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Spectral dilation of operator-valued measures and its application to infinite-dimensional harmonizable processes

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Abstract. The paper deals with the study of spectral dilation of operator-valued measures and its application to infinite-dimensional harmonizable processes. A necessary and sufficient condition for an operator-valued measure to be dilatable is given. A counterexample is constructed showing that in general an operator-valued measure need not admit a spectral dilation. For some special cases of interest more verifiable sufficient conditions are given. Applications to infinite-dimensional harmonizable processes are studied.

1. Introduction. To state the main objective of this paper let us briefly review the problem that led us to the study of this subject. To start let H and K be complex Hilbert spaces and let T be an additive function defined on an algebra Σ of subsets of an arbitrary set Ω with values in the space of continuous linear operators from H to K. We will be concerned with the problem of dilation of T in the form

$$T(\Delta) = SE(\Delta)R, \quad \Delta \in \Sigma, \tag{1.1}$$

where R and S are continuous linear operators from H to $\boldsymbol{K}$ and from $\boldsymbol{K}$ to K respectively, $\boldsymbol{K}$ is a Hilbert space and E is a spectral measure in $\boldsymbol{K}$.

The classical Naimark theorem (see [8], [17] for general theory of dilation of nonnegative operator-valued functions) states that for $H = K$ and $T(\Delta) \geqslant 0$, $\Delta \in \Sigma$, T admits a dilation (1.1). On the other hand, Niemi [19] proved that if $H = \boldsymbol{C}$ then each $K = L(\boldsymbol{C}, K)$-valued measure has the form (1.1). A vector-valued approach rather than a linear functional approach was used in [16] to prove the same result for the case $H = \boldsymbol{C}$. Later, Chatterji [3] noted that the existence of a dilation (1.1) is an algebraic property and that Niemi's result remains true for any bounded additive vector-valued function. In all these cases the notion of 2-majorizability plays an important role.

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In 1982, Rosenberg [26] stated the problem of existence of a dilation (1.1) for an operator-valued measure. He introduced the notion of 2-majorant of an operator-valued measure. Among his important results he showed that a weakly countably additive operator-valued measure admits a dilation (1.1) if and only if the measure has a 2-majorant. He also proved that if K or H is finite-dimensional then each weakly countably additive operator-valued function has the representation (1.1). Rosenberg asks the question whether the dilation (1.1) is valid in general.

The main purpose of this paper is to study the question of dilation of weakly countably additive operator-valued measure raised by Rosenberg [26]. Our study includes two types of results: one pertains to the necessary and sufficient conditions for the existence of a dilation. It also treats situations where only sufficient conditions are given (see Sections 4 and 5). The second deals with a negative result demonstrating that not all operator-valued measures are dilatable (Section 3). In applying the dilation results to harmonizable operator-valued processes we find that the standard notion of harmonizability must be replaced by a strengthened version which leads to fruitful results in the theory of harmonizable operator-valued processes (see Section 6).

Throughout the paper $\boldsymbol{N}$, $\boldsymbol{C}$ and $\boldsymbol{R}$ will stand for positive integers, complex numbers and real numbers, respectively. H, K, $\boldsymbol{K}$ will denote complex Hilbert spaces with inner product $(\cdot,\cdot)$. If X is a normed linear space, then X^* will denote its conjugate space and $\langle x, x^*\rangle$ will stand for the value of the functional $x^*\in X^*$ at the point $x\in X$. Recall that the mapping $H\ni y\to \bar{y}(\cdot)=(\cdot, y)\in H^*$ is an antilinear isometry from H onto H^*. By $L(X, Y)$ we will denote the set of all bounded linear operators between normed linear spaces X and Y equipped with the operator norm. If $X = Y$ then we will abbreviate $L(X, X)$ by $L(X)$. $L^+(H)$ will stand for the set of all positive operators in H, i.e., the set of all $T\in L(H)$ such that $(Tx, x)\geqslant 0$ for all $x\in H$. The direct sum of H and K will be denoted by $H\oplus K$, H^q will denote the direct sum of q copies of H, $1\leqslant q<\infty$, $l_q^2 = \boldsymbol{C}^q$. Operators from H^q into K^p will be identified with matrices with entries in $L(H, K)$. In particular, every operator

$$A\in L(l_p^2, l_q^2)$$

will be identified with its $q\times p$ complex matrix with respect to the standard bases. Unless otherwise stated, Σ will be an algebra of subsets of an arbitrary set Ω. Any additive function m from Σ into a normed space X will be referred to as a *finitely additive* (f.a.) *X-valued measure*. If Σ is a σ-algebra and m is countably additive then m will be called a *measure*. An $L(H, K)$-valued f.a. measure T defined on a σ-algebra Σ is said to be *weakly countably additive* (w.c.a.) if $(T(\cdot)x, y)$ is a complex measure for all $x\in H$ and $y\in K$. 1_Δ will denote the indicator of a set Δ. $\Pi(\Omega)$ will stand for the set of all finite Σ-partitions of a set Ω; the elements of $\Pi(\Omega)$ will be denoted by π. If Σ is a σ-algebra and X is a Banach space then by $B(X)$ we will denote the Banach space of all bounded Σ-measurable functions on Ω with values in X equipped with the norm

$$|f|_{\sup} = \sup\{|f(\omega)|:\ \omega\in\Omega\},\qquad f\in B(X).$$

2. Preliminaries. Let $(X, |\ |)$ be a normed linear space and let $m: \Sigma\to X$ be a f.a. measure defined on Σ, an algebra of subsets of a set Ω.

2.1. Definition. The number $\|m\|\in[0, \infty]$ defined by the formula

$$\|m\| = \sup\Big\{\Big|\sum_{\Delta\in\pi} m(\Delta)t_\Delta\Big|:\ t_\Delta\in \boldsymbol{C},\ |t_\Delta|\leqslant 1,\ \pi\in\Pi(\Omega)\Big\}$$

is said to be the *semivariation* of m.

The space of all X-valued f.a. measures on (Ω, Σ) with finite semivariation will be denoted by $\mathcal{M}(X)$. Clearly $(\mathcal{M}(X), \|\ \|)$ is a normed space (see [5], p. 53). In fact, if X is complete this becomes a Banach space.

Let $S(X)$ denote the set of all Σ-simple X-valued functions of the form

$$f = \sum_{j=1}^n 1_{\Delta_j}x_j,\qquad \text{where } \Delta_j\in\Sigma,\ x_j\in X,\ j = 1,\ldots, n.$$

For every $m\in\mathcal{M}(X^*)$ and $f = \sum_{j=1}^n 1_{\Delta_j}x_j\in S(X)$ define

$$\int_\Omega \langle f, dm\rangle = \sum_{j=1}^n \langle x_j, m(\Delta_j)\rangle \tag{2.2}$$

(cf. [5], Section 7). For every $f\in S(X)$ let

$$|f|_\infty = \sup\Big\{\Big|\int\langle f, dm\rangle\Big|:\ m\in\mathcal{M}(X^*),\ \|m\|\leqslant 1\Big\}. \tag{2.3}$$

2.4. Lemma. *Let X be a normed linear space. Then*

(a) *$(S(X), |\ |_\infty)$ is a normed space.*

(b) *The mapping*

$$\mathcal{M}(X^*)\ni m\to \int\langle\cdot, dm\rangle\in (S(X), |\ |_\infty)^*$$

is an antilinear isometry from $(\mathcal{M}(X^*), \|\ \|)$ onto $(S(X), |\ |_\infty)^*$.

Proof. (a) follows immediately from the definition of $|\ |_\infty$ and from the fact that $|f|_\infty<\infty$ for all $f\in S(X)$. Indeed, let $f\in S(X)$. Since $\int\langle f, dm\rangle$ does not depend on a representation of f, one can assume that $f = \sum_{j=1}^n 1_{\Delta_j}x_j$, where $\Delta_j = \{\omega : f(\omega) = x_j\}$ and $x_j \neq x_i$ provided $i \neq j$, $i, j = 1, \ldots, n$. Then

$$\Big|\int \langle f, dm\rangle\Big| = \Big|\sum_{j=1}^{n} \langle x_j, m(\Delta_j)\rangle\Big| \leqslant \sum_{j=1}^{n} |x_j| \cdot |m(\Delta_j)| \leqslant n\|m\| \sup\{|f(\omega)|: \omega \in \Omega\}$$

for all $m \in \mathcal{M}(X^*)$, so $|f|_\infty < \infty$.

(b) First observe that if $f = \sum_{k=1}^{n} 1_{\Delta_k} x t_k$, $t_k \in C$, $x \in X$, and $\Delta_1, \ldots, \Delta_n \in \Sigma$ are disjoint, then

$$\Big|\int \langle f, dm\rangle\Big| \leqslant |x| \Big|\sum_{k=1}^{n} m(\Delta_k) \bar{t}_k\Big| \leqslant |x| \|m\| \sup\{|t_k|: 1 \leqslant k \leqslant n\}$$

and so

$$|f|_\infty \leqslant |x| \sup\{|t_k|: 1 \leqslant k \leqslant n\}. \tag{2.5}$$

Let $m \in \mathcal{M}(X^*)$, $\|m\| \leqslant 1$. Then $\langle f, m\rangle \overset{\mathrm{df}}{=} \int \langle f, dm\rangle$, $f \in S(X)$, is a continuous linear functional on $(S(X), |\ |_\infty)$, and moreover

$$|\langle f, m\rangle| = \|m\| \Big|\int \langle f, d(m/\|m\|)\rangle\Big| \leqslant \|m\| |f|_\infty, \quad f \in S(X). \tag{2.6}$$

Conversely, let ν be a continuous linear functional on $S(X)$ and $\Delta \in \Sigma$ be fixed. Since by (2.5)

$$|\langle 1_\Delta x, \nu\rangle| \leqslant |\nu|_{S(X)^*} |1_\Delta x|_\infty \leqslant |\nu|_{S(X)^*} |x|, \quad x \in X,$$

there exists $m(\Delta) \in X^*$ such that

$$\langle 1_\Delta x, \nu\rangle = \langle x, m(\Delta)\rangle \quad \text{for all } x \in X.$$

Clearly $m(\cdot)$ is a f.a. measure on Σ and moreover from (2.5) it follows that

$$\begin{aligned}\|m\| &= \sup\Big\{\Big|\Big\langle x, \sum_{\Delta \in \pi} m(\Delta) \bar{t}_\Delta\Big\rangle\Big|: \pi \in \Pi(\Omega), |t_\Delta| \leqslant 1, |x| \leqslant 1\Big\} \\ &= \sup\Big\{\Big|\int \Big\langle \sum_{\Delta \in \pi} 1_\Delta x t_\Delta, dm\Big\rangle\Big|: \pi \in \Pi(\Omega), |t_\Delta| \leqslant 1, |x| \leqslant 1\Big\} \\ &\leqslant \sup\Big\{\Big|\int \langle f, dm\rangle\Big|: |f|_\infty \leqslant 1\Big\} = \sup\{|\langle f, \nu\rangle|: |f|_\infty \leqslant 1\} \\ &= |\nu|_{S(X)^*} < \infty.\end{aligned}$$

Thus $m \in \mathcal{M}(X^*)$, and by (2.6), $\|m\| = |\nu|_{S(X)^*}$. ∎

2.7. **Remark.** It is easy to see that for every $f \in S(X)$

$$\begin{aligned}|f|_{\sup} &\overset{\mathrm{df}}{=} \sup\{|f(\omega)|: \omega \in \Omega\} \\ &= \sup\Big\{\Big|\int \langle f, d\delta_\omega x^*\rangle\Big|: \omega \in \Omega, |x^*| \leqslant 1\Big\} \\ &\leqslant |f|_\infty,\end{aligned}$$

where $\delta_\omega(\Delta) = 1$ if $\omega \in \Delta$ and 0 otherwise. If X is a finite-dimensional space then both norms are equivalent. In fact, if $X = C^p$ then

$$\begin{aligned}\Big|\int (f, d\mu)\Big| &\leqslant \sum_{j=1}^{p} \Big|\int (f, e_j)(d\mu, e_j)\Big| \leqslant |f|_{\sup} \sum_{j=1}^{p} \|(\mu, e_j)\| \\ &\leqslant |f|_{\sup}\, p, \quad \text{provided } \|\mu\| \leqslant 1.\end{aligned}$$

Thus

$$|f|_\infty \leqslant p\,|f|_{\sup}.$$

The example below shows that if X is an infinite-dimensional space, then these norms need not be equivalent.

2.8. Example. Let $X = l^2$, $\Omega = N$, $\Sigma = 2^N$ and let

$$f_n = \sum_{k=1}^{n} 1_{\{k\}} \frac{e_k}{k^{1/3}}, \quad n \in N,$$

where $\{e_k: k = 1, 2, \ldots\}$ is the standard basis in l^2. Let μ denote the l^2-valued measure on Σ with $\|\mu\| = 1$ defined by the formula

$$\mu\{k\} = C \frac{e_k}{k^{2/3}}, \quad k = 1, 2, \ldots,$$

where $C = \big(\sum_{k=1}^{\infty} k^{-4/3}\big)^{-1/2}$. The sequence $\{f_n: n = 1, 2, \ldots\}$ is a Cauchy sequence in the sup-norm:

$$|f_{n+j} - f_n|_{\sup} = \Big|\sum_{k=n+1}^{n+j} 1_{\{k\}} \frac{e_k}{k^{1/3}}\Big|_{\sup} = \frac{1}{(n+1)^{1/3}} \to 0$$

as $n \to \infty$, but

$$|f_n|_\infty \geqslant \Big|\int (f_n, d\mu)\Big| = \Big|\sum_{k=1}^{n} (f_n(k), \mu\{k\})\Big| = C \sum_{k=1}^{n} \frac{1}{k} \to \infty.$$

In fact, the measure μ defined above belongs to $\mathcal{M}(l^2)$ but it is not a continuous functional on $(S(l^2), |\ |_{\sup})$, so

$$(S(l^2), |\ |_{\sup})^* = (B(l^2), |\ |_{\sup})^* \subsetneqq \mathcal{M}(l^2). \ ∎$$

Unless otherwise stated, $S(X)$ is assumed to be equipped with the $|\ |_\infty$-norm.

Now suppose that T is an $L(H, K)$-valued f.a. measure defined on Σ, where H and K are Hilbert spaces and $L(H, K)$ denotes the Banach space of all bounded linear operators from H into K equipped with the usual operator norm.

2.9. LEMMA. *Let H and K be Hilbert spaces and let T be an $L(H, K)$-valued f.a. measure defined on Σ. Then*

(a)
$$\begin{aligned}\|T\| &= \sup\{\|T(\cdot)x\|:\ x\in H,\ |x|\leqslant 1\}\\ &= \sup\{\|T(\cdot)^* y\|:\ y\in K,\ |y|\leqslant 1\}\\ &\leqslant 4\sup\{|T(\Delta)|:\ \Delta\in\Sigma\}\end{aligned}$$

where $\|T\|$, $\|T(\cdot)x\|$ and $\|T(\cdot)^* y\|$ denote the semivariations of T and of the K- and H-valued set functions $T(\cdot)x$ and $T(\cdot)^* y$, respectively.

