$= P \vee P_G^-$, $P \in B(G)$. This question has a positive answer if $h(G) = 0$, because in this case every perfect partition is the partition into points and it is sufficient to use Corollary 2 and Proposition 2. We have been unable to decide whether this question has a positive answer in the general case.

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On drop property

by

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Abstract. Let $(X, \| \ \|)$ be a Banach space. We say that the norm $\| \ \|$ has the drop property if for each closed set C disjoint with the closed unit ball $B = \{x: \|x\| \leqslant 1\}$, there is a point $a \in C$ such that $\operatorname{conv}(a \cup B) \cap C = \{a\}$.

We say that a Banach space $(X, \| \ \|)$ has the drop property if there is a norm $\| \ \|_1$ equivalent to the given one such that $\| \ \|_1$ has the drop property.

In the paper it is shown that each superreflexive space has the drop property and each space X which has the drop property is reflexive.

Let $(X, \| \ \|)$ be a Banach space. Let B denote the unit ball in X. By a *drop* induced by a point $a \notin B$ we mean the set

$$(1) \qquad D(a, B) = \operatorname{conv}(a, B).$$

Daneš [3] proved the following

THEOREM 1. (Drop theorem). *Let C be a closed set such that*

$$(2) \qquad \inf \{\|x\|: x \in C\} = R > 1.$$

Then there is a point $a \in C$ such that

$$(3) \qquad D(a, B) \cap C = \{a\}.$$

The drop theorem was used in various situations (see [1], [2], [4], [5], [10]).

Recently Penot [9] discussed the relations between the drop theorem and Ekeland's variational principle [7].

It is a natural question to ask when we can replace in the drop theorem assumption (2) by the weaker assumption that C is disjoint with B.

We shall say that the norm $\| \ \|$ has the *drop property* if the drop theorem holds under this weaker assumption. If there is a norm $\| \ \|_1$ equivalent to the norm $\| \ \|$ and having the drop property, then we say that the space X has the *drop property*.

In this paper we shall show that the uniformly convex norms have the drop property and that the spaces X with the drop property are reflexive.

Let $(X, \| \ \|)$ be a Banach space. We recall that the space $(X, \| \ \|)$ is called *uniformly convex* if there is an increasing positive function $\delta(\varepsilon)$ defined

for positive ε, such that $\|x\| = 1 = \|y\|$, $\|x - y\| \geqslant \varepsilon$ implies

(4)
$$\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta(\varepsilon).$$

PROPOSITION 1. *A Banach space* $(X, \|\ \|)$ *is uniformly convex if and only if there is a positive increasing function* $f(r)$ *defined for positive* r, *such that*

(5)
$$\lim_{r \to 0+0} f(r) = 0$$

and the diameter of the set $D(a, B) \backslash B$ *is not greater than* $f(\|a\| - 1)$,

(6)
$$\operatorname{diam}(D(a, B) \backslash B) \leqslant f(\|a\| - 1).$$

Proof. Necessity. Suppose that the space $(X, \|\ \|)$ is uniformly convex, i.e. (4) holds.

Let $a \notin B$. Let $x \in D(a, B) \backslash B$, $x \neq a$. Of course

(7)
$$1 < \|x\| < \|a\|.$$

Moreover,

(8)
$$1 < \left\|\frac{x+a}{2}\right\| = \|a\| \left\|\frac{1}{2}\left(\frac{x}{\|a\|} + \frac{a}{\|a\|}\right)\right\|$$
$$\leqslant \|a\| \left(\left\|\frac{1}{2}\left(\frac{x}{\|x\|} + \frac{a}{\|a\|}\right)\right\| + \frac{\|x\|}{2}\left|\frac{1}{\|x\|} - \frac{1}{\|a\|}\right|\right)$$
$$\leqslant \|a\| \left(1 - \delta\left(\left\|\frac{x}{\|x\|} - \frac{a}{\|a\|}\right\|\right)\right) + \|a\| \left(1 - \frac{1}{\|a\|}\right).$$

Hence

(9)
$$\frac{1}{\|a\|} \leqslant 1 - \delta\left(\left\|\frac{x}{\|x\|} - \frac{a}{\|a\|}\right\|\right) + \|a\|\left(1 - \frac{1}{\|a\|}\right)$$
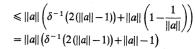
and

