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On generalized topological divisors of zero

by

W. ŻELAZKO (Warszawa)

Abstract. The main result of this paper states that a real topological algebra with unit element either possesses generalized topological divisors of zero or the group of its invertible elements is isomorphic to one of the following (multiplicative) groups: $\boldsymbol{R}\setminus\{0\}$, $\boldsymbol{C}\setminus\{0\}$, $\boldsymbol{Q}\setminus\{0\}$.

§ 1. Introduction. A *topological algebra* is a (nonzero) topological linear space together with an associative jointly continuous multiplication making of it an algebra over complex or real scalars. Denote by $\Phi(A)$ a basis of neighbourhoods of the origin in a topological algebra A. The joint continuity of the multiplication in A means that for each U in $\Phi(A)$ there is a neighbourhood V in $\Phi(A)$ such that

(1) $$V^2 \subset U.$$

A topological algebra is said to be a *locally convex algebra* if its underlying topological linear space is a locally convex space. A B_0-*algebra* is a completely metrizable locally convex algebra. A locally convex algebra A is said to be *locally multiplicatively convex* (shortly m-convex) if its topology can be given by means of a family $(\|x\|_\alpha)$ of submultiplicative seminorms, i.e. seminorms satisfying

(2) $$\|xy\|_\alpha \leqslant \|x\|_\alpha \|y\|_\alpha$$

for all x and y in A and all indices α. It is clear that relations (2) imply the joint continuity of the multiplication in A. Similarly, a *locally pseudoconvex algebra* is a topological algebra which is a locally pseudoconvex space. A *locally pseudoconvex space* X is a topological linear space whose topology is given by means of a family $(\|x\|_\alpha)$ of pseudonorms satisfying

(i) $\|x\|_\alpha \geqslant 0$, and if $\|x\|_\alpha = 0$ for all indices α then $x = 0$,

(ii) $\|x+y\|_\alpha \leqslant \|x\|_\alpha + \|y\|_\alpha$,

(iii) $\|\lambda x\|_\alpha = |\lambda|^{p_\alpha} \|x\|_\alpha$, where p_α is fixed with $0 < p_\alpha \leqslant 1$,

for all x, y in X, all scalars λ and all indices α.

A locally pseudoconvex algebra is said to be *m-pseudoconvex* if its topology can be given by means of a family of pseudonorms satisfying (i)–(iii) and (2). One can easily see that the completion of a locally pseudoconvex algebra is again such an algebra and the same holds true for m-pseudo-convex algebras.

A pair S_1 and S_2 of nonvoid subsets of a topological algebra A is said to be *generalized topological divisors of zero* if $0 \notin \bar{S}_1 \cup \bar{S}_2$ but $0 \in \overline{S_1 S_2}$. Here $S_1 S_2 = \{xy: x \in S_1, y \in S_2\}$ and $\bar{S}$ denotes the closure of S in A. If one of the sets S_i consists of a single nonzero element x, this element is said to be a (left or right) *topological divisor of zero*. In other words, a topological algebra A has generalized topological divisors of zero if there is a neighbourhood $U \in \Phi(A)$ and two nets (x_i), (y_i), $i \in I$, of elements of A such that $x_i \notin U$ and $y_i \notin U$ for all i in I, or for all sufficiently large i, and $\lim_I x_i y_i = 0$. Similarly, a nonzero element x in A is a *left* (*right*) *topological divisor of zero* if there is a U in $\Phi(A)$ and a net (z_i), $i \in I$, of elements of A such that $z_i \notin U$ for all i in I and $\lim_I xz_i = 0$ ($\lim_I z_i x = 0$). It is clear that if a topological algebra A has generalized topological divisors of zero, then these divisors can be chosen in any dense subset of A. In particular, a topological algebra has generalized topological divisors of zero if and only if its completion has such divisors.

Shilov's generalization of the Gelfand–Mazur theorem states that a real Banach algebra either possesses (two-sided) topological divisors of zero or is isomorphic to one of the three finite-dimensional division algebras over the reals: the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers, the division algebra $\boldsymbol{Q}$ of quaternions. A similar result fails for an arbitrary topological algebra and so in ([5], Definition 8.8) the concept of generalized topological divisors of zero was introduced in the hope that it would provide a proper notion for generalization of the above-mentioned result of Shilov. We shall need the following result which was proved in the same paper ([5], Theorem 8.9).

1.1. THEOREM. *Let A be a real topological algebra which is a division algebra. Then either A possesses generalized topological divisors of zero or it is isomorphic to one of the three finite-dimensional division algebras over $\boldsymbol{R}$: $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$.*

The above result also holds for a complex topological algebra A, except that then the algebras $\boldsymbol{R}$ and $\boldsymbol{Q}$ are excluded.