(b) *If moreover Σ is a σ-algebra and T is w.c.a. then*

$$\sup\{|T(\Delta)|:\ \Delta\in\Sigma\}<\infty.$$

Proof. First we note that

$$\begin{aligned}\|T\| &= \sup\Big\{\Big|\sum_{\Delta\in\pi} T(\Delta)t_\Delta\Big|:\ \pi\in\Pi(\Omega),\ |t_\Delta|\leqslant 1,\ t_\Delta\in \mathbf{C}\Big\}\\ &= \sup\Big\{\Big|\sum_{\Delta\in\pi}(T(\Delta)x, y)t_\Delta\Big|:\ \pi\in\Pi(\Omega),\ |t_\Delta|\leqslant 1,\ |x|\leqslant 1,\\ &\qquad\qquad |y|\leqslant 1,\ t_\Delta\in\mathbf{C},\ x\in H,\ y\in K\Big\}\\ &= \sup\Big\{\sup\Big\{\Big|\sum_{\Delta\in\pi}(T(\Delta)x, y)t_\Delta\Big|:\ \pi\in\Pi(\Omega),\ |t_\Delta|\leqslant 1\Big\}:\ |x|\leqslant 1, |y|\leqslant 1\Big\}\\ &= \sup\{\|(T(\cdot)x, y)\|:\ |x|\leqslant 1,\ |y|\leqslant 1\}.\end{aligned}$$

Thus

$$\begin{aligned}\|T\| &= \sup\{\sup\{\|(T(\cdot)x, y)\|:\ |y|\leqslant 1\}:\ |x|\leqslant 1\}\\ &= \sup\{\|T(\cdot)x\|:\ |x|\leqslant 1\},\quad\text{and}\\ \|T\| &= \sup\{\sup\{\|(x, T(\cdot)^* y)\|:\ |x|\leqslant 1\}:\ |y|\leqslant 1\}\\ &= \sup\{\|T(\cdot)^* y\|:\ |y|\leqslant 1\}.\end{aligned}$$

Moreover, by [6], III 1.4, Lemma 5,

$$\begin{aligned}\|T\| &= \sup\{\|(T(\cdot)x, y)\|:\ |x|\leqslant 1, |y|\leqslant 1\}\\ &\leqslant 4\sup\{|(T(\Delta)x, y)|:\ |x|\leqslant 1, |y|\leqslant 1, \Delta\in\Sigma\}\\ &\leqslant 4\sup\{|T(\Delta)|:\ \Delta\in\Sigma\}.\end{aligned}$$

If T is a w.c.a. measure defined on a σ-algebra Σ then

$$\sup\{|(T(\Delta)x, y)|:\ \Delta\in\Sigma\}=C(x, y)<\infty$$

for all $x\in H$ and $y\in K$ ([6], III 4.4). Applying the Banach–Steinhaus Theorem ([28], Th. 2.5) to the family of linear functionals $(T(\Delta)x, \cdot)$ and then once again using it to the family of bounded linear operators $T(\Delta)$, $\Delta\in\Sigma$, we conclude that $\sup\{|T(\Delta)|:\ \Delta\in\Sigma\}<\infty$. ∎

2.10. DEFINITION. Let T be a f.a. $L(H, K)$-valued measure defined on Σ. For each $f\in S(H)$, $f=\sum_{k=1}^n 1_{\Delta_k} y_k$, we denote

$$\int dTf=\sum_{k=1}^n T(\Delta_k)y_k\qquad\text{(cf. [5], Section 7).}$$

The linear operator Φ_T: $S(H)\to K$ defined by the formula

$$\Phi_T(f)=\int dTf,\qquad f\in S(H),$$

will be referred to as the *operator associated with* T.

2.11. LEMMA. *Let T be a f.a. $L(H, K)$-valued measure on Σ. Then the operator Φ_T from $(S(H), |\ |_\infty)$ into K is continuous if and only if $\|T\|<\infty$. Moreover, $|\Phi_T|=\|T\|$. In particular,*

$$\Big|\int dTf\Big|\leqslant|f|_\infty\|T\|\qquad\textit{for all } f\in S(H).$$

Proof. Let $f=\sum 1_{\Delta_k}x_k\in S(H)$. Then by Lemma 2.9

$$\begin{aligned}|\Phi_T(f)| &= \Big|\sum T(\Delta_k)x_k\Big| = \sup\Big\{\Big|\Big(\sum T(\Delta_k)x_k, y\Big)\Big|:\ |y|\leqslant 1\Big\}\\ &= \sup\Big\{\Big|\int(f, dT^* y)\Big|:\ |y|\leqslant 1\Big\}\\ &\leqslant |f|_\infty\sup\{\|T^*(\cdot)y\|:\ |y|\leqslant 1\}=|f|_\infty\|T\|.\end{aligned}$$

Thus $|\Phi_T|\leqslant\|T\|$.

Conversely, from Lemma 2.9 it follows that for every $\delta>0$ there exist $x\in H$, $|x|\leqslant 1$, $t_1,\ldots,t_n\in\mathbf{C}$, $|t_i|\leqslant 1$, $i=1,\ldots,n$, and disjoint sets $\Delta_1,\ldots,\Delta_n\in\Sigma$ such that

$$\|T\|<\Big|\sum_{j=1}^n T(\Delta_j)t_j x\Big|+\delta.$$

Let $f=\sum_{j=1}^n 1_{\Delta_j}t_j x$. Then

$$|\Phi_T(f)|=\Big|\sum_{j=1}^n T(\Delta_j)t_j x\Big|>\|T\|-\delta$$

and by (2.5), $|f|_\infty\leqslant 1$. Thus $|\Phi_T|\geqslant\|T\|$. ∎

2.12. Remark. The operator Φ_T need not be continuous if $S(H)$ is equipped with the sup-norm. To see this consider the spectral measure T: $\Sigma\to L(l^2)$ defined on $\Sigma=2^N$ as follows:

$$T(\Delta)=\text{orthogonal projection onto }\overline{\mathrm{sp}}\{e_k:\ k\in\Delta\}$$

where $\{e_k: k=1,2,\ldots\}$ is the standard orthonormal basis in l^2. Let $f_n = \sum_{k=1}^{n} 1_{\{k\}} e_k$. Then $|f_n|_{\sup} = 1$ for all $n \in N$, but

$$|\Phi_T(f_n)| = \Big|\sum_{k=1}^{n} T\{k\} e_k\Big| = \sqrt{n} \to \infty \quad \text{as } n \to \infty.$$

The set $\mathcal{M}(L(H, K))$ of all $L(H, K)$-valued f.a. set functions with finite semivariation is a Banach space. Since $L(H, K)$ is isometric to $(H \otimes K^*)^*$ where $H \otimes K^*$ denotes the tensor product of H and K^* equipped with the projective norm ([30], p. 190), Lemma 2.4 yields the following.

2.13. Lemma. *Let H and K be Hilbert spaces and let $H \otimes K^*$ denote the tensor product of H and K^* equipped with the projective norm $|\ |_\gamma$, i.e.,*

$$|x|_\gamma = \inf\Big\{\sum_{i=1}^{n} |x_i|\,|y_i|:\ x = \sum_{i=1}^{n} x_i \otimes y_i,\ x_i \in H,\ y_i \in K^*\Big\}.$$

A function ψ is a continuous linear functional on $\big(S(H \otimes K^*), |\ |_\infty\big)$ if and only if there exists an $L(H, K)$-valued f.a. measure $T \in \mathcal{M}\big(L(H, K)\big)$ such that for every $\varphi \in S(H \otimes K^*)$

$$\langle \varphi, \psi \rangle = \int \langle \varphi, dT \rangle.$$

Moreover, $|\psi| = \|T\|$.

2.14. Remark. The completion of $H \otimes K^*$ under the $|\ |_\gamma$-norm can be identified with the space $\operatorname{tr}(K, H)$ of all trace class operators from K to H equipped with the trace norm $|A|_\tau = \operatorname{trace}(\sqrt{A^* A})$, $A \in \operatorname{tr}(K, H)$ ([30], p. 63). Thus Lemma 2.13 says that a linear functional ψ on $S(\operatorname{tr}(K, H))$ has the form

$$\langle \varphi, \psi \rangle = \int \langle \varphi, dT \rangle = \int \operatorname{trace}(\varphi\, dT), \quad \varphi \in S\big(\operatorname{tr}(H, K)\big),$$

for some $L(H, K)$-valued f.a. measure T iff ψ is continuous in the $|\ |_\infty$-norm on $S(\operatorname{tr}(K, H))$.

2.15. Definition. (1) A f.a. $L(H)$-valued measure E defined on an algebra Σ is said to be a *f.a. spectral measure* in H if

(i) For every $\Delta \in \Sigma$, $E(\Delta)$ is an orthogonal projection operator in H.

(ii) $E(\Delta_1) E(\Delta_2) = 0$ provided $\Delta_1 \cap \Delta_2 = \emptyset$, $\Delta_1, \Delta_2 \in \Sigma$.

(iii) $E(\Omega) = I$.

If additionally Σ is a σ-algebra and E is w.c.a. on Σ then E is called a *spectral measure* in H.

(2) We say that a f.a. (w.c.a) measure $T: \Sigma \to L(H, K)$ defined on an algebra (σ-algebra) Σ has a *f.a. spectral dilation* (*spectral dilation*) if there exist a Hilbert space $\boldsymbol{K}$, a f.a. spectral measure (spectral measure) E in $\boldsymbol{K}$ and bounded linear operators $R \in L(H, \boldsymbol{K})$ and $S \in L(\boldsymbol{K}, K)$ such that for each $\Delta \in \Sigma$

$$T(\Delta) = S E(\Delta) R.$$

A quadruple $(S, \boldsymbol{K}, E, R)$ satisfying the condition above is called a *f.a.* (*w.c.a.*) *dilation of* T.

(3) A dilation $(S, \boldsymbol{K}, E, R)$ of T is called *quasi-isometric* if $S^* = J$ is an isometry from K into $\boldsymbol{K}$.

2.16. Remark. Since for any disjoint sets $\Delta_1, \ldots, \Delta_n \in \Sigma$, every collection of complex numbers $t_1, \ldots, t_n \in C$, $|t_i| \leqslant 1$, $i = 1, \ldots, n$, and every $x \in \boldsymbol{K}$

$$\Big|\sum_{j=1}^{n} E(\Delta_j) t_j x\Big|^2 = \sum_{j=1}^{n} |E(\Delta_j) t_j x|^2 \leqslant \sum_{j=1}^{n} |E(\Delta_j) x|^2 \leqslant |x|^2,$$

$\|E\| = \sup\{\|E(\cdot)x\|: |x| \leqslant 1\} \leqslant 1$. From $E(\Omega) = I$ we conclude that every (f.a.) spectral measure has the semivariation $\|E\| = 1$. Thus if a f.a. measure T has a f.a. dilation $(S, \boldsymbol{K}, E, R)$ then $\|T\|$ is necessarily finite; in fact

$$\|T\| \leqslant |S|\,|R|.$$

In particular, every f.a. $L^+(H)$-valued measure F defined on an algebra Σ has finite semivariation. In fact, from

$$|F(\Omega)| \leqslant \|F\| = \|R^* E(\cdot) R\| \leqslant |R^*|\,|R| = |R^* R| = |F(\Omega)|,$$

we have $\|F\| = |F(\Omega)|$.

2.17. Definition. We say that a f.a. (w.c.a.) measure T defined on an algebra (σ-algebra) Σ with values in $L(H, K)$ has a *f.a.* (*w.c.a.*) *2-majorant* F if F is a f.a. (w.c.a.) $L^+(H)$-valued measure on Σ such that

(2.18) For all $x_1, \ldots, x_n \in H$ and disjoint sets $\Delta_1, \ldots, \Delta_n \in \Sigma$

$$\Big|\sum_{j=1}^{n} T(\Delta_j) x_j\Big|^2 \leqslant \sum_{j=1}^{n} (F(\Delta_j) x_j, x_j) \quad \text{([26], p. 138).}$$

Let T be a f.a. measure defined on Σ and let us define

(2.19i) $C_S(T) = \inf\{|S|\,|R|: (S, \boldsymbol{K}, E, R) \text{ is a f.a. dilation of } T\}$,

(2.19ii) $C_I(T) = \inf\{|R|: (J^*, \boldsymbol{K}, E, R) \text{ is a f.a. quasi-isometric dilation of } T\}$,

(2.19iii) $C_M(T) = \inf\{\sqrt{|F(\Omega)|}: F \text{ is a f.a. 2-majorant of } T\}$;

by definition we let any of these be ∞ if the collection over which the inf is taken is empty.

If T is a w.c.a. measure on a σ-algebra Σ then in the same manner we define $C'_S(T)$, $C'_I(T)$ and $C'_M(T)$ by replacing f.a. by w.c.a. in (2.19i), (2.19ii), and (2.19iii) respectively.

The next theorem that relates these constants follows easily from Th. 2.9 and Lemma A.3 in [26]. We sketch the proof here for completeness.

2.20. THEOREM (cf. [26], Th. 2.9 and Lemma A.3). (a) *Let T be a f.a. measure defined on an algebra Σ. Then*

$$\|T\| \leqslant C_S(T) = C_I(T) = C_M(T).$$

In particular, if T has a 2-majorant then T has a f.a. spectral dilation.

(b) *If T is a w.c.a. measure defined on a σ-algebra Σ, then*

$$\|T\| \leqslant C_S(T) = C_I(T) = C_M(T) = C'_S(T) = C'_I(T) = C'_M(T).$$

Again, if T has a 2-majorant then T admits a spectral dilation.

Proof. (a) The inequality $C_S(T) \leqslant C_I(T)$ is obvious. We first prove that $C_M(T) \leqslant C_S(T)$. Suppose that $C_S(T) < \infty$ and let $T(\Delta) = SE(\Delta)R$, $\Delta \in \Sigma$, be a f.a. dilation of T. Then for all $x_1, \ldots, x_n \in H$ and disjoint $\Delta_1, \ldots, \Delta_n \in \Sigma$

$$\Big|\sum_{j=1}^{n} T(\Delta_j)x_j\Big|^2 = \Big|S\Big(\sum_{j=1}^{n} E(\Delta_j)Rx_j\Big)\Big|^2 \leqslant |S|^2 \sum_{j=1}^{n} (R^* E(\Delta_j)Rx_j, x_j).$$

Thus $F(\Delta) = |S|^2 R^* E(\Delta) R$ is a f.a. 2-majorant of T. Since $\sqrt{|F(\Omega)|} = |S|\,|R^* R|^{1/2} = |S|\,|R|$, $C_M(T) \leqslant C_S(T)$. Thus it remains to prove that $C_I(T) \leqslant C_M(T)$. If $C_M(T) = \infty$, then the inequality is obvious. Suppose that $C_M(T) < \infty$ and let F be a f.a. majorant of T. Then from [26], Th. 2.9(b), it follows that T has a f.a. quasi-isometric dilation (J^*, K, E, R) such that $F(\Delta) = R^* E(\Delta) R$, $\Delta \in \Sigma$ (note that the Equivalence Theorem ([26], p. 139) holds true for f.a. measures). Since $|R| = |R^* R|^{1/2} = \sqrt{|F(\Omega)|}$ we have $C_I(T) \leqslant C_M(T)$.

The inequality $\|T\| \leqslant C_S(T)$ follows from Remark 2.16.

(b) The same arguments as used in (a) show that

$$\|T\| \leqslant C'_S(T) = C'_I(T) = C'_M(T).$$

Since trivially $C_M(T) \leqslant C'_M(T)$, it suffices to prove that $C'_M(T) \leqslant C_M(T)$. Suppose that F is a f.a. measure with values in $L^+(H)$ such that

$$\Big|\sum_{j=1}^{n} T(\Delta_j)x_j\Big|^2 \leqslant \sum_{j=1}^{n} (F(\Delta_j)x_j, x_j)$$

for all disjoint $\Delta_1, \ldots, \Delta_n \in \Sigma$ and $x_1, \ldots, x_n \in H$. From [26], Lemma A.3, it follows that there exists a w.c.a. measure M on Σ such that for all $x \in H$ and $\Delta \in \Sigma$

$$(M(\Delta)x, x) = \inf\Big\{\sum_{j=1}^{\infty} (F(\Delta^j)x, x):\ \{\Delta^j: j = 1, 2, \ldots\} \text{ are countable partitions of } \Delta\Big\}.$$

Let $\Delta_1, \ldots, \Delta_n \in \Sigma$ be any fixed disjoint elements of Σ, $x_1, \ldots, x_n \in H$, $\delta > 0$ and let $\{\Delta_i^j: j = 1, 2, \ldots\}$ be a countable partition of Δ_i, $i = 1, \ldots, n$, such that

$$\sum_{j=1}^{\infty} (F(\Delta_i^j)x_i, x_i) \leqslant (M(\Delta_i)x_i, x_i) + \frac{\delta}{n}.$$

Since T is w.c.a., $T(\cdot)x$ is countably additive ([16], IV.10.1), and hence

$$\Big|\sum_{i=1}^{n} T(\Delta_i)x_i\Big|^2 = \lim_{k\to\infty} \Big|\sum_{i=1}^{n} T\Big(\bigcup_{j=1}^{k} \Delta_i^j\Big)x_i\Big|^2 \leqslant \lim_{k\to\infty} \sum_{i=1}^{n} \Big(F\Big(\bigcup_{j=1}^{k} \Delta_i^j\Big)x_i, x_i\Big)$$
$$= \sum_{i=1}^{n} \lim_{k\to\infty} \sum_{j=1}^{k} (F(\Delta_i^j)x_i, x_i) \leqslant \sum_{i=1}^{n} (M(\Delta_i)x_i, x_i) + \delta.$$

Thus M is a w.c.a. 2-majorant of T. Since $(M(\Omega)x, x) \leqslant (F(\Omega)x, x)$ for all $x \in H$, $C'_M(T) \leqslant C_M(T)$. ■

2.21. Remark. Note that in fact we have proved a little stronger result. Namely, we have proved that if F is a f.a. 2-majorant of a w.c.a. measure T: $\Sigma \to L(H, K)$ then there exists a w.c.a. quasi-isometric dilation (J^*, K, E, R) of T such that $|R| \leqslant \sqrt{|F(\Omega)|}$.

