. As shown in the proof of [12; Theorem VI. 4], f is a closed continuous finite-to-one mapping of Z onto X. Now, let F be a closed discrete subset of X. An argument similar to that used in the proof of the implication (a) $\to$ (b) of Theorem 2.5 shows that $\sup\{\operatorname{ord}_x f: x \in F\} < \infty$. This completes the proof of the "only if" part. On the other hand, by an argument similar to that used in the proof of the implication (c) $\to$ (a) of Theorem 2.5, we prove the "if" part of the theorem; but now we use [12; Theorem VI. 4] and [12; Theorem III. 8] instead of Theorem A and [12; Theorem II. 9] respectively.

3. Characterization of metrizable spaces which have strong large transfinite dimension.

The objective of this section is a characterization of the class of metrizable spaces which have strong large transfinite dimension. This class is narrower than the class considered in the previous section. We begin with definitions. For each ordinal number α, we write $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a nonnegative integer. For any normal space X and an nonnegative integer n, we put

$$P_n(X) = \bigcup \{U: U \text{ is an open set of } X \text{ such that } \operatorname{Ind} \overline{U} \leqslant n\} .$$

3.1. Definition ([2]). Let X be a normal space and let α be either an ordinal number $\geqslant 0$ or the integer -1. Then the *strong small transfinite dimension* sind of X is defined as follows:

(i) $\operatorname{sind} X = -1$ if and only if $X = \varnothing$.

(ii) $\operatorname{sind} X \leqslant \alpha$ if X is expressed in the form

$$X = \bigcup \{P_\xi\colon \xi < \alpha\}, \quad \text{where } P_\xi = P_{n(\xi)}(X - \bigcup \{P_\eta\colon \eta < \lambda(\xi)\})\,.$$

If there is an ordinal α such that $\operatorname{sind} X \leqslant \alpha$, we say that *$X$ has strong small transfinite dimension.*

3.2. Definition. We say that a normal space *has strong large transfinite dimension* if it has both large transfinite dimension and strong small transfinite dimension.

Spaces that have strong small transfinite dimension have been studied by P. Borst [2] and the author [6]. As noticed in [7], for a strongly hereditarily normal space X, X has strong small transfinite dimension if and only if $D(X) < \Delta$, where $D(X)$ is the D-dimension of X defined by D. W. Henderson [8]. Some characterizations of the class of metrizable spaces that have strong large transfinite dimension have been obtained by the author in [7]. We now give other characterizations of this class. A normal space X is called *strongly countable-dimensional* if X can be expressed as the union of countably many finite-dimensional (in the sense of dim) closed subsets of X.

3.3. Lemma ([7; Propositions 2.2 and 2.3]). *Let X be a metrizable space. Then the following conditions are equivalent:*

(a) *X has strong large transfinite dimension.*

(b) *X is a strongly countable-dimensional space satisfying the condition* (K).

(c) *X is a strongly countable-dimensional space which contains no discrete family $\{U_n\colon n \in N\}$ of open sets such that $\operatorname{Ind} U_n \geqslant n$ for each $n \in N$.*

The following lemma, which is a modification of Morita's theorem (see [12; Theorem III. 8]), can be easily verified.

3.4. Lemma. *A metrizable space X satisfies the inequality $\operatorname{Ind} X \leqslant n$ if and only if there are a zero-dimensional metrizable space Z and a closed continuous mapping f of Z onto X such that, for each $x \in X$, there is an open set U of Z with $U \cap f^{-1}(x) \neq \varnothing$ and $\operatorname{ord} f|U \leqslant n+1$.*

3.5. Theorem. *A metrizable space X has strong large transfinite dimension if and only if there are a zero-dimensional metrizable space Z and a closed continuous finite-to-one mapping f of Z onto X which satisfy the following condition:*