It was conjectured in ([5], Conjecture 8.10) that the above theorem is true for an arbitrary topological algebra A. The conjecture was proved in [6] for complex m-convex algebras and in [7] for real m-convex algebras. However, this conjecture fails in general: in [1], a commutative complex infinite-dimensional B_0-algebra without generalized topological divisors of zero was constructed. Nevertheless, some weaker form of this conjecture can be proved: a topological algebra with unit element either has generalized topological divisors of zero or its group of invertible elements is isomorphic to $\boldsymbol{C}\setminus\{0\}$ in the complex case and to one of the three groups $\boldsymbol{R}\setminus\{0\}$, $\boldsymbol{C}\setminus\{0\}$, $\boldsymbol{Q}\setminus\{0\}$ in the real case. Thus the triviality of an algebra has to be replaced by the triviality of the group of its invertible elements. Of course this is a weaker result—the triviality of an algebra excludes the existence of generalized topological divisors of zero while the triviality of the group of invertible elements does not (cf. Example 2.5 below). The aim of this paper is to prove this and related results. It extends the results obtained in [10], where it was shown that if a complex locally convex algebra A with unit element has no generalized topological divisors of zero then the only invertible elements in A are the scalar multiples of the unit. Here we give a positive answer to two problems posed in [10]: Problem 2, whether the same result holds true for an arbitrary complex topological algebra, and Problem 1, whether the nonexistence of generalized topological divisors of zero implies in the real case the triviality of the group of invertible elements. As corollaries we obtain new proofs of the main results of [6] and [7] in the more general setting of an m-pseudoconvex algebra (Corollaries 2.8, 2.11, 2.13). For more information on the above the reader is referred to [1], [3], [5]–[10].

§ 2. The results. Let A be a topological algebra over the field of scalars K ($K = \boldsymbol{R}$ or $K = \boldsymbol{C}$) and suppose that A has unit element denoted by e. Denote by $G(A)$ the multiplicative group of invertible elements in A and define the *spectrum* of an element x in A by

$$\sigma^K(x) = \{\lambda \in K: x - \lambda e \notin G(A)\};$$

if $K = \boldsymbol{C}$ we write $\sigma(x)$ instead of $\sigma^{\boldsymbol{C}}(x)$.

Note the following relations:

$$\sigma^K(x+\lambda e) = \sigma^K(x) + \lambda \quad (= \{\mu+\lambda: \mu \in \sigma^K(x)\}) \tag{3}$$

for all x in A and all λ in K, and

$$\sigma^K(x^{-1}) = [\sigma^K(x)]^{-1} \quad (= \{\lambda^{-1}: \lambda \in \sigma^K(x)\}) \tag{4}$$

for all x in $G(A)$.

The *spectral radius* $\varrho_K(x)$ of an element x in A is defined by

$$\varrho_K(x) = \sup\{|\lambda|: \lambda \in \sigma^K(x)\} \quad (\varrho_K(x) = -\infty \text{ if } \sigma^K(x) = \emptyset).$$

Call an element x in A a *constant element* if $x = \lambda e$, $\lambda \in K$.

For the sake of completeness we prove the following lemma (it follows immediately from Proposition 9.5 in [8]).

2.1. LEMMA. *Let A be a complete topological algebra with unit element e and suppose that A has no generalized topological divisors of zero. Then for each nonconstant element x in A its spectrum $\sigma^K(x)$ is an open subset of K.*

Proof. We have to show that if for some nonconstant element x_0 in A its spectrum is nonvoid and nonopen then A has generalized topological divisors of zero. So suppose that there is a scalar λ_0 in $\sigma^K(x_0)$ and a sequence of scalars (λ_n) with $\lambda_n \notin \sigma^K(x_0)$, $n = 1, 2, \ldots$, and with $\lambda_0 = \lim \lambda_n$. Consider the sequence $y_n = (x_0 - \lambda_n e)^{-1}$. First we show that this sequence cannot be bounded. Otherwise for each U in $\Phi(A)$ there is a positive ε_U such that $\lambda y_n \in U$ for all n and all λ with $|\lambda| < \varepsilon_U$. This implies that the sequence (y_n) is a Cauchy sequence in A. In fact, let U be a given element in $\Phi(A)$ and choose a V in $\Phi(A)$ which together with U satisfies relation (1). We can find an integer k_V so that for $m, n \geqslant k_V$, $|\lambda_m - \lambda_n|^{1/2} < \varepsilon_V$. We have now for $m, n \geqslant k_V$

$$\begin{aligned} y_n - y_m &= (x_0 - \lambda_n e)^{-1} - (x_0 - \lambda_m e)^{-1} \\ &= (\lambda_m - \lambda_n)^{1/2} (x_0 - \lambda_n e)^{-1} (\lambda_m - \lambda_n)^{1/2} (x_0 - \lambda_m e)^{-1} \in V^2 \subset U, \end{aligned}$$

with an arbitrary choice of $(\lambda_m - \lambda_n)^{1/2}$, and this means that (y_n) is a Cauchy sequence. Since A is complete there exists $y = \lim y_n$ in A. We have

$$e = y_n(x_0 - \lambda_n e) = \lim y_n \lim (x_0 - \lambda_n e) = y(x_0 - \lambda_0 e)$$