If Σ is a σ-algebra and ζ is a K-valued measure on Σ (i.e., ζ is countably additive) then the integral $\int f\,d\zeta$ is well defined for all bounded Σ-measurable complex-valued functions f ([6], IV.10). Thus, if T is a w.c.a. measure on Σ then by [6], IV.10.2, $T(\cdot)x$ is a K-valued measure for every $x \in H$ and one can define

$$\Big(\int f\,dT\Big)x = \int f\,d(Tx),$$

$x \in H$, $f \in B(C)$. The following two properties are immediate:

(2.22) *For every $f \in B(C)$, $\int f\,dT$ is a bounded linear operator from $S(H)$ into K and*

$$\Big|\int f\,dT\Big| \leqslant \|T\|\,|f|_{\sup}.$$

(2.23) *For any bounded linear operators $U \in L(K, \mathscr{K})$ and $V \in L(\mathscr{H}, H)$*

$$U\Big(\int f\,dT\Big)V = \int f\,d(UTV)$$

where UTV is an $L(\mathscr{H}, \mathscr{K})$-valued w.c.a. measure defined by the formula $(UTV)(\Delta) = UT(\Delta)V$. In particular,

$$\Big(\int f\,dTx, y\Big) = \int f\,d(Tx, y)$$

for all $x \in H$ and $y \in K$.

If $\Phi = [\varphi_{ij}]$ is a $p \times q$-matrix-valued function with bounded Σ-measurable entries φ_{ij}, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, and if T is a w.c.a. $L(H, K)$-valued measure defined on a σ-algebra Σ then we write

$$\int \Phi\,dT = \Big[\int \varphi_{ij}\,dT\Big] \quad \text{(cf. [25])}.$$

The integral $\int \Phi\, dT$ is a bounded linear operator from H^q into K^p. The following lemma follows immediately from ([25], Th. 4.10). It will be used in the next section in constructing the counterexample.

2.24. Lemma. *Let T be a w.c.a. $L(H, K)$-valued measure defined on a σ-algebra Σ. If T has a spectral dilation then*

(2.25) *There exists a constant C such that for every $1 \leqslant p, q < \infty$ and every bounded $p \times q$-matrix-valued Σ-measurable function Φ*

$$\left|\int \Phi\, dT\right| \leqslant C\, |\Phi|_{\sup},$$

where $|\Phi|_{\sup}$ denotes the sup-norm of Φ regarded as an $L(l_q^2, l_p^2)$-valued function.

Proof. Let (S, K, E, R) be a spectral dilation of T. Then from (2.23) it follows that

$$\int \Phi\, dT = \begin{bmatrix} S & & 0 \\ & \ddots & \\ 0 & & S \end{bmatrix} \left[\int \varphi_{ij}\, dE\right] \begin{bmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{bmatrix}, \qquad \Phi \in B\big(L(l_q^2, l_p^2)\big).$$

Thus from [25], Th. 4.10, we conclude that

$$\left|\int \Phi\, dT\right| \leqslant \left\| \begin{bmatrix} S & & 0 \\ & \ddots & \\ 0 & & S \end{bmatrix} \right\| \left|\int \Phi\, dE\right| \left\| \begin{bmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{bmatrix} \right\| \leqslant |S|\, |R|\, |\Phi|_{\sup},$$

for all $1 \leqslant p, q < \infty$ and $\Phi \in B\big(L(l_q^2, l_p^2)\big)$. ∎

2.26. Remark. The condition (2.25) is also sufficient for T to have spectral dilation. In fact, from a deep result by Wittstock (see [9] or [22]) it follows that if $T\colon \Sigma \to L(H)$ satisfies (2.25) then the mapping

$$B(C) \ni \varphi \to \int \varphi\, dT \in L(H)$$

is a linear combination of positive bounded mappings. Thus T is a linear combination of f.a. $L^+(H)$-valued measures. Since each f.a. $L^+(H)$-valued measure admits a f.a. spectral dilation, using the method of the proof of Cor. 2.10 in [26] it is easy to show that T has a f.a. spectral dilation and so, by Th. 2.20(b), T has a spectral dilation.

We are grateful to Professor S. Kwapień for pointing out to us the papers of Haagerup [9] and Loebl [13] from which we learned about the Wittstock Theorem and the example which we discuss in the next section.

3. Counterexample. In this section we state an example of a w.c.a. $L(H, K)$-valued measure which does not admit a spectral dilation. The example is a slight modification of the one given by Loebl ([13], Th. 2.2) and is based on the following lemma proved in [13] (see Lemma 2.1 and the proof of Th. 2.2 in [13]).

3.1. Lemma. *For every $n \geqslant 1$ there exist Hermitian $2^n \times 2^n$-matrices $A_1, \ldots, A_n$ having the following properties:*

(1) $\left|\sum_{i=1}^n \alpha_i A_i\right| \leqslant \sqrt{2}\sqrt{\sum_{i=1}^n |\alpha_i|^2}$ *for all* $\alpha_1, \ldots, \alpha_n \in C$.

(2) *There exists a unit vector $x \in l_{2^n}^2 \otimes l_{2^n}^2$ such that*

$$(A_i \otimes A_i)\, x = x$$

for all $i = 1, \ldots, n$, where $l_{2^n}^2 \otimes l_{2^n}^2$ denotes the Hilbertian tensor product and $A_i \otimes A_i$ is the tensor product of A_i with itself ([30], p. 183).

3.2. Remark. To translate condition (2) in Lemma 3.1, which uses the notion of Hilbertian tensors, to the language of operators on direct sum we proceed as follows:

First note that the mapping

$$\sum_{i=1}^m x_i \otimes e_i \xrightarrow{I} (x_i)_{i=1}^m, \qquad x_i \in l_m^2,$$

is an isometry between $l_m^2 \otimes l_m^2$ and $(l_m^2)^m$ ([30], p. 183), where $\{e_i\colon i = 1, \ldots, m\}$ is the standard basis in l_m^2. If A is an $m \times m$-matrix, then $A \otimes A$, defined by

$$(A \otimes A)\Big(\sum_{i=1}^n x_i \otimes y_i\Big) = \sum_{i=1}^n A x_i \otimes A y_i,$$

is a linear operator acting on $l_m^2 \otimes l_m^2$ and the corresponding operator $I(A \otimes A) I^{-1}$ in $(l_m^2)^m$ is given by the formula

$$\begin{aligned} I(A \otimes A) I^{-1} (x_j)_{j=1}^m &= I(A \otimes A)\Big(\sum_{j=1}^m x_j \otimes e_j\Big) \\ &= I\Big(\sum_{j=1}^m A x_j \otimes \Big(\sum_{k=1}^m a_{kj} e_k\Big)\Big) = I\Big(\sum_{k=1}^m \Big(\sum_{j=1}^m A a_{kj} x_j\Big) \otimes e_k\Big) \\ &= \Big(\sum_{j=1}^m A a_{kj} x_j\Big)_{k=1}^m = [A a_{kj}] \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}. \end{aligned}$$

Thus condition (2) in Lemma 3.1 can be written in the form

(3.3) *There exist $x_1, \ldots, x_{2^n} \in l_{2^n}^2$ such that $\sum_{k=1}^{2^n} |x_k|^2 = 1$ and such that for all $k = 1, \ldots, 2^n$ and $i = 1, \ldots, n$*

$$\sum_{j=1}^{2^n} A_i\, a_{kj}(i)\, x_j = x_k,$$

where $a_{kj}(i)$ denotes the (k, j)th entry of A_i.

3.4. COUNTEREXAMPLE. Let $\Omega = N$, $\Sigma = 2^N$, $H_n = l^2_{2^n}$, $n \in N$. Let $H = \bigoplus_{n=1}^{\infty} H_n$ and let for every $n = 1, 2, \ldots,$

$$A_1^{(n)}, A_2^{(n)}, \ldots, A_n^{(n)}$$

be the sequence of $2^n \times 2^n$-matrices satisfying (1) and (2) of Lemma 3.1. We regard $A_i^{(n)}$'s as elements of $L(H_n)$. Define

$$T\{1\} = P_1^* A_1^{(1)} P_1,$$
$$T\{2\} = 2^{-3/4} P_2^* A_1^{(2)} P_2, \qquad T\{3\} = 2^{-3/4} P_2^* A_2^{(2)} P_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T\left\{\frac{n(n-1)}{2}+i\right\} = n^{-3/4} P_n^* A_i^{(n)} P_n, \quad i = 1, \ldots, n, \ n \in N,$$

where P_n denotes the orthogonal projection from H onto H_n.

First we prove that for each $x \in H$ the series $\sum_{k=1}^{\infty} T\{k\}x$ converges unconditionally.

Let $\alpha_k \in C$, $|\alpha_k| \leqslant 1$, $k = 1, 2, \ldots, r$, $s \in N$, $r < s$,

$$n_r = \max\{n\colon 1+(n-1)n/2 \leqslant r\}, \qquad n_s = \min\{n\colon n(n+1)/2 \geqslant s\},$$

and let

$$I_k = \left\{\frac{k(k-1)}{2}+1, \frac{k(k-1)}{2}+2, \ldots, \frac{k(k+1)}{2}\right\}, \qquad k = 1, 2, \ldots$$

Since $T\{j\}H \perp T\{i\}H$ provided $i \in I_k$, $j \in I_l$, $k \neq l$, using property (1) in Lemma 3.1 and putting $\alpha'_j = \alpha_j$ if $r \leqslant j \leqslant s$ and 0 otherwise we have

$$\Big|\sum_{i=r}^{s} T\{i\}\alpha_i x\Big|^2 = \Big|\sum_{i=(n_r-1)n_r/2+1}^{(n_s+1)n_s/2} T\{i\}\alpha'_i x\Big|^2 = \sum_{k=n_r}^{n_s} \Big|\sum_{i \in I_k} T\{i\}\alpha'_i x\Big|^2$$
$$= \sum_{k=n_r}^{n_s} \Big|P_k^* k^{-3/4}\Big(\sum_{i \in I_k} A^{(k)}_{i-k(k-1)/2}\alpha'_i\Big) P_k x\Big|^2$$
$$\leqslant \sum_{k=n_r}^{n_s} k^{-3/2}\, 2 \sum_{i \in I_k} |\alpha_i|^2 |P_k x|^2 \leqslant \sum_{k=n_r}^{n_s} k^{-3/2}\, 2k\, |P_k x|^2$$
$$\leqslant 2 \sum_{k=n_r}^{\infty} |P_k x|^2 \to 0 \quad \text{as } r \to \infty.$$

For every $\Delta \in \Sigma$ let us define

$$T(\Delta)x = \sum_{k \in \Delta} T\{k\}x.$$

From the above considerations and the Banach–Steinhaus Theorem it follows that $T(\Delta) \in L(H)$ for every $\Delta \in \Sigma$ and that T is an $L(H)$-valued w.c.a. measure on Σ. We will prove that T does not satisfy the conclusion of Lemma 2.24 and so T cannot have a spectral dilation.

Let $\Phi_n(\omega) = \sum_{i=1}^{n} 1_{\{n(n-1)/2+i\}}(\omega) A_i^{(n)}$, $n = 1, 2, \ldots, \omega \in \Omega = N$, and let for each n

$$x_1^{(n)}, x_2^{(n)}, \ldots, x_{2^n}^{(n)} \in l^2_{2^n} = H_n \subset H$$

be the sequence satisfying (3.3). Then, by (1) in Lemma 3.1

$$|\Phi_n|_{\sup} = \sup\{|A_i^{(n)}|\colon i = 1, \ldots, n\} \leqslant \sqrt{2}.$$

But from (3.3) in Remark 3.2

$$\Big|\int \Phi_n\, dT\Big|^2 = \Big|\Big[\sum_{i=1}^{n} a_{kj}^{(n)}(i)\, T\{n(n-1)/2+i\}\Big]\Big|^2$$
$$\geqslant \left|\Big[\sum_{i=1}^{n} a_{kj}^{(n)}(i)\, P_n^* A_i^{(n)} P_n \cdot n^{-3/4}\Big] \begin{bmatrix} x_1^{(n)} \\ \vdots \\ x_{2^n}^{(n)} \end{bmatrix}\right|^2$$
$$= \sum_{k=1}^{2^n} \Big|\sum_{j=1}^{2^n} \sum_{i=1}^{n} a_{kj}^{(n)}(i)\, A_i^{(n)} x_j^{(n)}\Big|^2 n^{-3/2}$$
$$= \sum_{k=1}^{2^n} \Big|\sum_{i=1}^{n} \sum_{j=1}^{2^n} a_{kj}^{(n)}(i)\, A_i^{(n)} x_j^{(n)}\Big|^2 n^{-3/2}$$
$$= \sum_{k=1}^{2^n} \Big|\sum_{i=1}^{n} x_k^{(n)}\Big|^2 n^{-3/2}$$
$$= n^{1/2} \sum_{k=1}^{2^n} |x_k^{(n)}|^2 = n^{1/2} \to \infty \qquad \text{as } n \to \infty. \ \blacksquare$$

3.5. Remark. Observe that in view of Th. 2.20(b) the set function T constructed in 3.4 does not admit a f.a. spectral dilation either.

4. Spectral dilation. Since not every w.c.a. $L(H, K)$-valued measure has a spectral dilation it is of interest to characterize those measures that admit spectral dilations. For the case $H = C$, where $L(C, K)$ can be identified with K itself, the following characterization plays a central role.

4.1. THEOREM (see [20], Th. 2 and 10). *A f.a. K-valued measure T with $\|T\| < \infty$ defined on an algebra Σ has a f.a. spectral dilation if and only if*

(4.2) *There exists a constant C such that for any collection of scalar simple functions $f_1, \ldots, f_n \in S(C)$*

$$\sum_{k=1}^{n} \Big|\int f_k\, dT\Big|^2 \leqslant C \Big|\sum_{k=1}^{n} |f_k|^2\Big|_{\sup}.$$

Grothendieck's inequality guarantees that (4.2) is always satisfied with the constant $C = \frac{1}{2}\pi\|T\|$ in the real case and $C = (4/\pi)\|T\|$ in the complex case (see [21], Lemma 1). Hence with $H = C$, each f.a. or c.a. K-valued measure with finite semivariation admits a f.a. spectral dilation (or a spectral dilation).

In view of Remark 4.5, the theorem below is a generalization of Theorem 4.1 to the case of an arbitrary f.a. (w.c.a.) $L(H, K)$-valued measure T, where H and K are any Hilbert spaces.

