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On subsequences of the Haar basis in $H^1(\delta)$ and isomorphism between H^1-spaces

by

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Abstract. We classify and characterize the subspaces of $H^1(\delta)$ spanned by subsequences of the Haar basis. l_1 and $(\sum H_n^1)_{l^1}$ and $H^1(\delta)$ are the only isomorphic types which occur in this way. We also give a necessary and sufficient condition on an increasing sequence of fields $(\mathscr{F}_n)$ for $H^1((\mathscr{F}_n))$ to be linearly isomorphic to $H^1(\delta)$, thus verifying a conjecture of B. Maurey.

Introduction. To the pair (n, i), $n \in N$, $0 \leqslant i \leqslant 2^n - 1$, we associate the dyadic interval $(ni) = (2^{-n}i, 2^{-n}(i+1)]$ and the Haar function h_{ni} which is 1 on the left half of $(2^{-n}i, 2^{-n}(i+1)]$, -1 on the right half and zero elsewhere. The σ-algebra generated by the sets $\{(2^{-n}i, 2^{-n}(i+1)]: 0 \leqslant i \leqslant 2^n - 1\}$ is denoted by $\mathscr{E}_n$. Dyadic intervals are nested in the sense that if $I \cap J \neq \emptyset$ then either $I \subset J$ or $J \subset I$.

We will work in the following setting: Given $f = \sum_{(ni)} a_{ni} h_{ni}$ in $L^1(0, 1]$, we write

$$S(f) = \Big(\sum_{(ni)} a_{ni}^2 h_{ni}^2\Big)^{1/2} \quad \text{and} \quad \|f\|_{H^1(\delta)} = \int S(f),$$

$$H^1(\delta) = \{f \in L^1: \|f\|_{H^1(\delta)} < \infty\}.$$

H_n^1 denotes the subspace of $H^1(\delta)$ which is spanned by $\{h_{mj}: m \leqslant n, 0 \leqslant j \leqslant 2^m - 1\}$, and

$$\Big(\sum H_n^1\Big)_{l^1} = \{(f_n)_{n\in N}: f_n \in H_n^1 \text{ and } \sum \|f_n\| < \infty\}.$$

Given $f \in L^1(0, 1]$ and a dyadic interval I we write $f_I = |I|^{-1} \int_I f$ and

$$\|f\|_{\mathrm{BMO}(\delta)} = \sup\Big\{\Big(|I|^{-1} \int_I |f - f_I|^2\Big)^{1/2}: I \text{ a dyadic interval}\Big\},$$

$$\mathrm{BMO}(\delta) = \{f \in L^1: \textstyle\int f = 0 \wedge \|f\|_{\mathrm{BMO}(\delta)} < \infty\}.$$

The connection between $\mathrm{BMO}(\delta)$ and $H^1(\delta)$ is given by the following formula:

$$\|f\|_{H^1(\delta)} = \sup\Big\{\Big|\int fg\Big|: \|g\|_{\mathrm{BMO}} = 1 \wedge g \in L^\infty\Big\}.$$

We frequently use the fact that for $f=\sum a_{ni}h_{ni}$ we can express the BMO-norm of f by means of the coefficients. In fact,

$$\|f\|_{\mathrm{BMO}}=\sup_{(ni)}\Big(2^n\sum_{(mj)\subset(ni)}2^{-m}a_{mj}^2\Big)^{1/2}.$$

A subsequence of the Haar basis in $H^1(\delta)$ is given by a collection $\mathscr{B}$ of dyadic intervals. Let X denote a subspace of $H^1(\delta)$ spanned by an arbitrary subsequence of the Haar basis in $H^1(\delta)$. The fact that the Haar basis is unconditional in $H^1(\delta)$ implies that X is complemented in H^1. Theorem 1 says that the only spaces we can produce in this way are the obvious ones, namely l^1, $(\sum H_n^1)_{l^1}$, $H^1(\delta)$. In each case a geometric characterization in terms of $\mathscr{B}$ is given. To distinguish between the "small" spaces l^1, $(\sum H_n^1)_{l^1}$, a Carleson-measure-type condition is used.

In Section 2 we study general martingale $H^1((\mathscr{F}_n))$ spaces. Consider an increasing sequence $(\mathscr{F}_n)$ of finite fields on a probability space $(\Omega, \mathscr{F}, P)$ such that $\mathscr{F}$ is the σ-algebra generated by $\bigcup_n\bigcup\{A\colon A\in\mathscr{F}_n\}$. Given a P-integrable function f we set:

$$S(f)(t)=\Big(\sum_n\big(E(f|\mathscr{F}_n)-E(f|\mathscr{F}_{n-1})\big)^2\Big)^{1/2}(t),$$

$$f^*(t)=\sup_n E(f|\mathscr{F}_n)(t),$$

$$H^1((\mathscr{F}_n))=\{f\in L^1(\Omega, \mathscr{F}, P)\colon \|S(f)\|_{L^1}<\infty\},$$

$$\mathrm{BMO}((\mathscr{F}_n))=\{f\in L^1(\Omega, \mathscr{F}, P)\colon \sup_n\|E((f-f_{n-1})^2|\mathscr{F}_n)\|_\infty^{1/2}<\infty\}.$$

We use the following

THEOREM (Davis).

$$\frac{1}{c}\|f^*\|_{L^1}\leqslant\|S(f)\|_{L^1}\leqslant c\|f^*\|_{L^1}$$

for some constant c.

THEOREM (Maurey). *$H^1((\mathscr{F}_n))$ is isomorphic to a complemented subspace of $H^1(\delta)$.*

It is important to realize that this theorem holds without any further condition on $(\mathscr{F}_n)$.

Confirming a conjecture of B. Maurey, a necessary and sufficient condition on the fields $(\mathscr{F}_n)$ is given for $H^1((\mathscr{F}_n))$ to be isomorphic to $H^1(\delta)$.