and

$$e = (x_0 - \lambda_0 e) y.$$

Thus $y = (x_0 - \lambda_0 e)^{-1}$ and so $\lambda_0 \notin \sigma^K(x_0)$. This is a contradiction proving that the sequence (y_n) is unbounded. But then there is a neighbourhood U_0 in A such that for each natural n there is a natural k_n with $n^{-1} y_{k_n} \notin U_0$. Since the element x_0 is nonconstant we have $x_0 - \lambda_0 e \neq 0$ and we can assume $x_0 - \lambda_n e \notin U_0$ for large n, say $n \geqslant n_0$. Put $S_1 = S_2 = A \setminus U_0$, so that $0 \notin \bar{S}_1 \cup \bar{S}_2$. On the other hand, we have $n^{-1} y_{k_n} \in S_1$ and $(x_0 - \lambda_{k_n} e) \in S_2$ for $k_n \geqslant n_0$ and

$$n^{-1} e = n^{-1} y_{k_n} (x_0 - \lambda_{k_n} e) \in S_1 S_2.$$

Thus $0 \in \overline{S_1 S_2}$ and A has generalized topological divisors of zero. The conclusion follows.

Let A be a topological algebra with unit e and let x be an element of A. Denote by A_x the subalgebra of A generated by x, i.e. the closure in A of the set of all elements of the form $p = \sum_{i=0}^{n} \alpha_i x^i$, $\alpha_i \in K$, where $x^0 = e$.

2.2. Lemma. *Let A be a complete topological algebra with unit e and suppose that A has no generalized topological divisors of zero. Let x be a nonconstant element of A. Then for each λ in $\sigma^K(x)$ there is a K-valued continuous multiplicative linear functional f on A_x with $f(x) = \lambda$.*

Proof. Put $y = x - \lambda e$ and write any polynomial p in x in the form

$$p = \sum_{i=0}^{n} \alpha_i y^i, \tag{5}$$

$\alpha_i \in K$. First we show that the coefficients α_i are uniquely determined by the

element p of the form (5). In fact, if p has two different representations of the form (5), then there are coefficients β_i in K, $\sum |\beta_i| > 0$, with $q(y) = \sum_{i=0}^{k} \beta_i y^i = 0$, i.e. y is an algebraic element in A. If $K = \mathbf{C}$ we can write

$$q(y) = \lambda_0 (y - \lambda_1) \ldots (y - \lambda_k) = 0$$

and since A has no divisors of zero it follows that $y = \lambda_i e$ for some i, $1 \leqslant i \leqslant k$. This is impossible because x, and so y, is a nonconstant element in A. If $K = \mathbf{R}$ we can decompose

$$q(y) = q_1(y) \ldots q_m(y),$$

where q_i is a polynomial with real coefficients of degree at most two. As before, $q_i(y) = 0$ for some i and if q_i is of degree one we deal as before. Otherwise $y^2 = \alpha y + \beta e$ for some real α and β. If $\beta \neq 0$ we have $y(\beta^{-1} y - \beta^{-1} \alpha e) = e$ and so $y \in G(A)$. But formula (3) implies $0 \in \sigma^{\mathbf{R}}(y)$ and so y is a noninvertible element in A. Thus $\beta = 0$ and $y(y - \alpha e) = 0$, which implies that y is a constant element. The contradiction proves the uniqueness of the coefficients in (5).

We put now $f(p) = \alpha_0$. This is a multiplicative linear functional on the algebra of all polynomials of the form (5). We shall be done if we show that it is a continuous functional, since then we extend it by continuity onto the whole of A_x and the extension is again a continuous multiplicative linear functional. Suppose then that f is discontinuous on the set of all elements of the form (5) and try to get a contradiction. Thus for each U in $\Phi(A)$ we can find an element p_U of the form (5) with $p_U \in U$ and $f(p_U) = 1$. We can write

$$p_U = e - (x - \lambda e) q_U, \tag{6}$$

with a suitable q_U of the form (5). Consider the net (q_U), $U \in \Phi(A)$, where $\Phi(A)$ is ordered by inclusion. It either converges or diverges in A. If the net (q_U) converges in A, say to an element z, then formula (6) implies $e = (x - \lambda e) z = z(x - \lambda e)$ and so $\lambda \notin \sigma^K(x)$, which is a contradiction. Thus the net (q_U) diverges and since A is complete it is not a Cauchy net. This means that there is a neighbourhood U_0 in $\Phi(A)$ such that for each U in $\Phi(A)$ there is a $V(U) \in \Phi(A)$ with $V(U) \subset U$ and

$$r_U = q_{V(U)} - q_U \notin U_0. \tag{7}$$

By (6) we have

$$p_U - p_{V(U)} = (x - \lambda e) r_U.$$

But $p_U \in U$ and $V(U) \subset U$ and this means that $\lim_{\Phi(A)} p_U = \lim_{\Phi(A)} p_{V(U)} = 0$, so that the above relation implies

$$\lim_{\Phi(A)} (x - \lambda e) r_U = 0. \tag{8}$$

Relations (7) and (8) mean that $x-\lambda e$ is a topological divisor of zero in A. The contradiction proves that the functional f is continuous on A_x and the conclusion follows.