4.3. THEOREM. (a) *Let T be a f.a. $L(H, K)$-valued measure defined on an algebra Σ. The function T has a f.a. spectral dilation if and only if*

(4.4) *There exists a constant C such that for every collection of vectors x_j^k, $k = 1,\ldots, n$, $j = 1,\ldots, N$, and any disjoint sets $\Delta_j \in \Sigma$, $j = 1,\ldots, N$,*

$$\sum_{k=1}^{n} \Big|\sum_{j=1}^{N} T(\Delta_j)x_j^k\Big|^2 \leqslant C \sup\Big\{\Big|\sum_{k=1}^{n}\sum_{j=1}^{N} (F(\Delta_j)x_j^k, x_j^k)\Big|:\ F \in \mathcal{M}(L(H)),\ \|F\| \leqslant 1\Big\}.$$

(b) *If Σ is a σ-algebra and T is a w.c.a. measure on Σ then* (4.4) *is a necessary and sufficient condition for T to have a spectral dilation.*

The proof is given after the following remark.

4.5. Remark. One can reformulate condition (4.4) by considering the space of simple functions with values in the tensor product of H with itself endowed with the $|\ |_\infty$-norm. This reformulation is used in the proofs of Th. 4.3 and 5.4. It also demonstrates that in the infinite-dimensional case for a fruitful theory one must replace condition (4.2) involving the $|\ |_{\sup}$-norm with the similar one using the $|\ |_\infty$-norm.

For any two simple functions $f, g \in S(H)$,

$$f = \sum_{j=1}^{N} 1_{\Delta_j} x_j, \qquad g = \sum_{j=1}^{N} 1_{\Delta_j} y_j,$$

where $\Delta_1,\ldots,\Delta_N$ are disjoint sets in Σ (it is obvious that any two functions from $S(H)$ can be written in that form) let us denote by $f\circ\bar g$ the element of $S(H\otimes H^*)$ defined by the formula

$$(f\circ\bar g)(\omega) = f(\omega)\otimes\bar g(\omega) = \sum_{j=1}^{N} 1_{\Delta_j}(\omega)\, x_j\otimes\bar y_j, \qquad \omega\in\Omega,$$

where $\bar y$ is defined through $\bar y(h) = (h, y)$, $h, y \in H$. Note that from the definition of the $|\ |_\infty$-norm (2.3) and the fact that $(H\otimes H^*)^* = L(H)$ it follows that

$$|f\circ\bar g|_\infty = \sup\Big\{\Big|\sum_{j=1}^{N} (F(\Delta_j)x_j, y_j)\Big|:\ F\in\mathcal{M}(L(H)),\ \|F\|\leqslant 1\Big\},$$

$f = \sum_{j=1}^{N} 1_{\Delta_j} x_j$, $g = \sum_{j=1}^{N} 1_{\Delta_j} y_j$. Thus (4.4) can be written in the form

(4.6) *There exists a constant C such that for any collection $f_1,\ldots,f_n \in S(H\otimes H^*)$*

$$\sum_{k=1}^{n} \Big|\int dTf_k\Big|^2 \leqslant C\Big|\sum_{k=1}^{n} f_k\circ\bar f_k\Big|_\infty.$$

If $H = C$ then $f\circ\bar g = f\bar g$, $f, g\in S(C)$, and

$$\Big|\sum_{k=1}^{n} f_k\circ\bar f_k\Big|_\infty = \Big|\sum_{k=1}^{n} |f_k|^2\Big|_{\sup}$$

(see Remark 2.7). Thus in this case (4.6) (and (4.4)) reduces to (4.2). If H is infinite-dimensional then in view of Remark 2.12 the existence of a constant C in Th. 4.3 may fail if we replace $\big|\sum_{k=1}^{n} f_k\circ\bar f_k\big|_\infty$ by $\big|\sum_{k=1}^{n} f_k\circ\bar f_k\big|_{\sup}$.

Proof of Theorem 4.3. (a) If T has a f.a. dilation then from Th. 2.20(a) it follows that T has a f.a. 2-majorant F. Thus (4.4) holds with the constant

(4.7) $$C = \|F\| = |F(\Omega)| < \infty \qquad \text{(see Remark 2.16).}$$

Now, suppose that (4.4) holds. We shall prove that T has a f.a. 2-majorant F which in view of Th. 2.20(a) will complete the proof. The proof goes along the same lines as the proof of Pietsch's factorization theorem ([12], Prop. 3.1).

Consider the set W consisting of

$$\Big\{\sum_{k=1}^{n} f_k\circ\bar f_k:\ f_k\in S(H),\ k = 1,\ldots, n,\ \sum_{k=1}^{n} \Big|\int dTf_k\Big|^2 = 1,\ n\in N\Big\}.$$

From (4.6) it follows that

$$|\varphi|_\infty \geqslant 1/C$$

for every $\varphi\in W$. Since W is a convex set in $S(H\otimes H^*)$, from the Hahn–Banach Theorem ([28], Th. 3.4) and Lemma 2.13 it follows that there exist $\gamma\in(-\infty, +\infty)$ and a f.a. set function $G\in\mathcal{M}(L(H)) = (S(H\otimes H^*))^*$ with $\|G\| = 1$ such that for all $\varphi, \psi\in S(H\otimes H^*)$ with $\varphi\in W$ and $|\psi|_\infty\leqslant 1/C$ we have

$$\operatorname{Re}\langle\varphi, G\rangle \geqslant \gamma \geqslant \operatorname{Re}\langle\psi, G\rangle,$$

where $\langle\varphi, G\rangle = \int\langle\varphi, dG\rangle$ (see (2.2)) and Re z denotes the real part of a complex number z. Since for $\alpha = \overline{\langle\psi, G\rangle}/|\langle\psi, G\rangle|$, $|\langle\psi, G\rangle| = \operatorname{Re}\langle\alpha\psi, G\rangle$

and $|\psi|_\infty = |\alpha\psi|_\infty$ we have

$$\gamma \geqslant \sup\{\operatorname{Re}\langle\psi, G\rangle\colon |\psi|_\infty \leqslant 1/C\}$$
$$= \sup\{|\langle\psi, G\rangle|\colon |\psi|_\infty \leqslant 1/C\} = (1/C)\|G\| = 1/C.$$

Thus we conclude that

$$\operatorname{Re}\langle\varphi, G\rangle \geqslant 1/C \quad \text{for all } \varphi\in W.$$

Put $M(\Delta) = [G(\Delta)+G(\Delta)^*]/2$, $\Delta\in\Sigma$, and let $f_1,\ldots,f_n\in S(H)$ be such that $\sum_{k=1}^n |\int dTf_k|^2 > 0$. Then $M\in\mathscr{M}(L(H))$, $\|M\|\leqslant 1$ and

$$\int\langle(\sum_{k=1}^n |\int dTf_k|^2)^{-1}\sum_{k=1}^n f_k\circ\bar{f}_k, dM\rangle = \operatorname{Re}\int\langle(\sum_{k=1}^n |\int dTf_k|^2)^{-1}\sum_{k=1}^n f_k\circ\bar{f}_k, dG\rangle \geqslant 1/C.$$

Thus

$$(*)\qquad \sum_{k=1}^n |\int dTf_k|^2 \leqslant C\int\langle\sum_{k=1}^n f_k\circ\bar{f}_k, dM\rangle$$

whenever $f_1,\ldots,f_n\in S(H)$ and $\sum_{k=1}^n |\int dTf_k|^2 > 0$.

To eliminate the assumption $\sum_{k=1}^n |\int dTf_k|^2 > 0$ it suffices to prove that $M(\Delta)\geqslant 0$ for all $\Delta\in\Sigma$. Suppose that $(M(\Delta_1)x, x) = -a < 0$ and that there exist $\Delta_2\in\Sigma$ and $y = H$ such that $|T(\Delta_2)y| = 1$ (otherwise $T(\Delta) = 0$ for all $\Delta\in\Sigma$ and of course T has a spectral dilation). Let $f_1 = 1_{\Delta_1}xt$, $f_2 = 1_{\Delta_2}y$. If $|\int dTf_1| = |T(\Delta_1)x|^2 > 0$, then from $(*)$ it follows that

$$0 < |\int dTf_1|^2 \leqslant C\int\langle f_1\circ\bar{f}_1, dM\rangle = C(M(\Delta_1)xt, xt) = -Ca|t|^2 < 0.$$

Thus $|\int dTf_1| = 0$. Using $(*)$ once again we get

$$1 = |\int dTf_1|^2 + |\int dTf_2|^2 \leqslant C\int\langle f_1\circ\bar{f}_1 + f_2\circ\bar{f}_2, dM\rangle$$
$$= C(M(\Delta_1)xt, xt) + C(M(\Delta_2)y, y)$$
$$= -Ca|t|^2 + C(M(\Delta_2)y, y) < 0$$

for sufficiently large t, $t\in\boldsymbol{R}$.

Hence M is a positive f.a. measure with values in $L(H)$ and $(*)$ holds true for any sequence $f_1,\ldots,f_n\in S(H)$. In particular, if $n = 1$ and $f_1 = \sum_{j=1}^n 1_{\Delta_j}x_j$ then we get

$$|\sum_{j=1}^n T(\Delta_j)x_j|^2 = |\int dTf_1|^2 \leqslant C\int\langle f_1\circ\bar{f}_1, dM\rangle = C\sum_{j=1}^n (M(\Delta_j)x_j, x_j).$$

Therefore

(4.8) T has a f.a. majorant $F = CM$ with

$$|F(\Omega)| = C|M(\Omega)| \leqslant C\|M\| = C.$$

(b) If T is a w.c.a. measure defined on a σ-algebra Σ satisfying (4.4) then from (a) it follows that T has a f.a. majorant F and so by Th. 2.20(b), T has a w.c.a. quasi-isometric dilation (J, K, E, R) with $|R|\leqslant\sqrt{C}$ (see Remark 2.21). ∎

4.9. Definition. For a given f.a. measure T we let

$$C(T) = \inf\{\sqrt{C}\colon C \text{ satisfies (4.4)}\}$$

(by definition inf of an empty set is equal to $+\infty$).

The following result which generalizes Th. 4.7 from [26] to the infinite-dimensional case is an easy consequence of Th. 4.3.

4.10. Corollary. *Let T be a f.a. (w.c.a.) $L(H, K)$-valued measure defined on an algebra (σ-algebra) Σ. Then*:

(a) $\|T\|\leqslant C_S(T) = C_1(T) = C_M(T) = C(T)$ $(= C'_S(T) = C'_1(T) = C'_M(T))$.

(b) *If $C(T) < \infty$, then there exists a f.a. (w.c.a.) 2-majorant F of T such that $\sqrt{|F(\Omega)|} = C(T)$. As a consequence, there exists a f.a. (w.c.a.) quasi-isometric dilation (J^*, K, E, R) of T such that $|R| = C(T)$.*

Proof. (a) follows immediately from the proof of Th. 4.3 (see (4.7) and (4.8)) and from Th. 2.20.

(b) If $C(T) < \infty$ then (4.4) holds with $C = C(T)^2$. Thus from the proof of Th. 4.3 (see (4.8)) it follows that there exists a f.a. 2-majorant F of T such that $|F(\Omega)| = C(T)^2$. Hence in view of Remark 2.21 there exists a f.a. quasi-isometric dilation (J^*, K, E, R) of T such that $|R| = \sqrt{|F(\Omega)|} = C(T)$.

If T is additionally w.c.a., then from Remark 2.21 it follows that there exist a w.c.a. majorant F' of T and a w.c.a. quasi-isometric dilation (J'^*, K', E', R') of T such that

$$|R'| = \sqrt{|F'(\Omega)|} \leqslant \sqrt{|F(\Omega)|} = C(T).$$

As $C(T) = C'_1(T)\leqslant|R'|\leqslant C(T)$, we must have

$$|R'| = \sqrt{|F(\Omega)|} = C(T). \;∎$$

5. Some special cases. Let (Ω, Σ) be a fixed measurable space in the sense that Σ is an algebra (or a σ-algebra) of subsets of Ω and let T be an arbitrary f.a. (w.c.a.) $L(H, K)$-valued measure on Σ. We will say that T is *dilatable* if either

(a) T has a f.a. spectral dilation provided T is f.a., or

(b) T has a spectral dilation provided T is w.c.a.

Let us recall that from Th. 2.20(b) it follows that if a w.c.a. measure $T\colon \Sigma \to L(H, K)$ has a f.a. spectral dilation then it also has a spectral dilation, so we will drop the words f.a. or w.c.a. preceding the word "dilatable". Let Φ_T denote the operator associated with the measure T given by the formula

$$(5.1)\qquad \Phi_T(f) = \int dTf, \quad f \in S(H) \quad \text{(see Def. 2.10)}.$$

As we have seen in Lemma 2.11 the operator Φ_T from $(S(H), |\ |_\infty)$ into K is continuous if $\|T\| < \infty$.

In this section we state a series of sufficient conditions for a (f.a.) w.c.a. $L(H, K)$-valued measure T to have a (f.a.) spectral dilation. The first two sets of conditions deal with the properties of the operator Φ_T. We will prove that if Φ_T is 2-absolutely summing from $(S(H), |\ |_\infty)$ into K, then T is dilatable. We will also prove that the continuity of Φ_T with respect to the $|\ |_{\sup}$-norm ensures the existence of a spectral dilation of T. Lastly, we look upon T as a mapping from Σ into the Banach algebra $L(H)$. We point out that the factor that binds all these results is the main Th. 4.3.

5.2. Definition. A linear operator Φ from a normed linear space X into a Hilbert space H is said to be *2-absolutely summing* if there exists a constant C such that for all $x_1, \ldots, x_n \in X$

$$\sum_{i=1}^{n} |\Phi x_i|^2 \leqslant C \sup\Big\{\sum_{i=1}^{n} |\langle x_i, x^*\rangle|^2 \colon |x^*| \leqslant 1,\ x^* \in X^*\Big\}$$

(cf. [12], Def. 3.2).

5.3. Remark. Note that condition (4.2) in Th. 4.1 is equivalent to the 2-absolute summability of the operator $\Phi_T\colon S(C) \to K$. To see this it suffices to prove that for all $f_1, \ldots, f_n \in S(C)$

$$\Big|\sum_{k=1}^{n} |f_k|^2\Big|_{\sup} = \sup\Big\{\sum_{k=1}^{n} \Big|\int f_k\, d\mu\Big|^2 \colon \mu \in \mathscr{M}(C),\ \|\mu\| \leqslant 1\Big\}$$

because $S(C)^* = \mathscr{M}(C)$ (Lemma 2.4 and Remark 2.7). The inequality "$\leqslant$" is obvious by considering measures concentrated on singletons. Conversely, let $\|\mu\| \leqslant 1$ and let

$$f_k = \sum_{j=1}^{N} 1_{\Delta_j} a_j^k, \qquad k = 1, \ldots, n,$$

be any collection of Σ-simple complex functions, where $\Delta_1, \ldots, \Delta_N$ are disjoint. Then

$$\sum_{k=1}^{n} \Big|\int f_k\, d\mu\Big|^2 = \sum_{i,j=1}^{N} \mu(\Delta_i)\overline{\mu(\Delta_j)}(\mathbf{a}_i, \mathbf{a}_j) = \Big|\sum_{j=1}^{N} \mu(\Delta_j)\mathbf{a}_j\Big|^2$$

where $\mathbf{a}_j = (a_j^1, a_j^2, \ldots, a_j^n) \in l_n^2$, $j = 1, \ldots, N$, and from the triangle inequality we get

$$\begin{aligned}\sum_{k=1}^{n} \Big|\int f_k\, d\mu\Big|^2 &\leqslant \Big(\sum_{j=1}^{N} |\mathbf{a}_j|\,|\mu(\Delta_j)|\Big)^2 \leqslant \Big(\sup_{j\leqslant N} |\mathbf{a}_j| \cdot \sum_{j=1}^{N} |\mu(\Delta_j)|\Big)^2 \\ &\leqslant \sup_{j\leqslant N} |\mathbf{a}_j|^2 \cdot \|\mu\|^2 \leqslant \sup_{j\leqslant N} \sum_{k=1}^{n} |a_j^k|^2 = \Big|\sum_{k=1}^{n} |f_k|^2\Big|_{\sup},\end{aligned}$$

which completes the proof. Thus Th. 4.1 says that a $K = L(C, K)$-valued f.a. measure T is dilatable iff Φ_T is 2-absolutely summing.