The proofs of Theorems 1 and 2 below use Pełczyński's decomposition principle. In part c of Theorem 1 and in Section 2, Lyapunov's Theorem ([8], p. 159) on the range of a vector measure is repeatedly applied.

The works of Lindenstrauss–Pełczyński [7] and Enflo–Starbird [3] explain the use of Lyapunov's Theorem to construct functions which share the properties of Haar functions.

Throughout the paper we adopt the following convention: In a measure space (Ω, Σ, P), a system of sets (E_{ni}), $n\in N$, $0\leqslant i\leqslant 2^n-1$, is called a *tree* iff:

(1) There exists $c>0$ such that

$$\frac{1}{c}2^{-n}\leqslant P(E_{ni})\leqslant c2^{-n}\quad\text{for any } n\in N \text{ and } 0\leqslant i\leqslant 2^n-1.$$

(2) $E_{nj}\cap E_{ni}=\emptyset$ for $i\neq j$, $n\in N$.

(3) $E_{n+1,2i}\cup E_{n+1,2i+1}\subseteq E_{n,i}$ for $n\in N$ and $0\leqslant i\leqslant 2^n-1$.

c is called the *tree constant* of (E_{ni}).

Both Theorems 1 and 2 have their roots in the paper [4] of Gamlen and Gaudet. They classified the subspaces of L^p, $p>1$, which are spanned by a subsequence of the Haar basis. Hence the connection to Theorem 1 is obvious. In [9], p. 112, Maurey writes: "Cette conjecture est en partie inspirée par les résultats de Gamlen et Gaudet."

In Theorem 2 we prove the conjecture mentioned above, and this constitutes the second relation between [4] and our work here.

One more remark on the relation between [4] and this paper can be found at the end of Section 2.

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0. In several places of this paper we will be concerned with constructing complemented subspaces of $H^1((\mathscr{F}_n))$ which are isomorphic to $H^1(\delta)$.

In order to avoid unnecessary repetitions we formulate a general theorem which gives a criterion for a subspace of $H^1((\mathscr{F}_n))$ to be isomorphic to $H^1(\delta)$ and complemented there.

Theorem 0 is meant to be an auxiliary result.

THEOREM 0. *Suppose that $(\tilde{h}_{ni})_{n\in N,i=0}^{2^n-1}$ are functions in $H^1((\mathscr{F}_n))\cap L^2(\mathscr{F}, P)$, orthogonal in $L^2(\mathscr{F}, P)$ and such that the following holds:*

(1) *There exist an increasing sequence $k_n\in N$ and trees (A_{ni}), (B_{ni}) in (Ω, P) such that for given (ni) we have*

(1.a) $E(\tilde{h}_{ni}|\mathscr{F}_j)=\tilde{h}_{ni}$ *for* $j\geqslant k_{n+1}$.

(1.b) $E(\tilde{h}_{ni}|\mathscr{F}_j)=0$ *for* $j\leqslant k_n$.

(1.c) $\chi_{B_{ni}}\leqslant|\tilde{h}_{ni}|\leqslant 2\chi_{A_{ni}}$.

(2) *Take i such that $k_n \leqslant i < k_{n+1}$. Let B be an atom in $\mathcal{F}_i$. Then the following holds*:

$$\int_B |\tilde{h}_{mi}|^2(t)\,dP(t) \leqslant P(B)\,2^{-m+n} \quad \text{for } m > n.$$

Then $\overline{\operatorname{span}}\{\tilde{h}_{ni}: n\in N,\ i\leqslant 2^n-1\}$ *is a complemented subspace of* $H^1((\mathcal{F}_n))$ *which is isomorphic to* $H^1(\delta)$.

Proof. Fix a finite linear combination $f=\sum a_{mj}\tilde{h}_{mj}$. We have first to show that

$$(*) \qquad \frac{1}{C}\Big\|\sum a_{mj}h_{mj}\Big\|_{H^1(\delta)} \leqslant \|f\|_{H^1((\mathcal{F}_n))} \leqslant C\Big\|\sum a_{mj}h_{mj}\Big\|_{H^1(\delta)}$$

where C can be taken independent of (a_{mj}).

Define $K_j=\{m\in N:\ k_j\leqslant m<k_{j+1}\}$, $j\in N$. Fix $j\in K_n$; then (by property (1.a))

$$E(f|\mathcal{F}_j)(t) = E\Big(\sum_i a_{ni}\tilde{h}_{ni}\Big|\mathcal{F}_j\Big)(t) + \sum_{m<n} a_{mj}\tilde{h}_{mj}(t).$$

Hence we estimate

$$\int\sup_{j\in N}|E(f|\mathcal{F}_j)| \leqslant \int\sup_{n\in N}\sup_{j\in K_n}\Big|E\Big(\sum a_{ni}\tilde{h}_{ni}\Big|\mathcal{F}_j\Big)\Big| + \int\sup_n\Big|\sum_{\substack{m<n\\ 0\leqslant j\leqslant 2^m-1}} a_{mj}\tilde{h}_{mj}\Big|$$

$$\leqslant \int\sup_n\sum_{i=0}^{2^n-1}|a_{ni}|\,|\chi_{A_{ni}}|(t)\,dP(t) + C\Big\|\sum a_{mj}h_{mj}\Big\|_{H^1(\delta)}$$

$$\leqslant \int\Big(\sum_{n\in N}\sum_{i=0}^{2^n-1}a_{ni}^2|\chi_{A_{ni}}|\Big)^{1/2}dP(t) + C\Big\|\sum a_{mj}h_{mj}\Big\|_{H^1(\delta)}$$

$$\leqslant C\Big\|\sum a_{mj}h_{mj}\Big\|_{H^1(\delta)}.$$

This proves the right-hand inequality in $(*)$. The proof of the reverse inequality is easy and uses the fact that

$$\|f\|_{H^1((\mathcal{F}_n))} \geqslant \int\sup_n|E(f|\mathcal{F}_{k_n})| \geqslant C\Big\|\sum a_{ni}h_{ni}\Big\|_{H^1(\delta)}.$$

The constant C appearing in $(*)$ depends only on the tree constants of (A_{ni}) and (B_{ni}).