($*$) *For every closed discrete subset F of X, there are an open set U_F of Z and a natural number n_F such that $U_F \cap f^{-1}(x) \neq \varnothing$ for each $x \in F$ and $\operatorname{ord} f|U_F \leqslant n_F$.*

Proof. Let Z be a zero-dimensional metrizable space and f a closed continuous finite-to-one mapping of Z onto X, which satisfy the condition ($*$). By [14; Theorem 2], it follows that X is strongly countable-dimensional. Suppose that X contains a discrete family $\{U_n\colon n \in N\}$ of open sets such that $\operatorname{Ind} U_n \geqslant n$ for all $n \in N$. For each $n \in N$,

we put $f_n = f|f^{-1}(U_n)\colon f^{-1}(U_n) \to U_n$. By Lemma 3.4, there is a point $x_n \in U_n$ such that $\operatorname{ord} f_n|V_n > n+1$ for every open set V_n of $f^{-1}(U_n)$ with $V_n \cap f_n^{-1}(x_n) \neq \varnothing$. Put $F = \{x_n\colon n \in N\}$ and let V be an open set of Z with $V \cap f^{-1}(x) \neq \varnothing$ for any $x \in F$. For each $n \in N$, $V_n = V \cap f^{-1}(U_n)$ is an open of $f^{-1}(U_n)$ and $V_n \cap f_n^{-1}(x_n) \neq \varnothing$. Hence, $\operatorname{ord} f|V_n \geqslant \operatorname{ord} f_n|V_n > n+1$ for each $n \in N$. Therefore, F is a closed discrete subset of X such that $\operatorname{ord} f|V = \infty$. This is a contradiction. Therefore, by Lemma 3.3, X has strong large transfinite dimension. Conversely, let X be a metrizable space that has strong large transfinite dimension. By Lemma 3.3, there is a compact subset K of X such that $\operatorname{Ind} H < \infty$ for every closed set H of X which does not meet K. Let $K = \bigcup \{K_i\colon i \in N\}$, where each K_i is a finite-dimensional closed set of K such that $\operatorname{ord}\{K_i\colon i \in N\} \leqslant 2$. Put $H_1 = X - S(K, 1)$ and $H_i = \overline{S(K, 1/(i-1))} - S(K, 1/i)$ for $i \geqslant 2$. Furthermore, we put $X_{2i-1} = K_i$ and $X_{2i} = H_i$ for each $i \in N$. Then all X_i, $i \in N$, are closed finite-dimensional subsets of X, $X = \bigcup \{X_i\colon i \in N\}$ and $\operatorname{ord}\{X_i\colon i \in N\} \leqslant 2$. Let $\operatorname{Ind} X_i \leqslant n_i$. By [9; Theorem 12–9], there are locally finite closed covers $\mathscr{F}_i = \{F_\alpha\colon \alpha \in A_i\}$ of X, $i \in N$, and transformations $p_i^j\colon A_j \to A_i$, for $j > i$, which satisfy the following conditions:

(3.1) $\operatorname{mesh} \mathscr{F}_i \to 0$, as $i \to \infty$.

(3.2) $\operatorname{ord} \mathscr{F}_i|X_j \leqslant n_j + 1$ for each $i, j \in N$.

(3.3) For each $i, j \in N$ with $i < j$ and $\alpha \in A_i$, $F_\alpha = \bigcup \{F_\beta\colon p_i^j(\beta) = \alpha\}$.

For every $i \in N$, $\alpha \in A_i$ and $j \leqslant i$, we put $G_\alpha = F_\alpha$, $G_{(\alpha, j)} = F_\alpha \cap X_j$, $B_i = A_i \cup (A_i \times \{1, \ldots, i\})$ and $\mathscr{G}_i = \{G_\beta\colon \beta \in B_i\}$. For each $i, k \in N$ with $i < k$, we define a transformation $\pi_i^k\colon B_k \to B_i$ as follows: For any $\alpha \in A_k$,

$$\pi_i^k(\alpha) = p_i^k(\alpha)\,,$$

and

$$\pi_i^k((\alpha, j)) = \begin{cases} (p_i^k(\alpha), j) & \text{if } j \leqslant i\,, \\ p_i^k(\alpha) & \text{if } i < j \leqslant k\,. \end{cases}$$

Then, it follows that each $\mathscr{G}_i$ is a locally finite closed cover of X which satisfies the following conditions:

(3.4) $\operatorname{mesh} \mathscr{G}_i \to 0$, $i \to \infty$.