(10) $\delta\left(\left\|\dfrac{x}{\|x\|} - \dfrac{a}{\|a\|}\right\|\right) \leqslant \left(1 - \dfrac{1}{\|a\|}\right)(1 + \|a\|) = (\|a\| - 1)\left(1 + \dfrac{1}{\|a\|}\right) \leqslant 2(\|a\| - 1).$

Therefore

(11)
$$\left\|\frac{x}{\|x\|} - \frac{a}{\|a\|}\right\| \leqslant \delta^{-1}(2(\|a\| - 1)).$$

On the other hand,

(12)
$$\|x - a\| = \|a\| \left\|\frac{x}{\|a\|} - \frac{a}{\|a\|}\right\|$$
$$\leqslant \|a\| \left(\left\|\frac{x}{\|x\|} - \frac{a}{\|a\|}\right\| + \|x\|\left(\frac{1}{\|x\|} - \frac{1}{\|a\|}\right)\right)$$

$$\leqslant \|a\| \left(\delta^{-1}(2(\|a\| - 1)) + \|a\|\left(1 - \frac{1}{\|a\|}\right)\right)$$
$$= \|a\| (\delta^{-1}(2(\|a\| - 1)) + \|a\| - 1)$$

and $f(r) = 2(1 + r)(\delta^{-1}(2r) + r)$ is the required function.

Sufficiency. Suppose that a Banach space $(X, \|\ \|)$ is not uniformly convex. This implies that there are $\varepsilon > 0$ and sequences of elements $\{x_n\}$, $\{y_n\}$, $\|x_n\| = 1 = \|y_n\|$ such that

(13)
$$\|x_n - y_n\| \geqslant \varepsilon,$$

(14)
$$\left\|\frac{x_n + y_n}{2}\right\| > 1 - \frac{1}{n^2}.$$

Let $z_n = \dfrac{x_n + y_n}{2}$, $a_n = \left(1 + \dfrac{1}{n}\right)z_n$, $b_n = \dfrac{a_n + x_n}{2}$, $c_n = \dfrac{a_n + y_n}{2}$. Observe that

(15)
$$\frac{b_n + c_n}{2} = \frac{a_n}{2} + \frac{x_n + y_n}{4} = \left(\frac{1 + 1/n}{2} + \frac{1}{2}\right)z_n = \left(1 + \frac{1}{2n}\right)z_n$$

and by (14)

(16)
$$\left\|\frac{b_n + c_n}{2}\right\| > \left(1 + \frac{1}{2n}\right)\left(1 - \frac{1}{n^2}\right).$$

Therefore for $n \geqslant 3$, $\|(b_n + c_n)/2\| > 1$, and this implies that either $\|b_n\| > 1$ or $\|c_n\| > 1$. Say $\|b_n\| > 1$.

Observe that $b_n \in D(a_n, B) \backslash B$ and

$$\|a_n - b_n\| = \left\|\left(1 + \frac{1}{n}\right)\frac{x_n + y_n}{2} - \frac{(3 + 1/n)x_n + (1 + 1/n)y_n}{4}\right\|$$
$$= \left\|\frac{1 + 1/n}{4}y_n - \frac{1 - 1/n}{4}x_n\right\| \geqslant \tfrac{1}{4}\|y_n - x_n\| - \frac{1}{n}\|x_n + y_n\| > \frac{\varepsilon}{4} - \frac{1}{n} \to \frac{\varepsilon}{4}.$$

Thus the diameter of $D(a_n, B) \backslash B$ does not tend to 0.

PROPOSITION 2. *Let* $(X, \|\ \|)$ *be a Banach space. The norm* $\|\ \|$ *has the drop property if and only if each sequence* $\{x_n\}$ *such that*

(17)
$$x_{n+1} \in D(x_n, B) \backslash B$$

contains a convergent subsequence.

Proof. Sufficiency. Suppose that the norm $\|\ \|$ does not have the drop property. This means that there is a closed set C disjoint with the unit ball B such that for each $a \in C$

(18)
$$\inf\{\|x\|: x \in C \cap D(a, B)\} = 1.$$

Indeed, if (18) does not hold, i.e.

(19) $\inf\{\|x\|: x \in C \cap D(a, B)\} = r > 1$,

then using the classical drop theorem (Theorem 1) we deduce that there is $a_0 \in C \cap D(a, B)$ such that

(20) $C \cap D(a_0, B) = (C \cap D(a, B)) \cap D(a_0, B) = \{a_0\}$.