2.3. LEMMA. *Let A be a complete topological algebra with unit element. Suppose that A has no generalized topological divisors of zero. Then for each nonconstant element x in A we have either $\sigma^K(x) = \emptyset$ or $\sigma^K(x) = K$.*

Proof. Let x be a nonconstant element in A. Suppose that $\emptyset \neq \sigma^K(x) \neq K$ and try to get a contradiction. By Lemma 2.1 the set $\sigma^K(x)$ is a nonvoid open subset of K and its boundary $\partial\sigma^K(x)$ is nonvoid. Choose any λ in $\partial\sigma^K(x)$; we have $\lambda \notin \sigma^K(x)$. Put $y = (x-\lambda e)^{-1}$. By (3) and (4) we have $\varrho_K(y) = \infty$ and the spectrum $\sigma^K(y)$ is nonvoid and different from K. Again there is a λ_1 in $\partial\sigma^K(y)$ so that $\lambda_1 \notin \sigma^K(y)$. We put $z = y-\lambda_1 e$. By (3) we have $\varrho_K(z) = \infty$ while by (4), $\varrho_K(z^{-1}) = \infty$. Thus there are scalars α and β with $\alpha \in \sigma^K(z^{-1})$, $\beta \in \sigma^K(z)$, such that $|\alpha| \geqslant 1$, $|\beta| \geqslant 2$. By Lemma 2.2 there are continuous multiplicative linear functionals f_1 on $A_{z^{-1}}$ with $f_1(z^{-1}) = \alpha$ and f_2 on A_z with $f_2(z) = \beta$. Thus

(9) $$|f_1(z^{-n})| = |\alpha|^n \geqslant 1,$$

(10) $$f_2[(z/2)]^n = |\beta/2|^n \geqslant 1$$

for all natural n. Since the functionals f_1 and f_2 are continuous, there is a neighbourhood U in $\Phi(A)$ such that

(11) $$U \cap A_{z^{-1}} \subset \{x \in A_{z^{-1}}\colon\ |f_1(x)| < 1\},$$

(12) $$U \cap A_z \subset \{x \in A_z\colon\ |f_2(x)| < 1\}.$$

We put $S_1 = \{z^{-n}\colon\ n \in \boldsymbol{N}\}$, $S_2 = \{(z/2)^n\colon\ n \in \boldsymbol{N}\}$. By (11) and (12) we have $S_1 \cap U = \emptyset$ and $S_2 \cap U = \emptyset$ so that $0 \notin \bar{S}_1 \cup \bar{S}_2$. On the other hand we have

$$2^{-n}e = z^{-n}(z/2)^n \in S_1 S_2$$

and so $0 \in \overline{S_1 S_2}$. Thus A has generalized topological divisors of zero. The contradiction proves the lemma.

The following result gives a positive answer to Problem 2 of [10], mentioned in the introduction.

2.4. THEOREM. *Let A be a complex topological algebra with unit element and suppose that A has no generalized topological divisors of zero. Then every invertible element in A is a scalar multiple of the unit element.*

We do not give here a proof of this theorem since it follows immediately from a more general result given in Theorem 2.9 below.

The converse of the above result does not hold true. We give an example of a commutative complex B_0-algebra with unit element whose only invertible elements are scalar multiples of the unit and which has divisors of zero.

2.5. EXAMPLE. Let A_0 be the algebra of all entire functions x such that

$$|x|_n = \sup_{\lambda \in \boldsymbol{C}} |x(\lambda)\exp(-|\lambda|^{1/2}/n)| < \infty$$

for $n = 1, 2, \ldots$ Under the above seminorms and pointwise algebra operations it is a commutative complex B_0-algebra whose only invertible elements are the nonzero constant functions (cf. [4], § 3)(¹). Define

$$A = \{(x_1, x_2) \in A_0 \times A_0\colon\ x_1(0) = x_2(0)\}.$$

One can easily see that under the coordinatewise operations and the seminorms $\|x_1, x_2)\|_n = \max(|x_1|_n, |x_2|_n)$ it is a commutative B_0-algebra with unit element whose only invertible elements are the nonzero constant elements. For $z(\lambda) \equiv \lambda$ we have $z \in A_0$ and so the elements $(z, 0)$ and $(0, z)$ are in A. Since $(z, 0)(0, z) = (0, 0)$, the zero element of A, the algebra A has divisors of zero.

Theorem 2.4 immediately implies the following

2.6. COROLLARY. *Let A be as in Theorem 2.4. Then for each nonconstant element x in A we have*

$$\sigma(x) = \boldsymbol{C},$$

in particular the spectrum of an element of a complex topological algebra without generalized topological divisors of zero is never void.