Hence the closed linear hull of $\{\tilde{h}_{ni}:\ n\in N,\ 0\leqslant i\leqslant 2^n-1\}$ in $H^1((\mathcal{F}_n))$ is isomorphic to $H^1(\delta)$ and the mapping

$$i:\ H^1(\delta)\to H^1((\mathcal{F}_n)), \qquad h_{ni}\to\tilde{h}_{ni}$$

is an embedding. We must make sure that $i(H^1(\delta))$ is complemented in

$H^1((\mathcal{F}_n))$. Define the projection

$$P:\ H^1((\mathcal{F}_n))\to H^1((\mathcal{F}_n)), \qquad f\to\sum\Big(f,\frac{\tilde{h}_{ni}}{\|\tilde{h}_{ni}\|_2^2}\Big)\tilde{h}_{ni}.$$

By the orthogonality of $\tilde{h}_{ni}$ it is evident that P is bounded iff

$$(i^{-1}P)^*:\ \mathrm{BMO}(\delta)\to\mathrm{BMO}((\mathcal{F}_n)), \qquad h_{ni}\to\tilde{h}_{ni}$$

is bounded.

To show that $(i^{-1}P)^*$ is bounded, we take $f=\sum a_{mj}\tilde{h}_{mj}$. Fix $I\in\mathcal{F}_j$ and $J(\supset I)\in\mathcal{F}_{j-1}$. Take the largest m_1 such that $\tilde{h}_{m_1 i}$ is $\mathcal{F}_{j-1}$-measurable, and find i_1 such that $J\subset\operatorname{supp}\tilde{h}_{m_1 i_1}$. Therefore $j-1\in K_{m_1}$ and $j\in K_{m_1}\cup\{k_{m_1+1}\}$. Hence we estimate

$$E\big((f-f_{j-1})^2|\mathcal{F}_j\big)|_I = \frac{1}{P(I)}\int_I\Big(\sum_{m>m_1}a_{mi}\tilde{h}_{mi} - \Big(\sum_{m>m_1}a_{mi}\tilde{h}_{mi}\Big)_J\Big)^2$$

$$\leqslant \frac{c}{P(I)}\sum_{m>m_1}a_{mi}^2\int_I\tilde{h}_{mi}^2 + \frac{c}{P(J)}\sum_{m>m_1}a_{mi}^2\int_J\tilde{h}_{mi}^2$$

$$\leqslant 2c\sum_{\substack{m>m_1\\ (mi)\subset(m_1 i_1)}}a_{mi}^2\,2^{m_1-m} \leqslant 2c\Big\|\sum a_{mj}h_{mj}\Big\|_{\mathrm{BMO}(\delta)}^2.$$

1. Subsequences of the Haar basis in $H^1(\delta)$.

THEOREM 1. *Let $\mathcal{B}$ be an infinite collection of dyadic intervals. Let X be the closed linear span of $\{h_I:\ I\in\mathcal{B}\}$ in H^1 and $\sigma=\{t:\ t\in I$ for infinitely many $I\in\mathcal{B}\}$. Then*:

(a) *If* $|\sigma|=0$ *and* $\sup_I\big(|I|^{-1}\sum_{J\in\mathcal{B},\,J\subset I}|J|\big)<\infty$ *then X is isomorphic to* l^1.

(b) *If* $|\sigma|=0$ *and* $\sup_I\big(|I|^{-1}\sum_{J\in\mathcal{B},\,J\subset I}|J|\big)=\infty$ *then X is isomorphic to* $\big(\sum H_n^1\big)_{l_1}$.

(c) *If* $|\sigma|>0$ *then X is isomorphic to* $H^1(\delta)$.

Proof of Theorem 1, part (a). Suppose

$$\sup_I\frac{1}{|I|}\sum_{\substack{J\in\mathcal{B}\\ J\subset I}}|J| = M<\infty.$$

Then

$$\Big\|\sum_{I\in\mathcal{B}}a_I h_I\Big\|_{H^1} = \sup\Big\{\sum a_I b_I|I|:\ \Big\|\sum_{I\in\mathcal{B}}b_I h_I\Big\|_{\mathrm{BMO}}=1\Big\}$$

$$= \sup\Big\{\sum a_I b_I|I|:\ \sup_I\Big(|I|^{-1}\sum_{\substack{J\subset I\\ J\in\mathcal{B}}}b_J^2|J|\Big)^{1/2}=1\Big\}$$

$$\geqslant \sup\{\sum a_I b_I |I|:\ \sup_I \sup_{J\subset I} |b_J| (|I|^{-1} \sum_{\substack{J\subset I\\ J\in\mathscr{B}}} |J|)^{1/2} = 1\}$$

$$\geqslant \sup\{\sum a_I b_I |I|:\ \sup_I |b_I| M^{1/2} = 1\}$$

$$= M^{-1/2} \sum_{I\in\mathscr{B}} |a_I|\,|I|.$$

Thus $\{h_I |I|^{-1}:\ I\in\mathscr{B}\}$ is equivalent to the unit vector basis in l^1. ∎

The following lemmata and propositions are needed to prove part (b) of the theorem.