Basing on (18) we can choose a sequence $\{x_n\}$ such that (17) holds and moreover

(21) $\lim_{n \to \infty} \|x_n\| = 1$.

Then by our assumption the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$. We denote

(22) $x_0 = \lim_{k \to \infty} x_{n_k}$.

By (21), $\|x_0\| = 1$. Hence $x_0 \in B$. On the other hand, the set C is closed and $x_0 \in C$. This is a contradiction since B and C are disjoint.

Necessity. Suppose that there is a sequence $\{x_n\}$ such that (17) holds and the sequence $\{x_n\}$ does not contain any convergent subsequence. This means that the set C of all elements of the sequence $\{x_n\}$ is closed. By (17) the norm $\|\ \|$ does not have the drop property.

COROLLARY 1. *Let $(X, \|\ \|)$ be a Banach space. The norm $\|\ \|$ has the drop property if and only if for each separable subspace X_0 the norm $\|\ \|$ restricted to the subspace X_0 has the drop property.*

Proof. Suppose that the norm $\|\ \|$ has the drop property. Let C be a closed subset contained in the subspace X_0 disjoint with $B \cap X_0$. Then C is also disjoint with B and there is a point $a \in C$ such that

$$D(a, B) \cap C = D(a, B \cap X_0) \cap C = \{a\}.$$

On the other hand, if the norm $\|\ \|$ does not have the drop property, then there is a sequence $\{x_n\}$ satisfying (17) which does not contain any convergent subsequence. Let $X_0 = \overline{\lin\{x_n\}}$. The subspace X_0 is separable and the norm $\|\ \|$ restricted to the subspace X_0 does not have the drop property by Proposition 2.

Propositions 1 and 2 trivially imply

THEOREM 2. *Let $(X, \|\ \|)$ be a Banach space. Let the norm $\|\ \|$ be uniformly convex. Then the norm $\|\ \|$ has the drop property.*

Proof. Let $\{x_n\}$ be an arbitrary sequence satisfying (17). Of course $\|x_n\| \geq \|x_{n+1}\|$. We have two possibilities: either

(23) $\lim_{n \to \infty} \|x_n\| = r > 1$,

or

(24) $\lim_{n \to \infty} \|x_n\| = 1$.

In the first case using arguments similar to those given in the proof of Daneš' theorem [3] we conclude that $\{x_n\}$ is convergent. In the second case $\{x_n\}$ is convergent by Proposition 1.

There are Banach spaces $(X, \|\ \|)$ having the drop property which are not superreflexive, i.e., such that there is no uniformly convex norm $\|\ \|_1$ equivalent to $\|\ \|$. This follows from the following example given by P. Wojtaszczyk.

EXAMPLE 1. Let l_n^∞ denote the n-dimensional space $\mathbf{R}^n$ with the sup norm

$$\|y\|_\infty = \max_{1 \leq i \leq n} |y_i|$$

where $y = (y_1, \ldots, y_n)$. Let $X = (l_n^\infty)_{l_2}$ be the space of sequences $u = (u_n)$ where $u_n \in l_n^\infty$ and the norm in X is given by the formula

$$\|u\| = \Big(\sum_{n=1}^{\infty} \|u_n\|_\infty^2 \Big)^{1/2}.$$

It is known that X is a reflexive space which is not superreflexive.

We shall show that the norm $\|\ \|$ has the drop property. Suppose that the norm $\|\ \|$ does not have the drop property. Then by Proposition 2, there is a sequence $\{x_n\}$ satisfying (17) which does not contain any convergent subsequence. Thus there is a $\delta > 0$ and a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

(25) $\|x_{n_k} - x_{n_m}\| > 2\delta$ for $k \neq m$.

Without loss of generality we may assume

(24) $\lim_{k \to \infty} \|x_{n_k}\| = 1$

(compare the proof of Theorem 2).

The sequence $\{x_{n_k}\}$ is bounded and the space X is reflexive. Thus there is a subsequence $\{x_{n_{k_m}}\}$ tending weakly to some $x_0 \in B$. Let

(26) $y_m = x_{n_{k_m}} - x_0$.

Of course the sequence $\{y_m\}$ tends weakly to 0. Observe that by (25), $\|y_m\| > \delta$ for all m except at most one.