We say that an entire function $\varphi(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *operates on an element* x of a topological algebra A if the series $\sum_{n=0}^{\infty} \alpha_n x^n$ converges in A. This also makes sense for a real algebra if the function φ has real Taylor coefficients. In the following result we limit ourselves to pseudoconvex algebras and we do not know whether it can be extended to arbitrary topological algebras.

2.7. PROPOSITION. *Let A be a complex pseudoconvex algebra with unit element. If the exponential function $\exp\lambda = \sum_{n=0}^{\infty} \lambda^n/n!$ operates on some nonconstant element x in A, then A has generalized topological divisors of zero.*

Proof. Suppose that A has no generalized topological divisors of zero and try to get a contradiction. Without loss of generality we can assume that A is a complete algebra. Let $(\|x\|_\alpha)$ be a family of pseudonorms defining the topology of A. Since the series $\sum_{n=0}^{\infty} x^n/n!$ converges in A, we have $\lim \|x^n/n!\|_\alpha = 0$ for all indices α. This implies $\|x^n/n!\|_\alpha \leqslant M_\alpha$ for some constant M_α and

(¹) As A_0 we can also take the algebra constructed in [1]; it consists of entire functions, contains all polynomials, and by Theorem 2.4 its only invertible elements are nonzero scalar multiples of the unit element.

all n, and so

$$\|2^{-n}x^n/n!\|_\alpha \leqslant M_\alpha 2^{-p_\alpha n}$$

for $n = 1, 2, \ldots$ Since $1 < 2^{p_\alpha} \leqslant 2$, we see that $\sum \|2^{-n}x^n/n!\|_\alpha < \infty$ for all indices α, which means that the series $\exp(x/2)$ and $\exp(-x/2)$ are absolutely convergent in A. A routine calculation shows that the sum of one series is the inverse of the other. Since x is a nonconstant element, Corollary 2.6 implies that $\sigma(x) = \boldsymbol{C}$. By Lemma 2.2 we can see that the set $\exp \boldsymbol{C} = \boldsymbol{C}\backslash\{0\}$ is the spectrum of $\exp(x/2)$ in A_x. Thus $\exp(x/2)$ is a nonconstant invertible element in A_x and so in A. Together with Theorem 2.4 this gives the desired contradiction. The conclusion follows.

Relations (2) show that if A is a complete m-pseudoconvex algebra then all entire functions operate on all elements of A. Together with the fact that a topological algebra has generalized topological divisors of zero if and only if its completion has such divisors, this gives the following corollary to Proposition 2.7:

2.8. Corollary. *If A is a complex m-pseudoconvex algebra with unit element and $A \neq \boldsymbol{C}$, then A has generalized topological divisors of zero.*

The following result gives a positive answer to Problem 1 in [10]; it is the main result of this paper.

2.9. Theorem. *Let A be a real topological algebra with unit element e. Suppose that A has no generalized topological divisors of zero. Then the group $G(A)$ of all invertible elements in A is isomorphic either to $\boldsymbol{R}\backslash\{0\}$ or to $\boldsymbol{C}\backslash\{0\}$ or to $\boldsymbol{Q}\backslash\{0\}$.*

Proof. Without loss of generality we can assume A to be complete. First we show that $A_0 = G(A) \cup \{0\}$ is a subalgebra of A. Since the product of two elements in A_0 is again in A_0 and all constant elements are in A_0 it is sufficient to show that $x, y \in A_0$ implies $x + y \in A_0$. This is clearly true if either x or y is zero. If $x \neq 0 \neq y$ then $x + y = x(e + x^{-1}y)$ and if $x^{-1}y$ is nonconstant Lemma 2.3 implies $\sigma^{\boldsymbol{R}}(x^{-1}y) = \emptyset$ since it is an invertible element. This implies $e + x^{-1}y \in G(A)$ and so $x + y = x(e + x^{-1}y) \in G(A) \subset A_0$. If $x^{-1}y = \lambda e$, then $x + y = (\lambda + 1)x$ and also belongs to A_0. But now Theorem 1.1 implies that A_0 is isomorphic to either $\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{Q}$. The conclusion follows.

As is shown in Example 2.5 the triviality of $G(A)$ does not imply the nonexistence of generalized topological divisors of zero.

2.10. Proposition. *Let A be a real pseudoconvex algebra with unit element and suppose that the exponential function $\exp \lambda$ operates on some nonconstant element x with a nonvoid spectrum $\sigma^{\boldsymbol{R}}(x)$. Then A has generalized topological divisors of zero.*

Proof. Suppose that A has no generalized topological divisors of zero. By Lemma 2.3 we have $\sigma^{\boldsymbol{R}}(x) = \boldsymbol{R}$ and then we proceed in exactly the same way as in the proof of Proposition 2.7 obtaining a contradiction which proves our conclusion.

Since all entire functions with real Taylor coefficients operate on real complete m-pseudoconvex algebras, and since such an algebra either is a division algebra or has a nonconstant element with a nonvoid spectrum, using Theorem 1.1, just as in Corollary 2.8 we obtain the following.