DEFINITION 2. Let $\mathscr{B}$ be a collection of dyadic intervals. Let $I\in\mathscr{B}$. Put

$$G_1(I) = \{J\in\mathscr{B}:\ J\subset I,\ J \max\},$$

$$G_n(I) = \bigcup_{J\in G_{n-1}(I)} G_1(J).$$

We enumerate the intervals of $G_n(I)$ in such a way that

$$|I_1| \geqslant |I_2| \geqslant |I_3| \ldots$$

Let $k(\varepsilon)$ be the smallest integer such that

$$\sum_{k=1}^{k(\varepsilon)} |I_k| \geqslant (1-\varepsilon) \sum_{k=1}^{\infty} |I_k|.$$

Then we set

$$G_{n,n}^{\varepsilon}(I) = \{I_k:\ k\leqslant k(\varepsilon)\},$$

$$G_{p,n}^{\varepsilon}(I) = \{K\in G_p(I):\ \exists J\in G_{p+1,n}^{\varepsilon}(I) \wedge K\supset J\}, \qquad p<n.$$

Remark. We will use the fact that $G_{p,n}^{\varepsilon}(I)$ is a finite subset of $G_p(I)$ such that

$$\sum_{J\in G_{p,n}^{\varepsilon}} |J| > (1-\varepsilon) \sum_{J\in G_n(I)} |J|.$$

LEMMA 1 ([5], Ch. XI, Lemma 3.2). *Let $\mathscr{B}$ be a collection of dyadic intervals and $K\in\mathscr{B}$. If $n\in N$ and $\gamma<1$ are given, then*

$$\frac{1}{|K|} \sum_{\substack{J\subset K\\ J\in\mathscr{B}}} |J| > \frac{n}{1-\gamma}$$

implies that there exists $I_0\in\mathscr{B}$, $I_0\subset K$, such that

$$\frac{1}{|I_0|} \sum_{J\in G_n(I_0)} |J| \geqslant \gamma.$$

Proof. (a) Suppose this is false. Then, for any $I\in\mathscr{B}$ with $I\subset K$,

$$\frac{1}{|I|} \sum_{J\in G_n(I)} |J| < \gamma.$$

But this implies

$$\frac{1}{|K|} \sum_{\substack{J\subset K\\ J\in\mathscr{B}}} |J| = \frac{1}{|K|} \sum_{r=1}^{n} \sum_{m\in N} \sum_{J\in G_{nm+r}(K)} |J|$$

$$\leqslant \frac{1}{|K|} \sum_{r=1}^{n} \sum_{m\in N} \gamma^m |K| = \frac{n}{1-\gamma};$$

a contradiction.

MAIN LEMMA 2. *Let $\mathscr{B}$ be a collection of dyadic intervals. Suppose that there exist $I_0\in\mathscr{B}$ and $\varepsilon>0$ such that*

$$\frac{1}{|I_0|} \sum_{J\in G_{n,n}^{\varepsilon}(I_0)} |J| > \gamma_n, \qquad 1-4^{-n} < \gamma_n < 1.$$

Then there exists a subspace Y_n which is contained in span $\{h_I:\ I\in G_{p,n}^{\varepsilon},\ 0\leqslant p\leqslant n\}$, *4-complemented in $H^1(\delta)$, and 4-isomorphic to H_n^1.*

Proof.

Step 1. $\tilde{h}_{00} = h_{I_0}$.

Step 2. $E_0^+ = E(h_{I_0} = 1)$, $E_0^- = E(h_{I_0} = -1)$,

$$\tilde{h}_{10} = \sum_{J\in G_{1,n}^{\varepsilon}(I_0)\cap E_0^+} h_J, \qquad \tilde{h}_{11} = \sum_{J\in G_{1,n}^{\varepsilon}(I_0)\cap E_0^-} h_J.$$

We observe that

$$|\operatorname{supp}\tilde{h}_{10}| \geqslant (\gamma_n - \tfrac{1}{2})|I_0| \quad \text{and} \quad |\operatorname{supp}\tilde{h}_{11}| \geqslant (\gamma_n - \tfrac{1}{2})|I_0|.$$

Step 3. $j\in\{0,1\}$, $E_{1j}^+ = E(\tilde{h}_{1j} = 1)$, $E_{1j}^- = E(\tilde{h}_{1j} = -1)$,

$$\tilde{h}_{2,2j} = \sum_{J\in G_{2,n}^{\varepsilon}(I_0)\cap E_{1j}^+} h_J, \qquad \tilde{h}_{2,2j+1} = \sum_{J\in G_{2,n}^{\varepsilon}(I_0)\cap E_{1j}^-} h_J,$$

and we observe that

$$|\operatorname{supp}\tilde{h}_{2,2j}| \geqslant |\operatorname{supp}\tilde{h}_{1j}| - \tfrac{1}{4}|I_0| \geqslant (\gamma_n - \tfrac{1}{2} - \tfrac{1}{4})|I_0|,$$

$$|\operatorname{supp}\tilde{h}_{2,2j+1}| \geqslant (\gamma_n - \tfrac{1}{2} - \tfrac{1}{4})|I_0|.$$

At *step* m we are given $\tilde{h}_{mj}$, $0\leqslant j\leqslant 2^m-1$. We put

$$E_{mj}^+ = E(\tilde{h}_{mj} = 1), \qquad E_{mj}^- = E(\tilde{h}_{mj} = -1),$$

$$\tilde h_{m+1,2j} = \sum_{J\in G^{\varepsilon}_{m+1,n}(I_0)\cap E^{+}_{mj}} h_J, \qquad \tilde h_{m+1,2j+1} = \sum_{J\in G^{\varepsilon}_{m+1,n}(I_0)\cap E^{-}_{mj}} h_J,$$

and we get, for $k\in\{2j, 2j+1\}$,

$$|\operatorname{supp}\tilde h_{m+1,k}| \geqslant |\operatorname{supp}\tilde h_{m,j}| - \frac{1}{2^{m+1}}|I_0| \geqslant \Big(\gamma_n - \frac{1}{2} - \frac{1}{4} - \ldots - \frac{1}{2^{m+1}}\Big)|I_0|,$$

As the space Y_n we will take

$$\operatorname{span}\{\tilde h_{mj}\colon 0\leqslant m\leqslant n,\ 0\leqslant j\leqslant 2^m-1\}.$$

We must show that

(a) Y_n is isomorphic to H^1_n with constant 4.