In view of this remark it is of interest to find out the connection between the existence of a (f.a.) spectral dilation of T and the 2-absolute summability of Φ_T in the general case.

5.4. Theorem. (a) *Let T be a f.a. (w.c.a.) $L(H, K)$-valued measure defined on an algebra (σ-algebra) Σ. If Φ_T from $(S(H), |\ |_\infty)$ into K is 2-absolutely summing, then T is dilatable.*

(b) *There exist a measurable space (Ω, Σ), a Hilbert space H and an $L(H)$-valued dilatable w.c.a. measure T defined on Σ such that Φ_T from $(S(H), |\ |_\infty)$ into H is not 2-absolutely summing.*

Proof. (a) Suppose that Φ_T is 2-absolutely summing, i.e., there exists a constant C such that for all $f_1, \ldots, f_n \in S(H)$

$$(5.5)\qquad \sum_{k=1}^{n} \Big|\int dTf_k\Big|^2 \leqslant C \sup\Big\{\sum_{k=1}^{n} \Big|\int \langle f_k, dm\rangle\Big|^2 \colon m \in \mathscr{M}(H),\ \|m\| \leqslant 1\Big\}.$$

We shall prove that

$$\begin{aligned}\sup\Big\{\sum_{k=1}^{n} \Big|\int \langle f_k, dm\rangle\Big|^2 \colon m \in \mathscr{M}(H),\ \|m\| \leqslant 1\Big\} &\leqslant \Big|\sum_{k=1}^{n} f_k \circ f_k\Big|_\infty \\ &= \sup\Big\{\Big|\sum_{k=1}^{n} \int \langle f_k \circ f_k, dM\rangle\Big|^2 \colon M \in \mathscr{M}(L(H)),\ \|M\| \leqslant 1\Big\},\end{aligned}$$

which in view of Th. 4.3 (see also (4.6)) will complete the proof.

Let $m \in \mathscr{M}(H)$, $\|m\| \leqslant 1$. From 4.10(b) and the paragraph following Th. 4.1 it follows that there exist a Hilbert space $\mathbf{K}$, an isometry $J \in L(H, \mathbf{K})$, a f.a. spectral measure E in $\mathbf{K}$ and $y_0 \in \mathbf{K}$ with $|y_0| \leqslant \pi/2$ such that

$$m(\Delta) = J^* E(\Delta) y_0, \qquad \Delta \in \Sigma.$$

Let $f = \sum_{j=1}^{N} 1_{\Delta_j} x_j \in S(H)$, where $\Delta_1, \ldots, \Delta_N$ are disjoint, $\Delta_j \in \Sigma$, $x_j \in H$, $j = 1, \ldots, N$, and let $M(\Delta) = J^* E(\Delta) J$, $\Delta \in \Sigma$. Then $M \in \mathscr{M}(L(H))$,

$\|M\| \leqslant |J^*||J| = 1$ (see Remark 2.16) and

$$\left|\int \langle f, dm\rangle\right|^2 = \left|\sum_{j=1}^{N} (x_j, m(\Delta_j))\right|^2 = \left|\left(\sum_{j=1}^{N} E(\Delta_j) J x_j, y_0\right)\right|^2$$
$$\leqslant |y_0|^2 \left|\sum_{j=1}^{N} E(\Delta_j) J x_j\right|^2 = |y_0|^2 \sum_{j=1}^{N} (M(\Delta_j) x_j, x_j)$$
$$\leqslant \tfrac{1}{2}\pi \int \langle f \circ \bar{f}, dM\rangle.$$

Thus for any $f_1, \ldots, f_n \in S(H)$ and $m \in \mathcal{M}(H)$, $\|m\| \leqslant 1$,

$$\sum_{k=1}^{n} \left|\int \langle f_k, dm\rangle\right|^2 \leqslant \tfrac{1}{2}\pi \int \left\langle \sum_{k=1}^{n} f_k \circ \bar{f}_k, dM \right\rangle \leqslant \tfrac{1}{2}\pi \left|\sum_{k=1}^{n} f_k \circ \bar{f}_k\right|_{\infty},$$

since $\|M\| \leqslant 1$.

(b) Let $\Omega = N$, $\Sigma = 2^N$, $H = l^2$ and let for every $\Delta \in \Sigma$, $T(\Delta)$ be the orthogonal projection in l^2 onto $\overline{\mathrm{sp}}\{e_k: k \in \Delta\}$, where $\{e_k: k = 1, 2, \ldots\}$ denotes the standard orthonormal basis in l^2. Then T is a spectral measure itself so it is dilatable. But the operator Φ_T associated with T is not 2-absolutely summing. To see this consider the sequence of functions $f_k = 1_{\{k\}} e_k \in S(H)$, $k = 1, 2, \ldots$ We have

$$\sum_{k=1}^{n} |\Phi_T(f_k)|^2 = \sum_{k=1}^{n} \left|\int dT f_k\right|^2 = \sum_{k=1}^{n} |e_k|^2 = n \to \infty$$

whereas for every $m \in \mathcal{M}(H)$ with $\|m\| \leqslant 1$

$$\sum_{k=1}^{n} \left|\int \langle f_k, dm\rangle\right|^2 = \sum_{k=1}^{n} |(e_k, m\{k\})|^2 \leqslant \sum_{k=1}^{n} |m\{k\}|^2 \leqslant \|m\|^2 \leqslant 1,$$

where in the middle inequality we make use of the fact that if m is an H-valued f.a. measure defined on an algebra Σ with $\|m\| < \infty$ then for all disjoint $\Delta_1, \ldots, \Delta_N$ in Σ

$$\sum_{k=1}^{N} |m(\Delta_k)|^2 \leqslant \|m\|^2. \tag{5.6}$$

Indeed, let $\{r_j(t): j = 1, 2, \ldots\}$ be the Rademacher system in $L^2([0, 1], dt)$ where dt is the Lebesgue measure (i.e., r_j is a sequence of independent random variables on $[0, 1]$ taking values ± 1 with probability $\frac{1}{2}$) and let $\Delta_1, \ldots, \Delta_N$ be a collection of disjoint elements of Σ. Then

$$\sum_{j=1}^{N} |m(\Delta_j)|^2 = \int_0^1 \left|\sum_{j=1}^{N} m(\Delta_j) r_j(t)\right|^2 dt \leqslant \int_0^1 \|m\|^2\, dt = \|m\|^2. \ \blacksquare$$

5.7. Theorem. *Let T be a f.a. (w.c.a) $L(H, K)$-valued measure defined on an algebra (σ-algebra) Σ. If for every orthonormal basis $e = \{e_k: k = 1, 2, \ldots\}$ in H*

$$\sum_k \|T(\cdot) e_k\|^2 < \infty,$$

then T is dilatable.

Proof. Consider the measure $T(\cdot)^*: \Sigma \to L(K, H)$ and let Φ_{T^*} denote the operator associated with T^* by formula (5.1). Then using the notation from [29], p. 384, we deduce that for every orthonormal basis $e = \{e_k: k = 1, 2, \ldots\}$ in H

$$\sigma_e^2(\Phi_{T^*}) \overset{\mathrm{df}}{=} \sum_k \sup\{|(\Phi_{T^*} f, e_k)|^2: f \in S(K), |f|_\infty \leqslant 1\}$$
$$= \sum_k \sup\left\{\left|\left(\int dT^* f, e_k\right)\right|^2: f \in S(K), |f|_\infty \leqslant 1\right\}$$
$$= \sum_k \sup\left\{\left|\int \langle f, dTe_k\rangle\right|^2: f \in S(K), |f|_\infty \leqslant 1\right\}$$
$$\leqslant \sum_k \sup\{|f|_\infty^2 \|T(\cdot) e_k\|^2: f \in S(K), |f|_\infty \leqslant 1\}$$
$$= \sum_k \|T(\cdot) e_k\|^2 < \infty.$$

Thus from [29], Prop. 1, it follows that there exist a Hilbert space $\mathscr{H}$, a Hilbert–Schmidt operator R from $\mathscr{H}$ to H and a bounded linear operator $V \in L(S(K), \mathscr{H})$ with $|V| = 1$ such that

$$\Phi_{T^*} = RV.$$

Since every Hilbert–Schmidt operator is 2-absolutely summing ([12], Th. 6.3), Φ_{T^*} is 2-absolutely summing. Thus, by Th. 5.4, $T(\cdot)^*$ is dilatable and so is T. $\blacksquare$

5.8. Theorem. *If a f.a. (w.c.a.) $L(H, K)$-valued measure T defined on an algebra (σ-algebra) Σ has the form*

$$T(\Delta) = RS(\Delta), \quad \Delta \in \Sigma,$$

where R is a Hilbert–Schmidt operator from $\mathscr{H}$ to K, $\mathscr{H}$ is a Hilbert space and S is a f.a. $L(H, \mathscr{H})$-valued measure on Σ, with $\|S\| < \infty$, then T is dilatable.

Proof. Note that Φ_T has the factorization

$$\Phi_T = R\Phi_S$$

where $\Phi_S \in L(S(H), \mathscr{H})$ (Lemma 2.11). Thus Φ_T is 2-absolutely summing and by Th. 5.4, T is dilatable. $\blacksquare$

5.9. Remark. Since $T(\cdot)$ is dilatable if and only if $T(\cdot)^*$ is dilatable, it follows immediately from Theorem 5.7 that if H or K is finite-dimensional then every $L(H, K)$-valued f.a. measure with finite semivariation is dilatable. In particular, if H or K is finite-dimensional then every $L(H, K)$-valued w.c.a. measure T has a spectral dilation (see Lemma 2.9(b)).

Having proved Th. 5.4, 5.7 and 5.8 using the $|\ |_\infty$-norm we now proceed to endow $S(H)$ with the $|\ |_{\sup}$-norm and to investigate its implications.

5.10. Definition (cf. [31], (15)). For any f.a. $L(H, K)$-valued measure defined on an algebra Σ let us denote by $|||T|||$ the number in $[0, \infty]$ defined by the formula

$$|||T||| = \sup\{|\sum_{\Delta\in\pi} T(\Delta)x_\Delta|:\ \pi\in\Pi(\Omega),\ x_\Delta\in H,\ |x_\Delta|\leqslant 1\}.$$

Note that

$$|||T||| = \sup\{|\Phi_T(f)|:\ f\in S(H),\ |f|_{\sup}\leqslant 1\}.$$

Thus

(5.11) *$|||T|||<\infty$ if and only if the operator Φ_T associated with T by formula (5.1) is a bounded operator from $(S(H), |\ |_{\sup})$ into K.*

From [31] one can deduce that any Hilbert–Schmidt-operator-valued measure T satisfying $|||T|||<\infty$ admits a spectral dilation. Th. 5.18 below extends this to the $L(H, K)$-valued case.

This shows that if Φ_T is continuous from $(S(H), |\ |_{\sup})$ into K then T admits a spectral dilation. The connection between the continuity of the operator Φ_T with respect to the $|\ |_{\sup}$-norm and the 2-absolute summability of Φ_T with respect to the $|\ |_\infty$-norm is not quite clear to us. It needs further investigation.

In the proof of the next theorem we will use the following slight generalization of [24], Lemma 5, dealing with the case $H = \boldsymbol{R}$ (see also [31] for general H).

5.12. Lemma. *Let H be a Hilbert space. There exists a constant C such that for all $n, m\in N$, $t_1,\ldots,t_m\in l_n^2$, $x_1,\ldots,x_m\in H$*

$$\sum_{i,j=1}^{m}(t_i, t_j)(x_i, x_j)\leqslant C\int_{S_n}\Big|\sum_{j=1}^{m}\overline{\operatorname{sgn}(s, t_j)}\,|t_j|\,x_j\Big|^2\mu_n(ds),$$

where S_n is the unit sphere in l_n^2, μ_n is the normalized rotationally invariant measure on S_n and $\operatorname{sgn} z = z/|z|$ if $z\neq 0$ and 0 otherwise, $z\in C$. Moreover, $C\leqslant\frac{1}{2}\pi$ (more precisely: $C = 4/\pi$ if H and l_n^2 are complex spaces and $C=\frac{1}{2}\pi$ if H is a real Hilbert space and $l_n^2 = \boldsymbol{R}^n$ with Euclidean norm).

Proof. The proof proceeds along the same lines as in [24]. For the benefit of the readers we state the main steps of the proof. We will consider only the complex case.

Step 1. Integration in polar coordinates yields (see [21], p. 183)

$$\int_{S_n}(u, t)\overline{(w, t)}\,\mu_n(dt) = \frac{1}{n}(u, w),$$

(5.13)

$$\|w\|\int_{S_n}(u, t)\overline{\operatorname{sgn}(w, t)}\,\mu_n(dt) = \frac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+\frac{1}{2})}(u, w),$$

where $u, w\in l_n^2$ and Γ is the gamma function.

Step 2. Consider the two operators P_n and Q_n acting in the space of all complex-valued μ_n-square-integrable functions $L^2(S_n, \mu_n; C)$ defined by the formulas

$$(5.14)\qquad (P_n f)(s) = n\int_{S_n} f(t)\overline{(s, t)}\,\mu_n(dt),\qquad f\in L^2(S_n, \mu_n; C),$$

$$(5.15)\qquad (Q_n f) = \int_{S_n} f(t)\overline{\operatorname{sgn}(s, t)}\,\mu_n(dt),\qquad f\in L^2(S_n, \mu_n; C).$$

Following the proof of Lemma 4 in [24] and using relations (5.13) one can show that for every $f\in L^2(S_n, \mu_n; C)$

$$(Q_n^2 f, f)\geqslant C_n^2(P_n f, f)$$

where $C_n = \dfrac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+\frac{1}{2})}$.

Step 3. Let H be a separable Hilbert space, let $L^2(S_n, \mu_n; H)$ denote the Hilbert space of all strongly measurable μ_n-Bochner square-integrable functions on S_n with values in H (see [10], 3.5 and 3.8) and let P_n and Q_n be the operators acting in $L^2(S_n, \mu_n; H)$ defined by the formulas (5.14) and (5.15) with $f\in L^2(S_n, \mu_n; C)$ replaced by $f\in L^2(S_n, \mu_n; H)$. Then using the fact that $f(t) = \sum_{k=1}^{\infty}(f(t), e_k)e_k$, $f\in L^2(S_n, \mu_n; H)$, where $\{e_k:\ k = 1, 2\ldots\}$ is an orthonormal basis in H and the series converges μ_n-a.e. and in $L^2(S_n, \mu_n; H)$ one can easily verify that

$$(5.16)\qquad (Q_n^2 f, f)\geqslant C_n^2(P_n f, f)\quad \text{for every } f\in L^2(S_n, \mu_n; H).$$

Step 4. Let $n, m\in N$, $t_j\in l_n^2$, $|t_j| = 1$, $x_j\in H$, $j = 1,\ldots,m$, be fixed (one can assume that H is separable, in fact for fixed m one can consider $H^1 = \overline{\operatorname{sp}}\{x_j:\ 1\leqslant j\leqslant m\}$ so one can assume that H is even finite-dimensional). Consider a sequence of functions $f_p\in L^2(S_n, \mu_n; H)$ defined by the formula

$$f_p(t) = \sum_{j=1}^{m} f_{p,j}(t)\,x_j,\qquad p = 1, 2, \ldots,$$

where $f_{p,j}(t) = \dfrac{1}{\mu_n(K_p(t_j))}1_{K_p(t_j)}(t)$, $j = 1,\ldots,m$, and

$$K_p(u) = \{s\in S_n:\ |s-u| < 1/p\},\qquad p\in \boldsymbol{N}.$$

Applying (5.16) to the function f_p we obtain

$$\sum_{i,j=1}^{m}\int_{S_n}\int_{S_n}\overline{(s,t)}\,f_{p,i}(t)\overline{f_{p,j}(s)}\,\mu_n(dt)\,\mu_n(ds)\cdot(x_i,x_j)$$
$$\leqslant\frac{1}{nC_n^2}\sum_{i,j=1}^{m}(x_i,x_j)\cdot\int_{S_n}\int_{S_n}\int_{S_n}\operatorname{sgn}(s,u)\overline{\operatorname{sgn}(s,t)}f_{p,i}(t)\overline{f_{p,j}(u)}\,\mu_n(ds)\,\mu_n(dt)\,\mu_n(du).$$