(3.5) For each $i, j \in N$, $\operatorname{ord} \mathscr{G}_i|X_j \leqslant 3(n_j + 1)$.

(3.6) For each $i \in N$, $\beta \in B_i$ and $k > i$, $G_\beta = \bigcup \{G_\gamma\colon \pi_i^k(\gamma) = \beta\}$.

Let Y be the inverse limit of the inverse sequence $\{B_i, \pi_i^k\}_{i,k \in N}$. Let $Z = \{(\beta_1, \beta_2, \ldots) \in Y\colon \bigcap \{G_{\beta_i}\colon i \in N\} \neq \varnothing\}$ be a subspace of Y. We define a mapping f of Z to X as follows: $f((\beta_1, \beta_2, \ldots)) = \bigcap \{G_{\beta_i}\colon i \in N\}$. It is well known that f is a closed continuous finite-to-one mapping of Z onto X (cf. [9; Theorem 12–9]). Furthermore, we shall show that

(3.7) for each $i \in N$ and $x \in X_i$, there is an open set V_x of Z such that $V_x \cap f^{-1}(x) \neq \varnothing$ and $\operatorname{ord} f|V_x \leqslant n_i + 1$.

To show (3.7), let $i \in N$ and $x \in X_i$. There is an $\alpha \in A_i$ such that $x \in F_\alpha$.

Then $x \in F_\alpha \cap X_i = G_{(\alpha, i)}$. Put $V_x = \pi_i^{-1}((\alpha, i)) \cap Z$, where $\pi_i\colon Y \to B_i$ is the canonical mapping. Then (3.7) follows from (3.2) and the definition of π_i^k. Now, let F be a closed discrete subset of X. Put $F_1 = F \cap K$ and $F_2 = F - K$. Put $U_1 = \bigcup \{V_x\colon x \in F_1\}$, where V_x is given in (3.7). Clearly, $U_1 \cap f^{-1}(x) \neq \emptyset$ for each $x \in F_1$, and since F_1 consists of at most finitely many points,

$$\operatorname{ord} f | U_1 \leqslant \sum_{x \in F_1} (n_{i(x)} + 1),$$

where $i(x)$ is a natural number with $x \in X_{i(x)}$. Since F_2 is a closed set of X which does not meet K, there is a natural number i_0 such that $F_2 \subset \bigcup \{H_i\colon i = 1, \ldots, i_0\}$. For each $i \leqslant i_0$, we put $F_2^i = F_2 \cap H_i = F_2 \cap X_{2i}$. Let $\{W_x\colon x \in F_2\}$ be a discrete family of open sets of X such that $x \in W_x$ for each $x \in F_2$. Let V_x be an open set of Z described in (3.7). We put $U_x = V_x \cap f^{-1}(W_x)$. Then U_x is an open set of Z such that $U_x \cap f^{-1}(x) \neq \emptyset$ and $\operatorname{ord} f | U_x \leqslant n_{2i} + 1$ for every point $x \in F_2^i$. Hence $U_2^i = \bigcup \{U_x\colon x \in F_2^i\}$ satisfies $U_2^i \cap f^{-1}(x) \neq \emptyset$ for each $x \in F_2^i$ and $\operatorname{ord} f | U_2^i \leqslant n_{2i} + 1$. Therefore, the open set $U_F = U_1 \cup \bigcup \{U_2^i\colon i = 1, \ldots, i_0\}$ of Z and the natural number $n_F = \sum_{x \in F_1} (n_{i(x)} + 1) + \sum_{i=1}^{i_0} (n_{2i} + 1)$ are the desired ones. This completes the proof.