(b) Y_n is complemented in $H^1(\delta)$ and the norm of the projection is less than 4.

Ad (a). Observe that $\operatorname{supp}\tilde h_{m+1,2j}\cup\operatorname{supp}\tilde h_{m+1,2j+1}\subset\operatorname{supp}\tilde h_{m,j}$, take $(a_{mj})^{n,2^m-1}_{m=0,j=0}$ and estimate:

$$\Big\|\sum a_{mj}\tilde h_{mj}\Big\|_{H^1} = \int\Big(\sum a^2_{mj}S^2(\tilde h_{mj})\Big)^{1/2} \geqslant \sum_{i=0}^{2^n-1}\int\Big(\sum_{(mj)\supset(ni)} a^2_{mj}\Big)^{1/2}\chi_{\operatorname{supp}\tilde h_{ni}} \geqslant \sum_{i=0}^{2^n-1}\Big(\sum_{(mj)\supset(ni)} a^2_{mj}\Big)^{1/2}(\gamma_n-1+2^n)|I_0|.$$

On the other hand,

$$\Big\|\sum a_{mj}\tilde h_{mj}\Big\|_{H^1}\leqslant\sum_{i=0}^{2^n-1}\Big(\sum_{(mj)\supset(ni)} a^2_{mj}\Big)^{1/2}2^{-n}|I_0|.$$

Hence

$$i_n\colon H^1_n\to Y_n,\qquad h_{mj}\to\tilde h_{mj}\frac{1}{|I_0|}$$

is an isomorphism with

$$\|i_n\|\cdot\|i_n^{-1}\|\leqslant\frac{2^{-n}}{\gamma_n-1+2^{-n}}\leqslant 4.$$

Ad (b). Y_n is complemented in $H^1(\delta)$ by means of the following projection:

$$P_n\colon H^1(\delta)\to Y_n,\qquad f\to\sum\Big(f,\frac{\tilde h_{mj}}{\|\tilde h_{mj}\|_2^2}\Big)\tilde h_{mj}.$$

By orthogonality we see that P_n is bounded iff

$$(i_n^{-1}P_n)^*\colon \mathrm{BMO}_n\to\mathrm{BMO},\qquad h_{ni}\to\tilde h_{ni}$$

is bounded.

Take any dyadic interval $K\subset[0,1]$.

Case 1: $I_0\subset K$. Then

$$\frac{1}{|K|}\int_K\big((i^{-1}P)^*f-((i^{-1}P)^*f)_K\big)^2=\frac{1}{|K|}\int|(i^{-1}P)^*f|^2 = \frac{|I_0|}{|K|}\int_{[0,1]}|f|^2\leqslant\frac{|I_0|}{|K|}\|f\|^2_{\mathrm{BMO}}.$$

Case 2: $I_0\cap K=\emptyset$ *or* $K\subset K_0\in G_{n+1}(I_0)$. Then

$$\frac{1}{|K|}\int_K\big((i^{-1}P)^*f-((i^{-1}P)^*f)_K\big)^2=0.$$

Case 3: $\exists m_0\ \exists K_1\in G^{\varepsilon}_{m_0,n}(I_0)$ with $K_1\supset K\wedge\forall K_2\in G^{\varepsilon}_{m_0+1,n}(I_0)$, $K_2\cap K\neq\emptyset\Rightarrow K_2\subset K$. By the construction of $(\tilde h_{mj})$ we get the following:

(1) $\tilde h_{mj}|_K=\text{const}$ for $m_0\geqslant m-1$.

(2) $\int_K\tilde h^2_{mj}\leqslant|K|\,2^{m_0-m}$ for $m\geqslant m_0$.

(3) $\int_K\tilde h_{mj}=0$ for $m\geqslant m_0+1$.

Now we estimate as in Section 0.

PROPOSITION 3. *Let* $\mathscr{B}$ *be a collection of dyadic intervals so that*

(1) $|\sigma|=0$.

(2) $\sup_{I\in\mathscr{B}}|I|^{-1}\sum_{J\in\mathscr{B}\cap I}|J|=\infty$.

Then for any $\varepsilon_n>0$ *and* $\gamma_n<1$, $\mathscr{B}$ *can be decomposed into* $\mathscr{B}^1$ *and* $\mathscr{B}^2$ *so that for* $j\in\{1,2\}$ *we have*

(1) $\mathscr{B}^j=\bigcup_{k=1}^{\infty}\mathscr{A}^j_k$.

(2) $\mathscr{A}^j_k$ *are finite and pairwise disjoint.*

(3) *Any* $\mathscr{A}^j_k$ *contains a dyadic interval* I *such that* $G^{\varepsilon_k}_{p,k}(I)\subset\mathscr{A}^j_k$ *for* $p\leqslant k$ *and*

$$\frac{1}{|I|}\sum_{J\in G^{\varepsilon_k}_{k,k}(I)}|J|\geqslant\gamma_k(1-\varepsilon_k).$$

(4) *For any* $\mathscr{A}^j_k$ *and any* $I\in\mathscr{A}^j_k$ *we have*

$$\Big|I\ \bigcup_{l\geqslant k+1}\ \bigcup_{J\in\mathscr{A}^j_l}J\Big|\geqslant|I|/2.$$

Proof. Define $\sigma_m = \{I: |I| = 2^{-m}, I \in \mathscr{B}\}$, $\bar{\sigma}_m = \{\bigcup I: I \in \sigma_m\}$, $\tau_m = \bigcup_{n=m}^{\infty} \bar{\sigma}_m$. We will repeatedly use the following two observations:

(1) $\sigma = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \bar{\sigma}_m$ which implies (by the hypothesis on σ) that $|\bigcup_{m=n}^{\infty} \sigma_m|$ tends to zero when n goes to infinity.