Since for every bounded measurable complex function g on S_n and every $j=1,\ldots,m$

$$g(t_j)=\lim_{p\to\infty}\int_{S_n}g(t)f_{p,j}(t)\,dt$$

provided g is continuous at t_j, and since $g_s(\cdot)=\operatorname{sgn}(s,\cdot)$ is continuous at t_j, $j=1,\ldots,m$, for μ_n-almost each $s\in S_n$, using the Bounded Convergence Theorem we obtain

$$(5.17)\qquad \sum_{i,j=1}^{m}(t_i,t_j)(x_i,x_j)\leqslant\frac{1}{nC_n^2}\sum_{i,j=1}^{m}\int_{S_n}\operatorname{sgn}(s,t_j)\overline{\operatorname{sgn}(s,t_i)}(x_i,x_j)\,\mu_n(ds)$$
$$=\frac{1}{nC_n^2}\int_{S_n}\Big|\sum_{j=1}^{m}\overline{\operatorname{sgn}(s,t_j)}\,x_j\Big|^2\mu_n(ds).$$

If t_j's do not satisfy the condition $|t_j|=1$, then one can consider $t_j'=t_j/|t_j|$ and $x_j'=|t_j|\,x_j$, $j=1,\ldots,m$ (we ignore those t_j's which are zero). Thus since $1/(nC_n^2)\to 4/\pi$ ([21], p. 106), consideration of (5.17) completes the proof. ∎

5.18. THEOREM. *Let T be a f.a. (w.c.a.) $L(H,K)$-valued measure defined on an algebra (σ-algebra) Σ. If $|||T|||<\infty$, then T is dilatable.*

Proof. We shall show that T satisfies (4.4). Let Δ_j, $j=1,\ldots,N$, be disjoint nonempty subsets in Σ, $x_j^k\in H$, $k=1,\ldots,n$, $j=1,\ldots,N$, and let $e_1,\ldots,e_r$ be an orthonormal basis in $\operatorname{sp}\{x_j^k: j=1,\ldots,N,\ k=1,\ldots,n\}$. Then

$$\sum_{k=1}^{n}\Big|\sum_{j=1}^{N}T(\Delta_j)x_j^k\Big|^2=\sum_{k=1}^{n}\Big|\sum_{j=1}^{N}T(\Delta_j)\sum_{p=1}^{r}(x_j^k,e_p)e_p\Big|^2$$
$$=\sum_{i,j=1}^{N}\sum_{p,q=1}^{r}\big(T(\Delta_i)e_p,\,T(\Delta_j)e_q\big)\sum_{k=1}^{n}(x_i^k,e_p)\overline{(x_j^k,e_q)}$$
$$=\sum_{i,j=1}^{N}\sum_{p,q=1}^{r}\big(T(\Delta_i)e_p,\,T(\Delta_j)e_q\big)(t_{i,p},t_{j,q})_{l_n^2}$$

where $t_{i,p}=\big((x_i^1,e_p),\ldots,(x_i^n,e_p)\big)\in l_n^2$, $i=1,\ldots,N$, $p=1,\ldots,r$. Applying Lemma 5.12 we obtain

$$(5.19)\qquad \sum_{k=1}^{n}\Big|\sum_{j=1}^{N}T(\Delta_j)x_j^k\Big|^2\leqslant C\int_{S_n}\Big|\sum_{j=1}^{N}\sum_{q=1}^{r}\overline{\operatorname{sgn}(s,t_{j,q})}\,|t_{j,q}|\,T(\Delta_j)e_q\Big|^2\mu_n(ds)$$
$$=C\int_{S_n}\Big|\sum_{j=1}^{N}T(\Delta_j)\Big(\sum_{q=1}^{r}\overline{\operatorname{sgn}(s,t_{j,q})}\,|t_{j,q}|\,e_q\Big)\Big|^2\mu_n(ds).$$

Let $y_i(s)=\sum_{p=1}^{r}|t_{i,p}|\,\overline{\operatorname{sgn}(s,t_{i,p})}\,e_p$, $i=1,\ldots,N$. Then

$$|y_i(s)|^2\leqslant\sum_{p=1}^{r}|t_{i,p}|^2=\sum_{p=1}^{r}\sum_{k=1}^{n}|(x_i^k,e_p)|^2=\sum_{k=1}^{n}|x_i^k|^2$$

and

$$\Big|\sum_{j=1}^{N}T(\Delta_j)y_j(s)\Big|^2\leqslant|||T|||^2\sup\Big\{\sum_{k=1}^{n}|x_j^k|^2:\ j=1,\ldots,N\Big\}.$$

Thus from (5.19) we obtain

$$\sum_{k=1}^{n}\Big|\sum_{j=1}^{N}T(\Delta_j)x_j^k\Big|\leqslant C\,|||T|||\sup\Big\{\sum_{k=1}^{n}|x_j^k|^2:\ j=1,\ldots,N\Big\}$$
$$=C\,|||T|||\sum_{j=1}^{N}\sum_{k=1}^{n}\big(I\delta_{j_0}(\Delta_j)x_j^k,\,x_j^k\big)$$
$$\leqslant C\,|||T|||\sup\Big\{\Big|\sum_{j=1}^{N}\sum_{k=1}^{n}\big(F(\Delta_j)x_j^k,\,x_j^k\big)\Big|:\ F\in\mathscr{M}(L(H)),\ \|F\|\leqslant 1\Big\}$$

where $I\in L(H)$ is the identity, $j_0\in N$ is such that

$$\sum_{k=1}^{n}|x_{j_0}^k|^2=\sup\Big\{\sum_{k=1}^{n}|x_j^k|^2:\ j=1,\ldots,N\Big\},$$

and δ_{j_0} is a positive measure on Σ such that $\delta_{j_0}(\Delta_{j_0})=1$ and $\delta_{j_0}(\Omega\setminus\Delta_{j_0})=0$. From Th. 4.3 it follows that T is dilatable. ∎

The example given in Remark 2.12 shows that $|||T|||<\infty$ is not necessary for T to have a spectral dilation.

The inspiration for the following theorem comes partially from [13], Th. 4.4, and partially from the fact that any selfadjoint T is dilatable if and only if T has a Jordan decomposition, which is a fact already observed by Rosenberg [26], Cor. 2.10.

5.20. THEOREM. *Let T be a f.a. (w.c.a.) $L(H, K)$-valued measure defined on an algebra (σ-algebra) Σ. If $T(\Delta) = T(\Delta)^*$ for all $\Delta \in \Sigma$ and*

$$V(T) = \sup\{|\sum_{\Delta\in\pi} \sqrt{(T(\Delta))^2}|:\ \pi \in \Pi(\Omega)\} < \infty,$$

then T is dilatable.

Proof. We shall prove that T satisfies (4.4). Let $\Delta_1, \ldots, \Delta_N \in \Sigma$ be disjoint nonempty subsets of Ω. For each $k = 1, \ldots, N$ let us define

$$U(\Delta_k) = \sqrt{T(\Delta_k)^2} = \int_{\boldsymbol{R}} |\lambda|\, dE_k(\lambda) \quad \text{and}$$

$$M(\Delta_k) = U(\Delta_k) - T(\Delta_k) = \int_{\boldsymbol{R}} (|\lambda| - \lambda)\, dE_k(\lambda)$$

where $E_k(\lambda)$ is the spectral decomposition of $T(\Delta_k)$, $k = 1, \ldots, N$ ([28], p. 309). Then $U(\Delta_k) \geqslant 0$,

$$0 \leqslant M(\Delta_k) \leqslant 2U(\Delta_k)$$

and $T(\Delta_k) = U(\Delta_k) - M(\Delta_k)$ for each $k = 1, \ldots, N$. Let

$$\Omega_N = \bigcup_{k=1}^{N} \Delta_k, \qquad \Sigma_N = \{\bigcup_{j\in I} \Delta_j:\ I \subset \{1, \ldots, N\}\}.$$

For any $\Delta = \bigcup_{j\in I} \Delta_j \in \Sigma_N$, $I \subset \{1, \ldots, N\}$, we define

$$U(\Delta) = \sum_{j\in I} U(\Delta_j), \qquad M(\Delta) = \sum_{j\in I} M(\Delta_j).$$

Since U and M are $L^+(H)$-valued measures defined on (Ω_N, Σ_N), there exist Hilbert spaces $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, spectral measures E_1 in $\boldsymbol{K}_1$ and E_2 in $\boldsymbol{K}_2$ and bounded linear operators $R \in L(H, \boldsymbol{K}_1)$ and $S \in L(H, \boldsymbol{K}_2)$ such that for every $\Delta \in \Sigma_N$

$$U(\Delta) = R^* E_1(\Delta) R, \qquad M(\Delta) = S^* E_2(\Delta) S.$$

Note that

$$|R|^2 = |R^* R| = |U(\Omega_N)| = |\sum_{k=1}^{N} \sqrt{T(\Delta_k)^2}| \leqslant V(T) \quad \text{and}$$

$$|S|^2 = |S^* S| = |M(\Omega_N)| \leqslant 2|U(\Omega_N)| \leqslant 2V(T).$$

Let $\omega_k \in \Delta_k$, $k = 1, \ldots, N$, and let for each $\Delta \in \Sigma$,

$$F(\Delta) = \frac{1}{3V(T)} \sum_{k=1}^{N} [R^*, S^*] \begin{bmatrix} E_1(\Delta_k) & 0 \\ 0 & E_2(\Delta_k) \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} \cdot \delta_{\omega_k}(\Delta)$$

where $\delta_\omega(\Delta) = 1$ if $\omega \in \Delta$ and 0 otherwise, $\begin{bmatrix} R \\ S \end{bmatrix}$ denotes the bounded operator from H into $\boldsymbol{K}_1 \oplus \boldsymbol{K}_2$ defined by the formula

$$\begin{bmatrix} R \\ S \end{bmatrix} x = \begin{pmatrix} Rx \\ Sx \end{pmatrix}$$

and

$$[R^*, S^*] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R^* x_1 + S^* x_2, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_1 \oplus K_2.$$

Then F is an $L^+(H)$-valued w.c.a. measure on (Ω, Σ),

$$\|F\| = |F(\Omega)| = \frac{1}{3V(T)} \left| [R^*, S^*] \begin{bmatrix} I_{\boldsymbol{K}_1} & 0 \\ 0 & I_{\boldsymbol{K}_2} \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} \right|$$

$$= \frac{1}{3V(T)} \left\| \begin{bmatrix} R \\ S \end{bmatrix} \right\|^2 \leqslant \frac{1}{3V(T)} (|R|^2 + |S|^2) \leqslant 1,$$

and for arbitrary $x_1^k, \ldots, x_N^k \in H$, $k = 1, \ldots, n$,

$$\sum_{k=1}^{n} |\sum_{j=1}^{N} T(\Delta_j) x_j^k|^2 = \sum_{k=1}^{n} \left| \sum_{j=1}^{N} [R^*, -S^*] \begin{bmatrix} E_1(\Delta_j) & 0 \\ 0 & E_2(\Delta_j) \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} x_j^k \right|^2$$

$$\leqslant |[R^*, -S^*]|^2 \sum_{k=1}^{n} \sum_{j=1}^{N} \left| \begin{bmatrix} E_1(\Delta_j) & 0 \\ 0 & E_2(\Delta_j) \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} x_j^k \right|^2$$

$$\leqslant 3V(T) \cdot 3V(T) \cdot \sum_{j=1}^{N} \sum_{k=1}^{n} (F(\Delta_j) x_j^k, x_j^k),$$

since $\begin{bmatrix} E_1(\cdot) & 0 \\ 0 & E_2(\cdot) \end{bmatrix}$ is a spectral measure on Σ_N. Thus (4.4) holds with $C = 9V(T)^2$. ∎

6. Application to harmonizable processes. In this section $\boldsymbol{R}$ is the real line and $\mathscr{B}$ is the σ-algebra of Borel subsets of $\boldsymbol{R}$. Let $\mathscr{B} \times \mathscr{B} = \{\Delta \times \Delta' \subset \boldsymbol{R} \times \boldsymbol{R}:\ \Delta, \Delta' \in \mathscr{B}\}$.

6.1. DEFINITION. $\nu(\ ,\ )$: $\mathscr{B} \times \mathscr{B} \to \boldsymbol{C}$ is said to be a *bimeasure* if it is separately countably additive in each variable, i.e. for all $\Delta, \Delta' \in \mathscr{B}$, $\nu(\cdot, \Delta')$ and $\nu(\Delta, \cdot)$ are complex measures.

For a discussion of bimeasures and related material see Niemi [18a] and Rao [23] and references therein.

6.2. LEMMA. *Suppose that there exist a Hilbert space K and K-valued measures ζ, η defined on $\mathscr{B}$ such that $\nu(\Delta, \Delta') = (\eta(\Delta), \zeta(\Delta))$ for all $\Delta, \Delta' \in \mathscr{B}$. Then for any two bounded $\mathscr{B}$-measurable scalar-valued functions f, g on $\boldsymbol{R}$ the* *iterated integrals*

$$\int f(s)\int g(t)\,\nu(dt, ds) \quad \text{and} \quad \int g(t)\int f(s)\,\nu(dt, ds)$$

exist and are equal.

Proof. Note that by [6], IV 10.8(f),

$$\int g(t)\,\nu(dt,\cdot) = \left(\int g(t)\,\eta(dt),\ \zeta(\cdot)\right), \qquad \int f(s)\,\nu(\cdot, ds) = \left(\eta(\cdot),\ \int \bar{f}(s)\,\zeta(ds)\right)$$

are complex measures. Thus both iterated integrals exist and

$$\int f(s)\int g(t)\,\nu(dt, ds) = \left(\int g(t)\,\eta(dt),\ \int \bar{f}(s)\,\zeta(ds)\right) = \int g(t)\int f(s)\,\nu(dt, ds). \ \blacksquare$$

6.3. Definition. If the conclusion of Lemma 6.2 is satisfied then by

$$\iint f(s)\,g(t)\,\nu(dt, ds)$$

we will denote either one of the iterated integrals $\int f(s)\int g(t)\,\nu(dt, ds)$, $\int g(t)\int f(s)\,\nu(dt, ds)$.

6.4. Remark. One can extend the function $\nu(\cdot,\cdot)$ to the algebra $\mathscr{A}(\mathscr{B}\times\mathscr{B})$ generated by the rectangles $\mathscr{B}\times\mathscr{B}$. Denote the extension by $\tilde{\nu}$. $\tilde{\nu}$ is a f.a. measure on $\mathscr{A}(\mathscr{B}\times\mathscr{B})$. If $\tilde{\nu}$ is bounded on $\mathscr{A}(\mathscr{B}\times\mathscr{B})$ then it is easy to show that for any two $\mathscr{B}$-measurable scalar functions f and g, $f(s)g(t)$ is integrable with respect to $\tilde{\nu}$ in the sense of [6], III.2, and the Dunford–Schwartz integral

$$\int_{R^2} f(s)\,g(t)\,\tilde{\nu}(dt\times ds)$$

and the integral $\iint f(s)\,g(t)\,\nu(dt, ds)$ introduced in (6.3) coincide.

6.5. Definition. A function $\alpha(\ ,\ )\colon \mathscr{B}\times\mathscr{B}\to C$ is said to be *positive-definite* (abbreviated as p.d.) if

$$\sum_{i,j=1}^{m} \alpha_i\bar{\alpha}_j\,\nu(\Delta_i, \Delta_j) \geqslant 0$$

for all $\alpha_1,\ldots,\alpha_n\in C$, $n\in N$.

By the Aronszajn Theorem (see e.g. [14]) a function $\nu\colon \mathscr{B}\times\mathscr{B}\to C$ is p.d. and additive if and only if there exist a Hilbert space K and a K-valued f.a. measure ζ on $\mathscr{B}$ such that

$$\nu(\Delta, \Delta') = (\zeta(\Delta), \zeta(\Delta')) \tag{6.6}$$

for all $\Delta, \Delta'\in\mathscr{B}$.