2.11. Corollary. *Let A be a real m-pseudoconvex algebra with unit element. Then either A has generalized topological divisors of zero or it is isomorphic to one of the three finite-dimensional division algebras: $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$.*

We shall now consider algebras without unit elements.

Let A be a topological algebra over K ($K = \boldsymbol{C}$ or $K = \boldsymbol{R}$) without unit element. The *unitization* A_1 of A is the direct sum $A_1 = A \oplus Ke$, where Ke is the one-dimensional linear space over K spanned by a vector e. A_1 becomes an algebra over K with unit element e if we define there the multiplication by setting

$$(x + \lambda e)(y + \mu e) = xy + \lambda y + \mu x + \lambda\mu e,$$

where $x, y \in A$, , $\mu \in K$. It is a topological algebra containing topologically A if we define there a basis of neighbourhoods of the origin setting

$$\tilde{U} = N(U, \varepsilon) = \{x + \lambda e \in A_1 \colon x \in U, |\lambda| < \varepsilon\}, \tag{13}$$

where $U \in \Phi(A)$ and $0 < \varepsilon \leqslant 1$. In fact, for a neighbourhood $N(U, \varepsilon)$ of the form (13) we can find V and V_1 in $\Phi(A)$ so that $V_1 + V_1 + V_1 \subset U$, $V \subset V_1$, and $V^2 \subset V_1$. Without loss of generality we can also assume $\lambda V \subset V$ for all λ in K with $|\lambda| \leqslant 1$. Now for $x + \lambda e, y + \mu e \in N(V, \varepsilon)$ we have $x, y \in V$ and so xy, μx, λy are in V_1, hence $xy + \mu x + \lambda y \in U$, while $|\lambda\mu| < \varepsilon^2 \leqslant \varepsilon$. Thus $N(V, \varepsilon)^2 \subset N(U, \varepsilon)$ and relations (1) are satisfied if we set $\Phi(A_1) = \{N(U, \varepsilon) \colon U \in \Phi(A), 0 < \varepsilon \leqslant 1\}$.

The following proposition extends Lemma 3.6 of [10] to the more general setting of an arbitrary topological algebra.

2.12. Proposition. *Let A be a topological algebra without unit. If A has no generalized topological divisors of zero, then its unitization A_1 also has no such divisors.*

Proof. Without loss of generality we can assume that A, and so A_1, is a complete algebra. Suppose that the algebra A_1 has generalized topological divisors of zero S_1 and S_2. By the assumption there is a neighbourhood $\tilde{U}_0 = N(U_0, \varepsilon_0)$ with $S_1 \cap \tilde{U}_0 = \emptyset$ and $S_2 \cap \tilde{U}_0 = \emptyset$ such that for each $\tilde{U}$ in $\Phi(A_1)$ there are elements $x_{\tilde{U}} - \lambda_{\tilde{U}} e \in S_1$ and $y_{\tilde{U}} - \mu_{\tilde{U}} e \in S_2$ with

$$(x_{\tilde{U}} - \lambda_{\tilde{U}} e)(y_{\tilde{U}} - \mu_{\tilde{U}} e) \in \tilde{U}. \tag{14}$$

Without loss of generality we can assume $|\lambda_{\tilde{U}}| \leqslant 1$ and $|\mu_{\tilde{U}}| \leqslant 1$, for whenever $|\lambda_{\tilde{U}}| > 1$ we can replace $x_{\tilde{U}} - \lambda_{\tilde{U}} e$ by $|\lambda_{\tilde{U}}|^{-1}(x_{\tilde{U}} - \lambda_{\tilde{U}} e)$. The new element lies outside $\tilde{U}_0$ and relation (14) is again satisfied since we can assume $\lambda\tilde{U} \subset \tilde{U}$ for $|\lambda| \leqslant 1$. Thus the nets $(\lambda_{\tilde{U}})$, $(\mu_{\tilde{U}})$, $\tilde{U} \in \Phi(A_1)$, are bounded, and passing if necessary to finer nets $x_i - \lambda_i e$ and $y_i - \mu_i e$, $i \in I$, we can assume that the limits $\lambda = \lim_I \lambda_i$ and $\mu = \lim_I \mu_i$ exist. Relations (14) imply now

$$\lim_I (x_i - \lambda_i e)(y_i - \mu_i e) = 0, \tag{15}$$

so that also $\lim \lambda_i \mu_i = 0$, which, in turn, implies either $\lambda = 0$ or $\mu = 0$. By symmetry we can assume $\lambda = 0$.