(2) If $\sup_{I \in \mathscr{B}} |I|^{-1} \sum_{J \in \mathscr{B} \cap I} |J| = \infty$ then for $\mathscr{B}_k = \mathscr{B} \setminus \bigcup_{m=0}^{k} \sigma_m$ we also have

$$\sup_{I \in \mathscr{B}_k} \frac{1}{|I|} \sum_{J \in \mathscr{B}_k \cap I} |J| = \infty.$$

We start the iteration with

Step 0. Set $m_0 = 0$, $\sigma_0 = [0, 1]$, then find $m_1 \in N$ such that $|\tau_{m_1}| \leqslant \frac{1}{2}$ and set $\mathscr{B}_1 = \mathscr{B} \setminus \bigcup_{m=0}^{m_1} \sigma_m$.

Step 1. Find $I \in \mathscr{B}_1$ such that

$$\frac{1}{|I|} \sum_{J \in \mathscr{B}_1 \cap I} |J| \geqslant \frac{1}{1-\gamma_1}.$$

Then apply Lemma 1 and the Remark after Definition 2. We get $I_1 \in \mathscr{B}_1$ such that

$$\frac{1}{|I_1|} \sum_{J \in G_{1,1}^{\varepsilon_1}(I_1)} |J| \geqslant \gamma_1 (1-\varepsilon_1).$$

Then choose $m_2 \in N$ so large that $J \in G_{1,1}^{\varepsilon_1}(I_1)$ implies $|J| > 2^{-m_2}$ and such that for $I \in \bigcup_{m=0}^{m_1} \sigma_m$ we get $|\tau_{m_2}| \leqslant |I|/2$. Set $\mathscr{B}_2 = \mathscr{B}_1 \setminus \bigcup_{m=m_1}^{m_2} \sigma_m$. Now we continue and arrive at

Step n. First find $I \in \mathscr{B}_n$ such that

$$\frac{1}{|I|} \sum_{J \in \mathscr{B}_n \cap I} |J| \geqslant \frac{n}{1-\gamma_n}.$$

Again apply Lemma 1 and the Remark after Definition 2 to get $I_n \in \mathscr{B}_n$ such that

$$\frac{1}{|I_n|} \sum_{J \in G_{n,n}^{\varepsilon_n}(I_n)} |J| \geqslant \gamma_n (1-\varepsilon_n).$$

Then choose $m_{n+1} \in N$ such that

(1) $I \in G_{n,n}^{\varepsilon_n}(I_n) \Rightarrow |I| \geqslant 2^{-m_{n+1}}$.

(2) $I \in \bigcup_{m=1}^{m_n} \sigma_m \Rightarrow |\tau_{m_{n+1}}| \leqslant |I|/2$.

Summing up, we have:

$$\mathscr{A}_k = \bigcup_{j=m_{k-1}}^{m_k - 1} \sigma_j; \quad (\mathscr{A}_k)_{k \in N} \text{ is a partition of } \mathscr{B};$$

$$\mathscr{A}_k^1 = \mathscr{A}_{2k} \quad \text{and} \quad \mathscr{B}^1 = \bigcup_k \mathscr{A}_k^1,$$

$$\mathscr{A}_k^2 = \mathscr{A}_{2k+1} \quad \text{and} \quad \mathscr{B}^2 = \bigcup_k \mathscr{A}_k^2.$$

So by construction, (1)–(3) are satisfied and we only have to check (4).

Set $j = 1$. Fix $k \in N$, and choose $I \in \mathscr{A}_k^1$.

$$|I \setminus \bigcup_{l=k+1}^{\infty} \bigcup_{J \in \mathscr{A}_l^1} J| \geqslant |I| - |\tau_{2k+1}| \geqslant |I|/2.$$

For $j = 2$ we get the same estimate.

Proof of Theorem 1, part (b). The proof is divided into two steps:

(1) By using mainly property (4) of Proposition 3, we show that X is isomorphic to a complemented subspace of $(\sum H_n^1)_{l^1}$.

(2) By using properties (4) and (3) of Proposition 3 and the Main Lemma 2 we show that X contains a complemented subspace isomorphic to $(\sum H_n^1)_{l^1}$.

Then, using the fact that $(\sum H_n^1)_{l^1}$ is isomorphic to its l^1 sum, we apply Pełczyński's decomposition method, and are done.

Choose $\varepsilon_k > 0$ and $\gamma_k < 1$ such that

$$1 - 4^{-k} < \gamma_k (1-\varepsilon_k).$$

Take the partition of $\mathscr{B}$ as obtained before. Put $X_k = \operatorname{span}\{h_I: I \in \mathscr{A}_k\}$ and $X = \operatorname{span}\{h_I: I \in \mathscr{B}\}$ and let $P_{\mathscr{A}_k^j}$ be the natural projection from H^1 onto $\operatorname{span}\{h_I: I \in \mathscr{A}_k^j\}$. Take $f \in X$. We first show that

$$(*) \qquad 4\|f\|_{H^1} \geqslant \sum_k \|P_{\mathscr{A}_k^1} f\|_{H^1} + \sum_k \|P_{\mathscr{A}_k^2} f\|_{H^1}.$$

To do so we first observe that

$$2\|f\|_{H^1} \geqslant \|P_{\mathscr{B}^1} f\|_{H^1} + \|P_{\mathscr{B}^2} f\|_{H^1}.$$

Take $g \in \operatorname{span}\{h_I: I \in \mathscr{B}^1\}$. Define $G_k = \bigcup_{J \in \mathscr{A}_k^1} J$.

$$\int S(g) = \int \Big(\sum_k \Big(\sum_{I \in \mathscr{A}_k^1} a_I^2 h_I^2\Big)\Big)^{1/2}$$

$$\geqslant \sum_k \Big(\sum_{I \in \mathscr{A}_k^1} a_I^2 \chi_{I \setminus \bigcup_{j \geqslant k+1} G_j}\Big)^{1/2}$$

$$\geq \tfrac{1}{2}\sum_{k} \int \Big(\sum_{I \in \mathscr{A}_k^1} a_I^2 \chi_I \Big)^{1/2} \quad \text{(by (4) of Prop. 3)}$$
$$= \tfrac{1}{2}\sum_{k} \|P_{\mathscr{A}_k^1} f\|_{H^1}.$$

For $g \in \operatorname{span}\{h_I\colon I \in \mathscr{B}^2\}$ the same estimate holds. Thus we have verified (*).