6.7. Lemma. *Suppose that ν is a p.d. function which is separately additive. Then the following are equivalent:*

(i) *There exists a K-valued measure ζ such that* (6.6) *holds.*

(ii) *ν is a bimeasure and* $\sup\{\nu(\Delta, \Delta)\colon \Delta\in\mathscr{B}\} = C < \infty$.

(iii) *ν is a bimeasure and*

$$\sup\Big\{\sum_{i,j=1}^{m}\alpha_i\bar{\alpha}_j\,\nu(\Delta_i, \Delta_j)\colon \alpha_i\in C,\ |\alpha_i|\leqslant 1, \text{ and } \Delta_i\text{'s are disjoint}\Big\} = C' < \infty.$$

(iv) *ν is a bimeasure and*

$$\sup\Big\{\Big|\sum_{i,j=1}^{m}\alpha_i\beta_j\,\nu(\Delta_i, \Delta_j')\Big|\colon |\alpha_i|\leqslant 1,\ |\beta_j|\leqslant 1,\ \alpha_i, \beta_j\in C,\ \Delta_i\text{'s are disjoint and so are } \Delta_j'\text{'s}\Big\} = C'' < \infty.$$

Proof. (ii)$\Rightarrow$(i). Using Aronszajn's Theorem mentioned just before Lemma 6.7, there exist a Hilbert space K and a K-valued function ζ such that $\nu(\Delta, \Delta') = (\zeta(\Delta), \zeta(\Delta'))$ for all $\Delta, \Delta'\in\mathscr{B}$. One can assume that $K = \overline{\mathrm{sp}}\,\{\zeta(\Delta)\colon \Delta\in\mathscr{B}\}$. Since ν is a bimeasure, the function

$$\Big(\zeta(\cdot),\ \sum_{j=1}^{n}\alpha_j\zeta(\Delta_j)\Big) = \sum_{j=1}^{n}\alpha_j\,\nu(\cdot, \Delta_j)$$

is countably additive for all $\alpha_1,\ldots,\alpha_n\in C$, $\Delta_1,\ldots,\Delta_n\in\Sigma$. Let $K_0 = \mathrm{sp}\,\{\zeta(\Delta)\colon \Delta\in\mathscr{B}\}$, $x\in K$, $\delta>0$ and $\Delta_n\in\Sigma$, $\Delta_n\searrow\emptyset$. Let $x_\delta\in K_0$ be such that $|x-x_\delta| < \delta/\sqrt{C}$. Then

$$\begin{aligned}|(\zeta(\Delta_n), x)| &\leqslant |(\zeta(\Delta_n), x_\delta)| + |(\zeta(\Delta_n), x-x_\delta)|\\ &\leqslant |(\zeta(\Delta_n), x_\delta)| + |\zeta(\Delta_n)|\,|x-x_\delta|\\ &\leqslant |(\zeta(\Delta_n), x_\delta)| + \sqrt{\sup_n|\zeta(\Delta_n)|^2}\cdot\frac{\delta}{\sqrt{C}}\\ &\leqslant |(\zeta(\Delta_n), x_\delta)| + \sqrt{C}\cdot\frac{\delta}{\sqrt{C}} \leqslant 2\delta\end{aligned}$$

for sufficiently large n, because $(\zeta(\cdot), x_\delta)$ is countably additive. Thus $\zeta(\cdot)$ is weakly c.a., so, by [6], IV.10.1, it is a measure.

(i)$\Rightarrow$(iv). For any fixed $\Delta, \Delta'\in\mathscr{B}$ the set functions $\nu(\cdot, \Delta') = (\zeta(\cdot), \zeta(\Delta'))$ and $\nu(\Delta,\cdot) = (\zeta(\Delta), \zeta(\cdot))$ are countably additive, so ν is a bimeasure. If $\alpha_1,\ldots,\alpha_n\in C$, $|\alpha_i|\leqslant 1$, $\beta_1,\ldots,\beta_n\in C$, $|\beta_i|\leqslant 1$, $\Delta_1,\ldots,\Delta_n\in\mathscr{B}$, $\Delta_i\cap\Delta_j=\emptyset$ provided $i\neq j$ and $\Delta_1',\ldots,\Delta_n'\in\mathscr{B}$, $\Delta_i'\cap\Delta_j'=\emptyset$ provided $i\neq j$, then

$$\begin{aligned}\Big|\sum_{i,j=1}^{n}\alpha_i\beta_j\,\nu(\Delta_i, \Delta_j')\Big| &= \Big|\Big(\sum_{i=1}^{n}\alpha_i\zeta(\Delta_i),\ \sum_{j=1}^{n}\bar{\beta}_j\zeta(\Delta_j')\Big)\Big|\\ &\leqslant \Big|\sum\alpha_i\zeta(\Delta_i)\Big|\cdot\Big|\sum\bar{\beta}_j\zeta(\Delta_j')\Big| \leqslant \|\zeta\|^2\to\infty,\end{aligned}$$

by Lemma 2.9 (see also [6], IV.10.4). The implications (iv)$\Rightarrow$(iii) and (iii)$\Rightarrow$(ii) are obvious. Note that $C\leqslant C' = C'' = \|\zeta\|^2\leqslant 4C$. $\blacksquare$

6.8. Remark. From Lemmas 6.7 and 6.2 it follows that if ν is a p.d. bimeasure then for all bounded scalar $\mathscr{B}$-measurable functions the integral

$$\iint f(s)g(t)\nu(dt, ds)$$

is well defined.

6.9. Definition. Let K be a Hilbert space. By a K-*valued stochastic process* we will mean any function $(x_t)_{t\in\boldsymbol{R}}$ from $\boldsymbol{R}$ into K.

A K-valued stochastic process $(x_t)_{t\in\boldsymbol{R}}$ is said to be:

(a) *harmonizable* ([27]) if its correlation function $\gamma(t, s) = (x_t, x_s)$, $t, s\in\boldsymbol{R}$, admits the representation

$$\gamma(t, s) = \iint e^{-i(tv-su)}\nu(dv, du)$$

where ν is a p.d. bimeasure such that $\sup\{\sum_{i,j=1}^{n} \alpha_i\bar{\alpha}_j\nu(\Delta_i, \Delta_j)\colon \alpha_i\in\boldsymbol{C}, |\alpha_i|\leqslant 1$, Δ_i's are disjoint in $\mathscr{B}\} < \infty$;

(b) *V-bounded* ([17]) if it is continuous and there exists a constant C such that for every $\varphi\in L^1(\boldsymbol{R}, \boldsymbol{C})$

$$\left|\int x_t\varphi(t)\,dt\right| \leqslant C|\hat{\varphi}|_{\sup},$$

where the integral on the left-hand side is in the sense of Bochner ([10], 3.7), $L^1(\boldsymbol{R}, \boldsymbol{C})$ is the space of Lebesgue integrable functions, and for every $\varphi\in L^1(\boldsymbol{R}, \boldsymbol{C})$, $\hat{\varphi}(t) = \int e^{-its}\varphi(s)\,ds$ is the Fourier transform of φ;

(c) *stationary* if its correlation function satisfies $\gamma(t, s) = (x_t, x_s) = \gamma(t-s, 0)$, $t, s\in\boldsymbol{R}$, and it is continuous.

The following result is well known. Because of its importance and because of its repeated use in this section we state it here as a theorem.

6.10. Theorem ([18], [16]). *For any K-valued stochastic process $(x_t)_{t\in\boldsymbol{R}}$ the following are equivalent*:

(i) $(x_t)_{t\in\boldsymbol{R}}$ *is harmonizable.*

(ii) $(x_t)_{t\in\boldsymbol{R}}$ *is V-bounded.*

(iii) *There exists a K-valued measure ζ on $\mathscr{B}$ such that for every $t\in\boldsymbol{R}$*

$$x_t = \int e^{-its}\zeta(ds).$$

(iv) *There exist a Hilbert space $\boldsymbol{K}$, an isometry J from K into $\boldsymbol{K}$ and a $\boldsymbol{K}$-valued stationary process $(y_t)_{t\in\boldsymbol{R}}$ such that for every $t\in\boldsymbol{R}$*

$$x_t = J^* y_t.$$

(See [1] & [2] for a stronger version of harmonizability and its connection with Th. 6.10.)

The purpose of this section is to establish the relations between harmonizability, V-boundedness and stationary dilations for operator-valued processes using the results of earlier sections.

The definition of $L(H, K)$-valued stationary processes (see for instance [7], [4], [15]) suggests the following definition.

6.11. Definition. Let H and K be Hilbert spaces. By an $L(H, K)$-*valued stochastic process* we will mean a function $(X_t)_{t\in\boldsymbol{R}}$ from $\boldsymbol{R}$ into $L(H, K)$. An $L(H, K)$-valued stochastic process $(X_t)_{t\in\boldsymbol{R}}$ is said to be:

(a) *weakly harmonizable* if for each $x\in H$ the K-valued process $(X_t x)_{t\in\boldsymbol{R}}$ is harmonizable (in the sense of 6.9(a));

(b) *stationary* if for each $x\in H$ the K-valued process $(X_t x)_{t\in\boldsymbol{R}}$ is stationary.

As it turns out, see theorem below, this definition of harmonizability, although natural, is too weak to ensure the existence of a stationary dilation. However, as Th. 6.17 and 6.20 suggest, an additional condition may be imposed to guarantee the existence of stationary dilations of weakly harmonizable processes.

6.12. Theorem. (A) *Let $(X_t)_{t\in\boldsymbol{R}}$ be an $L(H, K)$-valued stochastic process and let $\Gamma(t, s) = X_s^* X_t$ be its correlation function. The following are equivalent*:

(i) $(X_t)_{t\in\boldsymbol{R}}$ *is weakly harmonizable.*

(ii) *For every $x\in H$ the K-valued stochastic process $(X_t x)_{t\in\boldsymbol{R}}$ is V-bounded (in the sense of Def.* 6.9(b)).

(iii) *There exists a function $F\colon \mathscr{B}\times\mathscr{B}\to L(H)$ such that*

(a) *For all fixed $\Delta, \Delta'\in\mathscr{B}$, $F(\cdot, \Delta')$ and $F(\Delta, \cdot)$ are $L(H)$-valued w.c.a. measures.*

(b) $\sum_{i,j=1}^{n} \alpha_i\bar{\alpha}_j F(\Delta_i, \Delta_j) \geqslant 0$ *for all $\alpha_1, \ldots, \alpha_n\in\boldsymbol{C}$ and disjoint sets $\Delta_1, \ldots, \Delta_n\in\mathscr{B}$.*

(c) $\sup\{|F(\Delta, \Delta)|\colon \Delta\in\mathscr{B}\} < \infty$.

(d) *For all $x\in H$ and $t, s\in\boldsymbol{R}$*

$$(\Gamma(t, s)x, x) = \iint e^{-i(tv-su)}(F(dv, du)x, x)$$

(see Remark 6.8).

(iv) *There exists a w.c.a. $L(H, K)$-valued measure Z such that for every $t\in\boldsymbol{R}$*

$$X_t = \int e^{-its} Z(ds)$$

(for the definition of this integral see the paragraph following Remark 2.21).

(B) *There exist infinite-dimensional Hilbert spaces H, K and a weakly harmonizable $L(H, K)$-valued process $(X_t)_{t\in\boldsymbol{R}}$ such that $(X_t)_{t\in\boldsymbol{R}}$ does not admit the factorization*

$$X_t = RY_t, \quad t\in\boldsymbol{R},$$

for any Hilbert space $\boldsymbol{K}$, $R\in L(\boldsymbol{K}, K)$ and a stationary $L(H, \boldsymbol{K})$-valued process $(Y_t)_{t\in\boldsymbol{R}}$.

Proof. (A). (i) and (ii) are equivalent by Th. 6.10.

(ii) $\Rightarrow$ (iv). Suppose that for every $x \in H$, $X_t x$, $t \in \boldsymbol{R}$, is V-bounded. Then from Th. 6.10 it follows that there exists a K-valued measure ζ_x such that $X_t x = \int e^{-its} \zeta_x(ds)$, $t \in \boldsymbol{R}$. By [11], Lemma 2 and Th. 2,

$$\|\zeta_x\| = \sup\{|\int \varphi(s) X_s x\, ds|:\ \varphi \in L^1(\boldsymbol{R}, \boldsymbol{C}),\ |\hat{\varphi}|_{\sup} \leqslant 1\} \leqslant C_x < \infty.$$

Consider the family of continuous linear operators from H into K defined by the formula

$$T_\varphi x = \int_{\boldsymbol{R}} \varphi(s) X_s x\, ds, \qquad \varphi \in L^1(\boldsymbol{R}, \boldsymbol{C}),\ |\hat{\varphi}|_{\sup} \leqslant 1$$

(note that from the definition of V-boundedness it follows that $X_t x$ is bounded for each $x \in H$ so, by the Banach–Steinhaus Theorem, [28], Th. 2.5, $\sup\{|X_t|:\ t \in \boldsymbol{R}\} < \infty$). Since for every $x \in H$ the orbit

$$\{T_\varphi x:\ \varphi \in L^1(\boldsymbol{R}, \boldsymbol{C}),\ |\hat{\varphi}|_{\sup} \leqslant 1\}$$

is bounded, again by the Banach–Steinhaus Theorem it follows that

$$\sup\{\|\zeta_x\|:\ x \in H,\ \|x\| \leqslant 1\} \leqslant \sup\{|T_\varphi|:\ \varphi \in L^1(\boldsymbol{R}, \boldsymbol{C}),\ |\hat{\varphi}|_{\sup} \leqslant 1\} = C < \infty.$$

Thus $\|\zeta_x\| \leqslant C|x|$ for all $x \in H$. For each $\Delta \in \mathscr{B}$ we define $Z(\Delta)x = \zeta_x(\Delta)$, $x \in H$. Since ζ_x is uniquely determined and since for all $\Delta \in \mathscr{B}$ and $x \in H$, $|Z(\Delta)x| = |\zeta_x(\Delta)| \leqslant \|\zeta_x\| \leqslant C|x|$, $Z(\Delta)$ is a w.c.a. measure on $\mathscr{B}$ and

$$X_t x = \int e^{-its} Z(ds) x, \qquad t \in \boldsymbol{R},\ x \in H.$$

(iv) $\Rightarrow$ (iii). Let $F(\Delta, \Delta') = Z(\Delta')^* Z(\Delta)$, $\Delta, \Delta' \in \mathscr{B}$. Then $|F(\Delta, \Delta)| = |Z(\Delta)|^2 \leqslant \|Z\|^2 < \infty$. Obviously F satisfies (a), (b) and (d).

(iii) $\Rightarrow$ (i). Trivial by Lemma 6.7.

(B). Let H and the w.c.a. $L(H)$-valued measure T be as in (3.4). We consider T as a w.c.a. measure defined on $\mathscr{B}$ and concentrated on the set of positive integers. Let

$$X_t = \int e^{-its} T(ds), \qquad t \in \boldsymbol{R}.$$

By part (A), X_t is a weakly harmonizable $L(H)$-valued process. Suppose that $X_t = RY_t$, $t \in \boldsymbol{R}$, where $R \in L(K, H)$, Y_t is a stationary $L(H, K)$-valued process and K is a Hilbert space. From [4], [15], it follows that

$$Y_t = \int e^{-its} E(ds) Y_0, \qquad t \in \boldsymbol{R},$$

where E is a spectral measure in $M(Y) = \overline{\operatorname{sp}}\{Y_t x:\ t \in \boldsymbol{R},\ x \in H\} \subset \boldsymbol{K}$. Hence

$$X_t = \int e^{-its} T(ds) = \int e^{-its} R E(ds) Y_0, \qquad t \in \boldsymbol{R}.$$

Since all measures on $\mathscr{B}$ are regular and the Fourier transform uniquely determines a measure,

$$T(\Delta) = (RP_{M(Y)}) E(\Delta) Y_0, \qquad \Delta \in \mathscr{B},$$

so T has a spectral dilation, which is in contradiction with 3.4. ∎

6.13. Remark. Note that condition (iii)(c) in Th. 6.12(A) may be replaced by

(c′) $\sup\{|\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j F(\Delta_i, \Delta_j)|:\ |\alpha_j| \leqslant 1,\ \Delta_j\text{'s are disjoint},\ \Delta_j \in \mathscr{B}\} < \infty.$

In fact from the proof of (iv) $\Rightarrow$ (iii) in Th. 6.12(A) it follows that:

(c′) $\sup\{|\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j F(\Delta_i, \Delta_j)|:\ |\alpha_j| \leqslant 1,\ \Delta_j\text{'s are disjoint in } \mathscr{B}\} \leqslant \|Z\|^2 < \infty.$

(d′) *For all $x, y \in H$ the integral*

$$\iint e^{-i(tv - su)} (F(dv, du)x, y)$$

is well defined (see Lemma 6.2 and Def. 6.3) *and is equal to* $(\Gamma(t, s)x, y)$.