We shall now show that for each u in A, $u \neq 0$, there is a neighbourhood $U(u)$ in $\Phi(A)$ and an index $i(u)$ in I with

$$u(x_i - \lambda_i e) \notin U(u). \tag{16}$$

for all $i \geqslant i(u)$. To prove this suppose that (16) fails to be true. Thus there is a nonzero element u_0 in A such that for each U in $\Phi(A)$ and each index i in I there is an index j_i in I, $j_i \geqslant i$, with $u_0(x_{j_i} - \lambda_{j_i} e) \in U$. This implies $\lim_I u_0(x_{j_i} - \lambda_{j_i} e) = 0$. But $\lim_I \lambda_{j_i} = 0$ and so $\lim_I \lambda_{j_i} u_0 = 0$ which implies

$$\lim_I u_0 x_{j_i} = \lim u_0(x_{j_i} - \lambda_{j_i} e) + \lim \lambda_{j_i} u_0 = 0. \tag{17}$$

On the other hand, $x_{j_i} - \lambda_{j_i} e \notin \tilde{U}_0 = N(U_0, \varepsilon_0)$ and since for large i we have $|\lambda_{j_i}| < \varepsilon_0$, it follows that $x_{j_i} \notin U_0$ for, say, $i \geqslant i_0$. This together with (17) implies that u_0 is a left-topological divisor of zero in A. The contradiction proves formula (16).

We shall now show that there are an element v_0 in A and a neighbourhood V in $\Phi(A)$ such that for each i in I there is a j_i in I with $j_i \geqslant i$ and

$$(y_{j_i} - \mu_{j_i} e) v_0 \notin V \tag{18}$$

for all i in I. If $\mu = 0$ we proceed as in the proof of (16). Consider then the case $\mu \neq 0$ and suppose that relation (18) fails to be true. This means that

$$\lim_I (y_i - \mu_i e) v = 0$$

for all v in A, which, in turn, implies

$$\lim_I \mu^{-1} y_i v = v \tag{19}$$

for all v. Suppose first that the net $(\mu^{-1} y_i)$, $i \in I$, converges in A to some element u. Relation (19) implies $uv = v$ for all v in A, in particular $u^2 = u$ and $u \neq 0$. We cannot have $vu = v$ for all v in A, because then u would be a unit in A and there is none. Thus $wu \neq w$ for some w in A and setting $x = wu - w$ we have a nonzero element with $xu = w(u^2 - u) = 0$. Thus A has divisors of zero which is nonsense. This proves that the net $(\mu^{-1} y_i)$ diverges in A, and since A is complete it is not a Cauchy net. Thus there exists a neighbourhood V_1 in $\Phi(A)$ such that for each i in I there is a $k_i \in I$, $k_i > i$, with

$$r_i = \mu^{-1}(y_{k_i} - y_i) \notin V_1. \tag{20}$$

Relation (19) implies now

$$\lim_I r_i v = 0$$

for all v in A, which together with (20) shows that each nonzero element in A is a right topological divisor of zero. The contradiction proves formula (18).

We put $p_i = (y_{j_i} - \mu_{j_i} e) v_0$, where (j_i) and v_0 satisfy (18). We have $p_i \notin V$ for all i. Choose any nonzero element u in A and put $q_i = u(x_{j_i} - \lambda_{j_i}) e$. Relations (16) show that $q_i \notin U(u)$ for $i \geqslant i(u)$. Relation (15) implies now

$$\lim_I q_i p_i = \lim_I u(x_{j_i} - \lambda_{j_i} e)(y_{j_i} - \mu_{j_i} e) v_0 = 0$$

and so A has generalized topological divisors of zero. The contradiction proves the proposition.

The above proposition together with Corollaries 2.8 and 2.11 immediately implies the following corollary which generalizes the main results in [6] and [7].

2.13. Corollary. *Let A be an m-pseudoconvex algebra without generalized topological divisors of zero. If A is a complex algebra then A is isomorphic to $\boldsymbol{C}$, if it is a real algebra then it is isomorphic to one of the algebras $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$.*

Let A be an algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$ and suppose that A has no unit element. An element x in A is said to be a *quasi-inverse* of an element y in A if $e+x$ is the inverse of $e+y$ in the unitization A_1 of A. This implies in particular that x and y commute.

2.14. Proposition. *Let A be a real or complex topological algebra without unit element and suppose that A has no generalized topological divisors of zero. Then the only quasi-invertible element in A is the zero element.*

Proof. If A is a complex algebra the conclusion follows immediately from Theorem 2.4 and Proposition 2.12. If A is a real algebra and x is a quasi-invertible element in A, then, by Theorem 2.9, the element x lies in a subalgebra of the unitization A_1 of A isomorphic to a subalgebra of $\boldsymbol{Q}$ and so x is an algebraic element over $\boldsymbol{R}$ of order at most two. If $x^2 = \alpha x + \beta e$, $\alpha, \beta \in \boldsymbol{R}$, and $\beta \neq 0$, then the unit e of A_1 is in A, which is impossible. Thus $x(x - \alpha e) = 0$ and since $x \neq \alpha e$ we have $x = 0$. The conclusion follows.

§ 3. Final remarks and open problems. We do not know any example of a noncommutative complex topological algebra without generalized topological divisors of zero and we conjecture that the answer to the following problem is positive:

PROBLEM 1. Let A be a complex noncommutative topological algebra. Does it follow that A has generalized topological divisors of zero?