By the argument of A. Arhangel'skii ([1; Theorem 3.7]), it follows that a metrizable space X is strongly countable-dimensional if and only if there is an integer-valued function φ on X such that for every open cover $\mathscr{U}$ of X there is a σ-discrete open cover $\mathscr{V}$ such that $\mathscr{V}$ is a refinement of $\mathscr{U}$ and $\operatorname{ord}_x \mathscr{V} \leqslant \varphi(x)$ for each $x \in X$. (The reader should be warned that the term "weakly countable-dimensional" is used instead of "strongly countable-dimensional" in [1].) Now, we give a similar characterization of the class of metrizable spaces that have strong large transfinite dimension. The following lemma is a direct consequence of [12; V 3 (A)].

3.6. Lemma. *Let X be a metrizable space and F a closed subset of X with* $\operatorname{Ind} F \leqslant n$. *Then for every open cover $\mathscr{U}$ of X there is a family $\mathscr{V}$ of open sets of X such that $\mathscr{V}$ is a refinement of $\mathscr{U}$ ($\mathscr{V}$ is not necessarily a cover of X), $F \subset \mathscr{V}^{\#}$ and $\mathscr{V}$ is the union of at most $(n+1)$ discrete subfamilies.*

3.7. Theorem. *A metrizable space X has strong large transfinite dimension if and only if there is an integer-valued function φ on X such that, for every open cover $\mathscr{U}$ of X, there is an open cover $\mathscr{V}$ of X such that $\mathscr{V}$ is a refinement of $\mathscr{U}$, $\mathscr{V}$ is the union of finitely many discrete subfamilies and* $\operatorname{ord}_x \mathscr{V} \leqslant \varphi(x)$ *for each* $x \in X$.

Proof. Let X be a metrizable space of strong large transfinite dimension. By Lemma 3.3, there is a compact subset K of X such that $\operatorname{Ind} H < \infty$ for every closed subset H of X which does not meet K. Let $K = \bigcup \{K_i\colon i \in N\}$, where each K_i is a closed set of K with $\operatorname{Ind} K_i \leqslant m_i < \infty$. For each $i \in N$, let $H_i = X - S(K, 1/i)$ and $\operatorname{Ind} H_i \leqslant n_i < \infty$. For each $x \in X$, we put

$$i(x) = \begin{cases} \min\{i\colon x \in K_i\} & \text{if } x \in K, \\ \min\{i\colon x \in H_i\} & \text{if } x \notin K. \end{cases}$$

We define an integer-valued function φ of X as follows:

$$\varphi(x) = \begin{cases} \sum_{i=1}^{i(x)} (m_i + 1) & \text{if } x \in K, \\ \sum_{i=1}^{i(x)} (m_i + n_i + 2) & \text{if } x \notin K. \end{cases}$$

Let $\mathscr{U}$ be an open cover of X. By use of Lemma 3.6, we can find sequences $\{\mathscr{B}_i\colon i \in N\}$ and $\{\mathscr{W}_i\colon i \in N\}$ of families of open sets of X such that, for each $i \in N$,

(3.8) $K_i - \bigcup \{K_j\colon j \leqslant i-1\} \subset \mathscr{B}_i^{\#} \subset S(K, 1/i)$,

(3.9) $H_i - \bigcup \{H_j\colon j \leqslant i-1\} \subset \mathscr{W}_i^{\#} \subset X - \overline{S(K, 1/i)}$,

(3.10) $\mathscr{B}_i^{\#} \cap (\bigcup \{K_j\colon j \leqslant i-1\}) = \emptyset$,

(3.11) $\mathscr{W}_i^{\#} \cap (\bigcup \{H_j\colon j \leqslant i-1\}) = \emptyset$,

(3.12) $\mathscr{B}_i$ and $\mathscr{W}_i$ are refinements of $\mathscr{U}$ and are the unions of finitely many discrete subfamilies.