The notion of a V-bounded $L(H, K)$-valued stochastic process can be defined pointwise in the same way as a weakly harmonizable $L(H, K)$-valued process. But the following definition seems to be more natural.

6.14. Definition. An $L(H, K)$-valued stochastic process $(X_t)_{t \in \boldsymbol{R}}$ is said to be *V-bounded* if for every $x \in H$ the function $\boldsymbol{R} \ni t \to X_t x \in K$ is continuous and there exists a constant $C < \infty$ such that for every $\varphi \in L^1(\boldsymbol{R}, H)$

$$|\int X_t \varphi(t)\, dt| \leqslant C |\hat{\varphi}|_{\sup},$$

where $L^1(\boldsymbol{R}, H)$ is the space of all H-valued Bochner integrable functions w.r.t. the Lebesgue measure, $\hat{\varphi}(s) = \int e^{-its} \varphi(t)\, dt$, $\varphi \in L^1(\boldsymbol{R}, H)$, and the integrals are in the sense of Bochner.

6.15. Remark. We first note that $X_t \varphi(t)$ is separably valued and weakly measurable for each $\varphi \in L^1(\boldsymbol{R}, H)$. So $X_t \varphi(t)$ is strongly measurable. Additionally, by the assumption in Def. 6.14, $X_t \varphi(t)$ is Bochner integrable, i.e. $\int |X_t \varphi(t)|\, dt < \infty$ for all $\varphi \in L^1(\boldsymbol{R}, H)$ ([10], 3.7.4). Setting $\varphi(t) = f(t)x$, $x \in H$, $f \in L^1(\boldsymbol{R}, \boldsymbol{C})$, we obtain

$$\int |X_t x|\, |f(t)|\, dt < \infty \quad \text{for all } f \in L^1(\boldsymbol{R}, \boldsymbol{C}).$$

Thus $\sup\{|X_t x|:\ t \in \boldsymbol{R}\} < \infty$ and by the Banach–Steinhaus Theorem $\sup\{|X_t|:\ t \in \boldsymbol{R}\} < \infty$.

6.16. Remark. Since the functions $\varphi(t) = \sum_{k=1}^{n} f_k(t) x_k$, $x_k \in H$, $f_k \in L^1(\boldsymbol{R}, \boldsymbol{C})$, form a dense subset of $L^1(\boldsymbol{R}, H)$, and the Fourier transforms of

$L^1(\boldsymbol{R}, \boldsymbol{C})$-functions are dense in $C_0(\boldsymbol{R}, \boldsymbol{C})$, and since for every $\varphi \in L^1(\boldsymbol{R}, H)$

$$|\hat{\varphi}|_{\sup} \leqslant |\varphi|_{L^1},$$

it follows that the set $\{\hat{\varphi}\colon \varphi \in L^1(\boldsymbol{R}, H)\}$ is dense in $C_0(\boldsymbol{R}, H)$.

6.17. THEOREM. *Let $(X_t)_{t\in\boldsymbol{R}}$ be an $L(H, K)$-valued stochastic process and let $\Gamma(t, s) = X_s^* X_t$ be its correlation function. Then the following are equivalent:*

(i) *$(X_t)_{t\in\boldsymbol{R}}$ is V-bounded (in the sense of 6.14).*

(ii) *There exists an $L(H, K)$-valued measure Z with $|||Z||| < \infty$ (see Def. 5.10) such that for every $t \in \boldsymbol{R}$*

$$X_t = \int e^{-its} Z(ds).$$

(iii) *There exists a function $F\colon \mathscr{B} \times \mathscr{B} \to L(H)$ such that*

(a) *For all fixed $\Delta, \Delta' \in \mathscr{B}$, $F(\cdot, \Delta')$ and $F(\Delta, \cdot)$ are w.c.a. $L(H)$-valued measures.*

(b) $\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j F(\Delta_i, \Delta_j) \geqslant 0$ *for all $|\alpha_j| \leqslant 1$, $\alpha_j \in \boldsymbol{C}$, and disjoint $\Delta_j \in \mathscr{B}$, $j = 1, \ldots, n$, $n \in \boldsymbol{N}$.*

(c) $\sup\{\sum_{i,j=1}^{n} (F(\Delta_i, \Delta_j) x_i, x_j)\colon x_j \in H,\ |x_j| \leqslant 1,\ \Delta_j$*'s are disjoint sets in $\mathscr{B}\} < \infty$.*

(d) *For all $x \in H$ and $t, s \in \boldsymbol{R}$*

$$(\Gamma(t, s)x, x) = \iint e^{-i(tv-su)} (F(dv, du)x, y)$$

(cf. Th. 6.12(A) and Remark 6.13).

Proof. (i) $\Rightarrow$ (ii). Suppose that $(X_t)_{t\in\boldsymbol{R}}$ is V-bounded. Then setting $\varphi(t) = f(t)x$, $f \in L^1(\boldsymbol{R}, \boldsymbol{C})$, $x \in H$, we obtain

$$\Big|\int X_t x f(t)\,dt\Big| \leqslant C|x|\,|\hat{f}|_{\sup},$$

thus $(X_t x)_{t\in\boldsymbol{R}}$ is a K-valued V-bounded stochastic process for every $x \in H$. From Th. 6.12(A) it follows that $X_t = \int e^{-its} Z(ds)$, $t \in \boldsymbol{R}$, for some w.c.a. $L(H, K)$-valued measure Z on $\mathscr{B}$. We shall prove that $|||Z||| < \infty$. Let

$$A = \Big\{\varphi \in C_0(\boldsymbol{R}, H)\colon \varphi(t) = \sum_{k=1}^{n} g_k(t) x_k,\ g_k \in C_0(\boldsymbol{R}, \boldsymbol{C}),\ x_k \in H\Big\}, \text{ and}$$

$$B = \Big\{\varphi \in C_0(\boldsymbol{R}, H)\colon \varphi(t) = \sum_{k=1}^{n} \hat{f}_k(t) x_k,\ f_k \in L^1(\boldsymbol{R}, \boldsymbol{C}),\ x_k \in H\Big\}.$$

Define the operator B_Z from B into K by the formula

$$B_Z\Big(\sum_{k=1}^{n} \hat{f}_k(t) x_k\Big) = \sum_{k=1}^{n} \int \hat{f}_k(t) Z(dt) x_k.$$

Since for every $y \in K$

$$\begin{aligned}
\Big(B_Z\Big(\sum_{k=1}^{n} \hat{f}_k(t) x_k\Big), y\Big) &= \sum_{k=1}^{n} \int \hat{f}_k(t) (Z(dt) x_k, y) \\
&= \sum_{k=1}^{n} \iint e^{-its} f_k(s)\,ds\,(Z(dt) x_k, y) \\
&= \sum_{k=1}^{n} \int f_k(s) \int e^{-ist} (Z(dt) x_k, y)\,ds \\
&= \sum_{k=1}^{n} \int f_k(s) \Big(\int e^{-ist} Z(dt) x_k, y\Big)\,ds \\
&= \int \Big(X_s\Big(\sum_{k=1}^{n} f_k(s) x_k\Big), y\Big)\,ds \\
&= \Big(\int X_s\Big(\sum_{k=1}^{n} f_k(s) x_k\Big)\,ds, y\Big)
\end{aligned}$$

(we have repeatedly used (2.23) and [10], Th. 3.7.12),

$$\Big|B_Z\Big(\sum_{k=1}^{n} \hat{f}_k(t) x_k\Big)\Big| = \Big|\int X_t\Big(\sum_{k=1}^{n} f_k(t) x_k\Big)\,dt\Big| \leqslant C\Big|\sum_{k=1}^{n} \hat{f}_k(t) x_k\Big|_{\sup}.$$

Thus B_Z is well defined with $|B_Z| \leqslant C$. B_Z can be extended to a bounded linear operator from $C_0(\boldsymbol{R}, H) = \bar{B}$ into K. We will denote the extension by the same letter B_Z. Let

$$\varphi = \sum_{k=1}^{n} g_k x_k \in A$$

and let

$$|\hat{f}_k^{(m)} - g_k|_{\sup} \to 0 \quad \text{as } m \to \infty$$

for each $k = 1, \ldots, n$. Then from [6], IV.10.8 (or 2.22),

$$B_Z(\hat{f}_k^{(m)} x_k) = \int \hat{f}_k^{(m)}(s) Z(ds) x_k \xrightarrow[m]{} \int g_k(s) Z(ds) x_k, \quad k = 1, \ldots, n.$$

Thus

$$B_Z\Big(\sum_{k=1}^{n} g_k x_k\Big) = \sum_{k=1}^{n} \int g_k(s) Z(ds) x_k,$$

for every $\varphi = \sum_{k=1}^{n} g_k x_k \in A$.

Let $x_1, \ldots, x_n \in H$, $|x_i| \leqslant 1$, and let $K_1, \ldots, K_n$ be disjoint compact subsets of $\boldsymbol{R}$. There exist sequences $\{g_k^{(m)}\}_{m=1}^{\infty} \subset C_0(\boldsymbol{R}, \boldsymbol{C})$, $k = 1, \ldots, n$, such that

(a) $g_i^{(m)}(t) \to 1_{K_i}$ for every $i = 1, \ldots, n$.

(b) $|g_i^{(m)}|_{\sup} \leqslant 1$ for every $i = 1, \ldots, n$ and $m \in N$.

(c) $g_1^{(m)}, \ldots, g_n^{(m)}$ have disjoint supports for every $m \in N$

(e.g., $g_i^{(m)}(t) = \max\left(1 - \frac{m}{\delta}\operatorname{dist}(t, K_i), 0\right)$ where

$$3\delta < \min\{\operatorname{dist}(K_i, K_j):\ i \neq j,\ i, j = 1, \ldots, n\}).$$

From the Dominated Convergence Theorem ([6], IV.10.10) we obtain

$$\Big|\sum_{i=1}^{n} Z(K_i)x_i\Big| = \lim_{m\to\infty}\Big|\sum_{i=1}^{n}\int g_i^{(m)}(t)Z(dt)x_i\Big| = \lim_{m\to\infty}\Big|B_Z\Big(\sum_{i=1}^{n} g_i^{(m)}(t)x_i\Big)\Big|$$

$$\leqslant |B_Z|\sup\Big\{\Big|\sum_{i=1}^{n} g_i^{(m)}x_i\Big|_{\sup}:\ m \in N\Big\} \leqslant C.$$

Since each measure $Z(\cdot)x_i$, $i = 1, \ldots, n$, is regular, we conclude that for all disjoint $\Delta_1, \ldots, \Delta_n \in \mathscr{B}$ and $x_1, \ldots, x_n \in H$ with $|x_i| \leqslant 1$, $i = 1, \ldots, n$,

$$\Big|\sum_{i=1}^{n} Z(\Delta_i)x_i\Big| \leqslant C.$$

Thus $|||Z||| \leqslant C < \infty$.

(ii)$\Rightarrow$(iii). Let $F(\Delta, \Delta') = Z(\Delta')^* Z(\Delta)$. Then for all $x_1, \ldots, x_n \in H$ and disjoint sets $\Delta_1, \ldots, \Delta_n \in \mathscr{B}$

$$\sum_{i,j=1}^{n}(F(\Delta_i, \Delta_j)x_i, x_j) = \Big|\sum Z(\Delta_i)x_i\Big|^2 \leqslant |||Z|||^2 < \infty,$$

provided $|x_i| \leqslant 1$. Statements (a), (b) and (d) are obvious.

(iii)$\Rightarrow$(ii). By Th. 6.12(A), $X_t = \int e^{-its}Z(ds)$ for some w.c.a. $L(H, K)$-valued measure Z. The same argument as in (ii)$\Rightarrow$(iii) shows that $|||Z|||^2 = \sup\{\sum(F(\Delta_i, \Delta_j)x_i, x_j):\ |x_j| \leqslant 1,\ \Delta_j\text{'s are disjoint on } \mathscr{B}\} < \infty$.

(ii)$\Rightarrow$(i). From the paragraph following Def. 5.10 it follows that for every simple H-valued function φ

$$\Big|\int dZ\varphi\Big| \leqslant |||Z|||\,|\varphi|_{\sup}, \tag{6.18}$$

so the integral $\int dZ\varphi$ can be extended to any separable-valued $\mathscr{B}$-measurable bounded function taking values in H. Let us verify that for every $\varphi \in L^1(\boldsymbol{R}, H)$

$$\int Z(ds)\Big(\int \varphi(t)e^{-ist}\,dt\Big) = \int X_t\varphi(t)\,dt. \tag{6.19}$$

Since $|X_t| = \left|\int e^{-its}Z(ds)\right| \leqslant |||Z|||$, both sides of (6.19) are linear and continuous operators on $L^1(\boldsymbol{R}, H)$ so it suffices to prove that (6.19) holds for functions of the form $\varphi(t) = f(t)x$, where $x \in H$, $f \in L^1(\boldsymbol{R}, \boldsymbol{C})$, which obviously follows as was indicated in the proof of the implication (i)$\Rightarrow$(ii). From (6.18) and (6.19) it follows that for every $\varphi \in L^1(\boldsymbol{R}, H)$

$$\Big|\int X_t\varphi(t)\,dt\Big| = \Big|\int dZ\hat{\varphi}\Big| \leqslant |||Z|||\,|\hat{\varphi}|_{\sup},$$

hence $(X_t)_{t\in\boldsymbol{R}}$ is V-bounded. ∎

In view of 6.17 we introduce the following definition.

6.20. Definition. Let $(X_t)_{t\in\boldsymbol{R}}$ be a weakly harmonizable $L(H, K)$-valued process, i.e., $X_t = \int e^{-its}Z(ds)$, $t \in \boldsymbol{R}$. If $|||Z||| < \infty$, then $(X_t)_{t\in\boldsymbol{R}}$ is called *harmonizable*.

We note that $(X_t)_{t\in\boldsymbol{R}}$ is harmonizable if and only if it is V-bounded.

The main theorem of this section is an immediate consequence of Th. 5.18.

6.21. Theorem. *Let $(X_t)_{t\in\boldsymbol{R}}$ be a harmonizable $L(H, K)$-valued stochastic process. Then there exist a Hilbert space $\boldsymbol{K}$, a stationary $L(H, \boldsymbol{K})$-valued process $(Y_t)_{t\in\boldsymbol{R}}$ and an isometry $J \in L(K, \boldsymbol{K})$ such that for every $t \in \boldsymbol{R}$*

$$X_t = J^* Y_t.$$

Proof. By definition, $X_t = \int e^{-its}Z(ds)$, where Z is a w.c.a. $L(H, K)$-valued measure with $|||Z||| < \infty$. Thus from Th. 5.18 it follows that there exist a Hilbert space $\boldsymbol{K}$, $R \in L(H, \boldsymbol{K})$, a spectral measure E in $\boldsymbol{K}$ and an isometry $J \in L(K, \boldsymbol{K})$ such that $Z(\Delta) = J^*E(\Delta)R$, $\Delta \in \mathscr{B}$. Let $Y_t = \int e^{-ist}E(ds)R$, $t \in \boldsymbol{R}$. Then $(Y_t)_{t\in\boldsymbol{R}}$ is stationary ([4], [15]) and by (2.23), $X_t = J^*Y_t$ for all $t \in \boldsymbol{R}$. ∎

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