To factor the identity on X over $(\sum H_n^1)_{l^1}$ we first find a sequence n_k such that

$$X_k \subset H_{n_k}^1$$

and define

$$i\colon X \to (\sum H_{n_k}^1)_{l^1}, \quad f \to (P_{\mathscr{A}_k} f)_{k \in N},$$
$$P\colon (\sum H_n^1)_{l^1} \to X, \quad (f_n) \to \sum P_{\mathscr{A}_k} f_k.$$

By the calculation above, we get $P \circ i = \mathrm{id}_X$ and $\|i\| \cdot \|P\| \leq C$. On the other hand we factor the identity on $(\sum H_n^1)_{l^1}$ over X. Using property (3) of $\mathscr{A}_k$ we see, by the Main Lemma 2, that there exist

$$i_n\colon H_n^1 \to X_n, \quad P_n\colon X \to i_n(H_n^1)$$

such that $i_n^{-1} P_n i_n = \mathrm{id}_{H_n^1}$ and $\|i_n\| \cdot \|i_n^{-1} P_n\| \leq 4$. Then define:

$$j\colon (\sum_n H_n^1)_{l^1} \to X, \quad (f_n)_{n \in N} \to \sum_n i_n f_n,$$
$$P\colon X \to (\sum H_n^1)_{l^1}, \quad f \to (i_n^{-1} P_n f)_{n \in N}.$$

Proof of Theorem 1, part (c). Define $C_{00} = \{I \in \mathscr{B}\colon I \text{ maximal}\}$. By a standard approximation argument we assume that C_{00} is finite. For $\tilde{h}_{00} = \sum_{I \in C_{00}} h_I$ we obtain

$$\chi_\sigma \leq |\tilde{h}_{00}| \leq \chi_{[0,1]}.$$

We set $\delta(0) = \inf\{|I|\colon I \in C_{00}\}$. A_0 denotes a covering of $[0, 1]$ by intervals of length $\delta(0)$. Consider the vector measure

$$\mu\colon [0, 1] \to R^{|A_0|+1}, \quad E \to (|E \cap \sigma|, |E \cap \sigma \cap I|;\ I \in A_0).$$

As an application of Lyapunov's theorem we get for $\varepsilon_1 > 0$ a natural number k_1 and disjoint sets E_{10}, E_{11} in $\mathscr{E}_{k_1}$ such that

$$E_{10} \cup E_{11} \subset [0, 1],$$
$$\tfrac{1}{2}(1-\varepsilon_1)|\sigma| \leq |E_{1j}| \leq \tfrac{1}{2}(1+\varepsilon_1)|\sigma|, \quad j \in \{1, 2\},$$
$$\tfrac{1}{2}(1-\varepsilon_1)|I \cap \sigma| \leq |E_{1j} \cap I| \leq \tfrac{1}{2}(1+\varepsilon_1)|I \cap \sigma|, \quad j \in \{1, 2\},\ I \in A_0.$$

At stage n we are given E_{nj} in $\mathscr{E}_{k_n}$. Define $C_{ni} = \{I \in \mathscr{B}\colon I \subset E_{ni},\ I \text{ maximal}\}$. We assume C_{ni} to be finite. For $\tilde{h}_{ni} = \sum_{I \in C_{ni}} h_I$ we obtain

$$\chi_{\sigma \cap E_{ni}} \leq |\tilde{h}_{ni}| \leq \chi_{E_{ni}}.$$

We set $\delta(n) = \inf\{|I|\colon I \in C_{ni}\}$. A_n denotes a covering of E_{ni} by intervals of length $\delta(n)$. Consider the vector measure

$$\mu\colon E_{ni} \to R^{|A_n|+1}, \quad E \to (|E \cap \sigma|, |E \cap \sigma \cap I|;\ I \in A_n).$$

As an application of Lyapunov's theorem we get for $\varepsilon_{n+1} > 0$ a natural number k_{n+1}, disjoint sets $E_{n+1,2i}$, $E_{n+1,2i+1}$ in $\mathscr{E}_{k_{n+1}}$ such that

$$E_{n+1,2i} \cup E_{n+1,2i+1} \subset E_{ni},$$
$$\frac{1-\varepsilon_{n+1}}{2}|E_{ni}| \leq |E_{n+1,2i+j}| \leq \frac{1+\varepsilon_{n+1}}{2}|E_{ni}|, \quad j \in \{1, 2\},$$
$$\frac{1-\varepsilon_{n+1}}{2}|I \cap \sigma| \leq |E_{n+1,2i+j} \cap I| \leq \frac{1+\varepsilon_{n+1}}{2}|I \cap \sigma|, \quad j \in \{1, 2\},\ I \in A_n.$$

This finishes the induction and we see that $Y = \{\tilde{h}_{ni}\colon n \in N,\ 0 \leq i \leq 2^n - 1\}$ is a subspace of X which (by Theorem 0) is isomorphic to $H^1(\delta)$ and complemented in $H^1(\delta)$.

Remark. Due to the fact that the orthogonal projections we use are bounded in L^2, we obtain by interpolation between $H^1(\delta)$ and L^2 (cf. [2]) the result of [4] for $p \leq 2$. Our projections can be dualized and this proves the result of [4] for $\infty > p > 2$.