Since K is compact and $\bigcup \{\mathscr{B}_i\colon i \in N\}$ covers K, there is a finite subfamily $\mathscr{V}_1$ of $\bigcup \{\mathscr{B}_i\colon i \in N\}$ such that $K \subset \mathscr{V}_1^{\#}$. Put $H = X - \mathscr{V}_1^{\#}$ and let i_0 be a natural number such that $H \subset H_{i_0}$. Let $\mathscr{V} = \mathscr{V}_1 \cup \bigcup \{\mathscr{W}_i\colon i = 1, \ldots, i_0\}$. Clearly, $\mathscr{V}$ is an open cover of X, which is the union of finitely many discrete subfamilies. On the other hand, it is easy to see that $\operatorname{ord}_x \mathscr{V} \leqslant \operatorname{ord}_x(\bigcup \{\mathscr{B}_i\colon i \in N\} \cup \bigcup \{\mathscr{W}_i\colon i \in N\}) \leqslant \varphi(x)$ (cf. the proof of [7; Theorem 3.12]). Conversely, let φ be an integer-valued function of X which satisfies the condition stated in the theorem. By [1; Theorem 3.7], X is a strongly countable-dimensional space. Suppose that X contains a discrete family $\{U_n\colon n \in N\}$ of open sets such that $\operatorname{Ind} U_n \geqslant n$ for each $n \in N$. Since $\operatorname{Ind} \overline{U}_n \geqslant n$, there is a family $\mathscr{U}_n$ of open sets of X such that $\overline{U}_n \subset \mathscr{U}_n^{\#}$ and $\operatorname{ord} \mathscr{V}_n > n$ for every open cover $\mathscr{V}_n$ of $\overline{U}_n$ which refines $\mathscr{U}_n | \overline{U}_n$. We can assume that $\mathscr{U}_m^{\#} \cap \mathscr{U}_n^{\#} = \emptyset$ if $m \neq n$. Put $\mathscr{U} = \bigcup \{\mathscr{U}_n\colon n \in N\} \cup \{X - (\bigcup \{\overline{U}_n\colon n \in N\})\}$ and let $\mathscr{V}$ be an open cover of X which refines $\mathscr{U}$. For each $n \in N$, $\mathscr{V} | \overline{U}_n$ is an open cover of $\overline{U}_n$ which refines $\mathscr{U}_n$. Thus $\operatorname{ord} \mathscr{V} | \overline{U}_n > n$ and hence $\mathscr{V}$ cannot be expressed as a union of finitely many discrete subfamilies. This is a contradiction. Therefore, by Lemma 3.3, X has strong large transfinite dimension. This completes the proof.

3.8. Remark. By the proof of Theorem 3.7, we have the following: A metrizable space X satisfies the condition (K) if and only if every open cover of X has an open refinement which is the union of finitely many discrete subfamilies.

References

[1] A. V. Arhangel'skii, *The ranks of systems of sets and dimension of spaces*, Fund. Math. 52 (1963), 257–275 (in Russian).

[2] P. Borst, *Infinite-dimension theory, Part II*, manuscript, (1981).

[3] R. Engelking, *Dimension Theory*, North-Holland, Amsterdam 1978.

[4] —, *Transfinite dimension*, Surveys in General Topology, Academic Press, New York 1980, 131–161.