This problem can be answered in the negative in the case of a real algebra. The counterexample is the Banach algebra Q. A positive answer to the following problem would imply a positive solution of Problem 1.

PROBLEM 2. Suppose that A is a complex topological algebra with the property that for arbitrary nets (x_i), (y_i), $i \in I$, of elements of A the condition $\lim_I x_i y_i = 0$ implies $\lim_I y_i x_i = 0$. Does it follow that A is a commutative algebra?

The positive answer to Problem 2 would give a generalization of the following result due to Le Page [2]: If A is a complex Banach algebra and there is a positive constant k such that $\|xy\| \leqslant k\|yx\|$ for all x and y in A then the algebra A is commutative. Using a technique similar to that of [2] one can obtain a positive solution to Problem 2 in the case when A is an m-pseudoconvex algebra.

PROBLEM 3. Suppose that a topological algebra A has generalized topological divisors of zero. Does there exist a commutative subalgebra of A also possessing generalized topological divisors of zero?

References

[1] H. Arizmendi and W. Żelazko, *A B_0-algebra without generalized topological divisors of zero*, Studia Math. 82 (1985), 191–198.
[2] C. Le Page, *Sur quelques conditions entraînant la commutativité dans les algèbres de Banach*, C. R. Acad. Sci. Paris 265 (1967), 235–237.
[3] E. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).
[4] B. S. Mityagin, S. Rolewicz and W. Żelazko, *Entire functions in B_0-algebras*, Studia Math. 21 (1962), 291–306.
[5] W. Żelazko, *Metric generalizations of Banach algebras*, Rozprawy Mat. (Dissertationes Math.) 47 (1965).
[6] —, *On generalized topological divisors of zero in m-convex locally convex algebras*, Studia Math. 28 (1966), 9–16.
[7] —, *On generalized topological divisors of zero in real m-convex algebras*, ibid. 28 (1967), 231–234.
[8] —, *Selected Topics in Topological Algebras*, Aarhus University Lecture Notes 31, 1971.
[9] —, *Banach Algebras*, Elsevier, Amsterdam 1973.
[10] —, *Concerning locally convex algebras without generalized topological divisors of zero*, Funct. Approx. Comment. Math. (in print).

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa, Poland

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Near isometries of spaces of weak* continuous functions, with an application to Bochner spaces

by

MICHAEL CAMBERN (Santa Barbara, Cal.)

Abstract. For a Banach dual E and a compact Hausdorff space X we denote by $C(X, E_{\sigma^*})$ the Banach space of continuous functions F from X to E when the latter space is provided with its weak* topology, normed by $\|F\|_\infty = \sup_{x \in X} \|F(x)\|$. Here we show that if X and Y are extremally disconnected compact Hausdorff spaces and E is a uniformly convex Banach space with $C(X, E_{\sigma^*})$ and $C(Y, E_{\sigma^*})$ nearly isometric, then X and Y are homeomorphic. The result has the following immediate consequence for Bochner spaces. If $(\Omega_i, \Sigma_i, \mu_i)$ are σ-finite measure spaces, $i = 1, 2$, and E a uniformly smooth Banach space such that $L^1(\mu_1, E)$ and $L^1(\mu_2, E)$ are nearly isometric or that $L^\infty(\mu_1, E^*)$ and $L^\infty(\mu_2, E^*)$ are nearly isometric, then $L^1(\mu_1, E)$ is isometric to $L^1(\mu_2, E)$ and $L^\infty(\mu_1, E^*)$ is isometric to $L^\infty(\mu_2, E^*)$.

0. Introduction. Throughout this paper the letter E stands for a Banach space, while X and Y denote compact Hausdorff spaces. U denotes the closed unit ball in E and S the surface of U. Interaction between elements of a Banach space and those of its dual will be denoted by $\langle \cdot, \cdot \rangle$. We will write $E_1 \cong E_2$ to indicate that the Banach spaces E_1 and E_2 are isometric.

Given X, assume that E is a Banach dual. Then $C(X, E_{\sigma^*})$ stands for the Banach space of continuous functions F on X to E when the latter space is provided with its weak* topology, normed by $\|F\|_\infty = \sup_{x \in X} \|F(x)\|$.

If (Ω, Σ, μ) is a positive measure space and E is any Banach space then, for $1 \leqslant p \leqslant \infty$, the Bochner spaces $L^p(\Omega, \Sigma, \mu, E)$ will be denoted by $L^p(\mu, E)$ when there is no danger of confusing the underlying measure spaces involved. For the definitions and properties of these spaces we refer to [10].

Following Banach [1, p. 242] we will call the Banach spaces E_1 and E_2 *nearly isometric* if $1 = \inf\{\|T\| \|T^{-1}\|\}$, where T runs through all isomorphisms of E_1 onto E_2. It is of course equivalent to suppose that $1 = \inf\{\|T\|\}$, where $\|T^{-1}\| = 1$ and hence T is a norm-increasing isomorphism of E_1 onto E_2. For if T is any continuous isomorphism of one Banach space onto

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