Since the sequence $\{y_m\}$ tends weakly to zero, by the specific form of the norm and by (24) for each $\varepsilon > 0$ we can find two elements of the sequence, say $y_{m_0}, y_{n_0}, m_0 > n_0$, such that

(27) $\|x_0\|^2 + \|y_{m_0}\|^2 \geqslant 1 + \varepsilon,$

(28) $\|x_0\|^2 + \|y_{n_0}\|^2 \geqslant 1 + \varepsilon,$

(29) $\|y_{n_0} + y_{m_0}\|^2 \leqslant (1+\varepsilon)(\|y_{n_0}\|^2 + \|y_{m_0}\|^2),$

(30) $\left\|x_0 + \dfrac{y_{n_0} + y_{m_0}}{2}\right\|^2 \leqslant (1+\varepsilon)\left(\|x_0\|^2 + \left\|\dfrac{y_{n_0}+y_{m_0}}{2}\right\|^2\right).$

Then

$$\left\|x_0 + \frac{y_{n_0}+y_{m_0}}{2}\right\|^2 \leqslant (1+\varepsilon)\left[\|x_0\|^2 + \tfrac{1}{4}(1+\varepsilon)(\|y_{n_0}\|^2 + \|y_{m_0}\|^2)\right]$$

$$= (1+\varepsilon)\left[\|x_0\|^2 + \tfrac{1}{2}(1+\varepsilon)(\|y_{n_0}\|^2 + \|y_{m_0}\|^2)\right]$$

$$-\tfrac{1}{4}(1+\varepsilon)^2(\|y_{n_0}\|^2 + \|y_{m_0}\|^2)$$

$$\leqslant (1+\varepsilon)\left[\|x_0\|^2 + \tfrac{1}{2}(1+\varepsilon)(\|y_{n_0}\|^2 + \|y_{m_0}\|^2)\right] - \tfrac{1}{4}\delta^2.$$

The arbitrariness of ε implies that for some m_0, n_0

$$\left\|x_0 + \frac{y_{n_0}+y_{m_0}}{2}\right\|^2 < 1 - \frac{\delta^2}{5}.$$

This leads to a contradiction since $x_0 + y_{m_0} \in D(x_0 + y_{n_0}, B)\backslash B$ and thus

$$\frac{(x_0 + y_{m_0}) + (x_0 + y_{n_0})}{2} \notin B.$$

Theorem 2 can be generalized in the following way. We recall that Kuratowski's index of noncompactness of a set A is

$$\alpha(A) = \inf\{r: \text{there is a finite system of sets}$$

$$A_1, \ldots, A_n \text{ such that } A \subset \bigcup_{i=1}^{n} A_i \text{ and the}$$

$$\text{diameter of } A_i \text{ is less than } r\}.$$

THEOREM 3. *If there is a continuous increasing function $f(r)$ such that $f(0) = 0$ and*

(31) $\alpha(D(a, B)\backslash B) \leqslant f(\|a\| - 1),$

then the norm $\|\ \|$ has the drop property.

Proof. Let $\{x_n\}$ be a sequence such that (17) holds. Then either $r = \lim_{n\to\infty}\|x_n\| > 1$ and by the Daneš theorem the sequence $\{x_n\}$ is convergent or $\lim_{n\to\infty}\|x_n\| = 1$ and by (31) it contains a convergent subsequence.

THEOREM 4. *Let $(X, \|\ \|)$ be a real Banach space. Suppose that there is a continuous linear functional f of norm one, $\|f\| = 1$, such that Kuratowski's index of noncompactness of the set*

$$G_\varepsilon = \{x: \|x\| \leqslant 1, f(x) \geqslant 1 - \varepsilon\}$$

does not tend to zero as $\varepsilon \to 0$,

(32) $\inf \alpha(G_\varepsilon) > 0.$

Then the norm $\|\ \|$ does not have the drop property.