[5] R. Engelking and R. Pol, *Some characterizations of spaces that have transfinite dimension*, Topology Appl. 15 (1983), 247–253.
[6] Y. Hattori, *On spaces related to strongly countable dimensional spaces*, Math. Japonica 28 (1983), 583–593.
[7] —, *Characterizations of certain classes of infinite dimensional metrizable spaces*, Topology Appl. 20 (1985), 97–106.
[8] D. W. Henderson, *D-dimension*, I, Pacific J. Math. 26 (1968), 91–107.
[9] K. Nagami, *Dimension Theory*, Academic Press, New York 1970.
[10] K. Nagami and J. H. Roberts, *A note on countable-dimensional metric spaces*, Proc. Japan Acad. 41 (1965), 155–158.
[11] J. Nagata, *On the countable sum of zero-dimensional metric spaces*, Fund. Math. 48 (1960), 1–14.
[12] —, *Modern Dimension Theory, revised and extended edition*, Heldermann Verlag, Berlin 1983.
[13] E. Pol, *The Baire-category method in some compact extension problems*, Pacific J. Math. 122 (1986) 197–210.
[14] J. W. Walker and B. R. Wenner, *Characterizations of certain classes of infinite-dimensional metric spaces*, Topology Appl. 12 (1981), 101–104.

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Received 24 June 1985

FUNDAMENTA
MATHEMATICAE
128 (1987)

On the relationships between shape properties of subcompacta of S^n and homotopy properties of their complements

by

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Abstract. Taking for the set of morphisms from X to Y the direct limit of the sets of all homotopy classes or all weak homotopy classes or all shapping between the n-fold suspension of X and Y we obtain (respectively) the stable homotopy category $\mathcal{S}$ or the stable weak homotopy category $\mathcal{S}w\mathcal{O}_n$ of open subsets of S^n or the stable shape category $\mathcal{S}\mathcal{S}h_n$ of subcompacta of S^n.

We prove that there exists an isomorphism $\mathcal{D}_n\colon \mathcal{S}\mathcal{S}h_n \to \mathcal{S}w\mathcal{O}_n$ such that $\mathcal{D}_n(X) = S^n\setminus X$. If we limit ourselves to movable compacta, then $\mathcal{S}w\mathcal{O}_n$ can be replaced by a suitable full subcategory of $\mathcal{S}$. These facts generalize the classical Spanier–Whitehead duality.

Applications to the ordinary shape theory are also given. In particular, if $1 < k \leqslant n$ and $X \subset S^n$ is an approximatively 1-connected continuum, then $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$ iff $S^n\setminus X$ and $S^n\setminus S^k$ are isomorphic in $\mathcal{S}$.

The relationships between shape properties of closed subsets of S^n and properties of their complements have been studied by many mathematicians ([Sh]). If $X, Y \subset S^n$ are compacta with sufficiently large codimension and X, Y satisfy some conditions concerning the way in which they are embedded in S^n, then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ iff $S^n\setminus X$ and $S^n\setminus Y$ are homeomorphic ([Sh]). In the case when $1 \neq k \leqslant n \geqslant 5$ and $Y = S^k$ (see [R]), or more generally, Y is an S^k-like continuum (see [V]), the assumption concerning the codimension of X and Y may be eliminated.

We begin with examples. They will illustrate and motivate some of the problems which will be discussed here.

The Alexander duality theorem states that the Čech cohomology groups (which belong to the most important invariants of the shape theory) of a closed subset X of S^n are uniquely determinated by the topological (homotopical) type of $S^n\setminus X$. Since the second suspension $\Sigma^2(X)$ of a compactum $X \neq \emptyset$ is an approximatively 1-connected continuum, $\mathrm{Fd}(\Sigma^2(X)) = \mathrm{Fd}(\Sigma^2(Y))$ if X and Y are subcontinua of S^n with homotopically equivalent complements ([N], p. 35).

Similarly, if $\Sigma^2(X) \in \mathrm{FANR}$ and $S^n\setminus X$ is homotopically equivalent to $S^n\setminus Y$ [G–L], then $\Sigma^2(Y) \in \mathrm{FANR}$.

On the other hand, if $S^3\setminus K$ is a 3-cube with a knotted channel joining two