Proof. It is easy to observe that for each $\varepsilon > 0$ and each finite-dimensional subspace L

(33) $\sup_{x\in G_\varepsilon}(\inf_{y\in L}\|x - y\|) \geqslant \tfrac{1}{2}\inf\alpha(G_\varepsilon).$

Let δ be an arbitrary positive number smaller than $\tfrac{1}{2}\inf\alpha(G_\varepsilon)$. Now we shall construct by induction a sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$ such that

(34)$_{(n)}$ $f(x_n) > 1,$

(35)$_{(n)}$ $\inf\{\|x_n - z\|: z \in \operatorname{lin}\{x_0, x_1, \ldots, x_{n-1}\}\} > \delta/2.$

Let x_0 be an arbitrary element such that $f(x_0) > 1$. Suppose that the elements $\{x_0, \ldots, x_n\}$ satisfying (34)$_{(i)}$ and (35)$_{(i)}$, $i = 1, 2, \ldots, n$, have been constructed. Take $\varepsilon < f(x_n) - 1$. Let $\bar{x}_{n+1}$ be an arbitrary element of G_ε such that

(36) $\inf\{\|\bar{x}_{n+1} - z\|: z \in \operatorname{lin}\{x_0, \ldots, x_n\}\} > \delta.$

Such an element exists by (33) and the definition of δ. Let

$$x_{n+1} = \frac{x_n + \bar{x}_{n+1}}{2}.$$

Then by the definition of ε

(34)$_{(n+1)}$ $f(x_{n+1}) = \tfrac{1}{2}f(x_n) + \tfrac{1}{2}f(\bar{x}_{n+1}) \geqslant \tfrac{1}{2}f(x_n) + \tfrac{1}{2}(1-\varepsilon) > 1.$

Moreover,

(35)$_{(n+1)}$ $\inf\{\|x_{n+1} - z\|: z \in \operatorname{lin}\{x_0, \ldots, x_n\}\}$

$$= \tfrac{1}{2}\inf\{\|\bar{x}_{n+1} - z\|: z \in \operatorname{lin}\{x_0, \ldots, x_n\}\} > \delta/2$$

by the definition of δ.

Let $C = \{x_0, x_1, \ldots, x_n, \ldots\}$. It is a closed set. By (34)$_{(n)}$, C is disjoint from B. Moreover, by the construction, $x_{n+1} \in D(x_n, B)$, hence the norm $\|\ \|$ does not have the drop property.

COROLLARY 2. *Let* $(X, \| \ \|)$ *be a real Banach space. For each* $\varepsilon > 0$ *there is a norm* $\| \ \|_\varepsilon$ *such that*

$$\|x\| \leqslant \|x\|_\varepsilon \leqslant (1+\varepsilon)\|x\|$$

and the norm $\| \ \|_\varepsilon$ *does not have the drop property.*

Proof. Let f_0 be an arbitrary linear continuous functional of norm $\|f_0\| = 1$. Let

$$\|x\|_\varepsilon = \max\left(\|x\|, (1+\varepsilon)f_0(x)\right).$$

The unit ball B_ε contains on its boundary a flat face

$$H_0 = B \cap \{x: f_0(x) = 1/(1+\varepsilon)\}.$$

Since H_0 is an open set in an infinite-dimensional linear manifold, its index of noncompactness is positive. Taking $f = (1+\varepsilon)f_0$ and constructing the sets G_ε we observe that $H_0 \subset G_\varepsilon$ for all $\varepsilon > 0$, and therefore $\inf \alpha(G_\varepsilon) > 0$. Thus by Theorem 3 the norm $\| \ \|_\varepsilon$ does not have the drop property.

As a consequence of Theorem 4 we obtain

THEOREM 5. *If a real Banach space* $(X, \| \ \|)$ *has the drop property, then* X *is reflexive.*

Proof. Without loss of generality we may assume that the norm $\| \ \|$ has the drop property. By Theorem 4 for each linear continuous functional f the index of noncompactness of the set

$$G_\varepsilon = \{x: \|x\| \leqslant 1, f(x) \geqslant 1-\varepsilon\}$$

tends to zero as $\varepsilon \to 0$.

By the properties of the index of noncompactness the set $G_0 = \bigcap_{\varepsilon > 0} G_\varepsilon$ is a compact nonempty set. Observe that $G_0 = \{x \in B: f(x) = 1\}$. Thus each linear continuous functional f of norm one has a point at which it attains its supremum. Therefore by James' theorem [8] the space X is reflexive.

Theorems 4 and 5 and Corollary 2 are also valid for complex Banach spaces provided we define the sets G_ε in the following way:

$$G_\varepsilon = \{x: \|x\| \leqslant 1, \operatorname{Re} f(x) \geqslant 1-\varepsilon\}.$$

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