2. General martingale H^1. In this section we will give a necessary and sufficient condition on $(\mathscr{F}_n)$ such that $H^1((\mathscr{F}_n))$ is isomorphic to $H^1(\delta)$. We thus prove a conjecture of B. Maurey.

Definition. $A_k^\varepsilon = \bigcup\{B\colon B \text{ is an atom in } \mathscr{F}_k \wedge P(B) < \varepsilon\}$,

$$A^\infty = \bigcap_{\varepsilon > 0} \bigcup_{k \in N} A_k^\varepsilon.$$

Theorem 2. *$H^1((\mathscr{F}_n))$ is linearly isomorphic to $H^1(\delta)$ if and only if $P(A^\infty) > 0$.*

Proof. We first show that $P(A^\infty) > 0$ is a sufficient condition. By the theorem of Maurey it is enough to find a complemented subspace in $H^1((\mathscr{F}_n))$ which is isomorphic to $H^1(\delta)$.

Step 1. Observe that $(\Omega \cap A^\infty, \mathscr{F}, P)$ is a nonatomic measure space. Fix a sequence (ε_j) such that $\prod(1+\varepsilon_j) < 1{,}5$. For $\varepsilon_1 > 0$ find $k_0 \in N$ and $A_{00} \in \mathscr{F}_{k_0}$ such that $A_{00} \supset A^\infty$ and $P(A_{00} \setminus A^\infty) < \varepsilon_1$. Set $C_{00} = \{B\colon B \text{ atom in } \mathscr{F}_{k_0} \text{ and } B \subset A_{00}\}$.

Step 2. Apply Lyapunov's theorem to the measure

$$\mu\colon (A^\infty, \mathscr{F}) \to \mathbf{R}^n, \quad E \to \big(P(E), P(B\cap E);\ B\in C_{00}\big)$$

and obtain, for $\varepsilon_2 > 0$, disjoint sets A_{10}, A_{11} and a natural number $k_1 > k_0$ such that

(1) $A_{10}, A_{11} \in \mathscr{F}_{k_1}$ and $A_{10}\cup A_{11} \subset A_{00}$.

(2) $P(A_{1j}\cap B) \overset{\varepsilon_2}{\approx} P(A^\infty \cap B)/2$, $j\in\{0, 1\}$, $B\in C_{00}$.

(3) $P(A_{1j}) \overset{\varepsilon_2}{\approx} P(A^\infty)/2$, $j\in\{0,1\}$.

We define the "Haar function":

$$h_{A_{00}} = \sum_{B\in C_{00}} \left(\frac{P(B\cap A^\infty)}{2P(B\cap A_{10})}\chi_{A_{10}\cap B} - \frac{P(B\cap A^\infty)}{2P(B\cap A_{11})}\chi_{A_{11}\cap B}\right).$$

We continue and arrive at

Step n. We are given A_{ni} in $\mathscr{F}_{k_n}$ and $C_{ni} = \{B\colon B\subset A_{ni} \wedge B \text{ atom in } \mathscr{F}_{k_n}\}$. Define a vector measure

$$\mu\colon (A^\infty\cap A_{ni}, \mathscr{F}) \to \mathbf{R}^{|C_{ni}|+1}, \quad E\to \big(P(E), P(B\cap E);\ B\in C_{ni}\big).$$

As an application of Lyapunov's theorem we get, for a given ε_{n+2}, disjoint sets $A_{n+1,2i}$, $A_{n+1,2i+1}$ and a natural number k_{n+1} such that

(1) $A_{n+1,2i}, A_{n+1,2i+1} \in \mathscr{F}_{k_{n+1}}$ and $A_{n+1,2i}\cup A_{n+1,2i+1} \subset A_{ni}$.

(2) $P(A_{n+1,2i+j}) \overset{\varepsilon_{n+2}}{\approx} \frac{1}{2} P(A_{ni}\cap A^\infty)$, $j\in\{0, 1\}$.

(3) $P(A_{n+1,2i+j}\cap B) \overset{\varepsilon_{n+2}}{\approx} \frac{1}{2}P(B\cap A^\infty)$, $B\in C_{ni}$, $j\in\{0,1\}$.

We use these sets to define

$$h_{A_{ni}} = \sum_{B\in C_{ni}} \left(\frac{P(B\cap A^\infty)}{2P(A_{n+1,2i}\cap B)}\chi_{A_{n+1,2i}\cap B} - \frac{P(B\cap A^\infty)}{2P(A_{n+1,2i+1}\cap B)}\chi_{A_{n+1,2i+1}\cap B}\right).$$

The subspace $Y = \overline{\operatorname{span}}\,\{h_{A_{ni}}\colon n\in\mathbf{N},\ 0\leqslant i\leqslant 2^n-1\}$ is isomorphic to $H^1(\delta)$ and complemented in $H^1((\mathscr{F}_n))$. Indeed, a glance at the construction shows that the $(h_{A_{ni}})$ satisfy the conditions in Theorem 0.

To show that $P(A^\infty) > 0$ is a necessary condition, we prove simply that $P(A^\infty) = 0$ implies that l^2 does not embed in $H^1((\mathscr{F}_n))$.

Let e_i be equivalent to the unit vector basis of l^2 in $H^1((\mathscr{F}_n))$. e_i tends to zero in the $\sigma(H^1, \mathrm{BMO})$ topology. By taking a subsequence if necessary, we may suppose that for any sequence (λ_n)

$$\Big\|\sum \lambda_i e_i\Big\|_{H^1((\mathscr{F}_n))} \geqslant \tfrac{1}{4}\int \big(\sum \lambda_i^2 |e_i|^2(t)\big)^{1/2}\, dP(t).$$

We claim that for some $\delta > 0$ the numbers $P(E(e_i > \delta))$, $P(E(e_i < -\delta))$ tend to zero as i tends to infinity. Indeed, choose $\mathscr{B}_i \in \bigcup \mathscr{F}_n$ such that $A, B\in\mathscr{B}_i$, $A\neq B \Rightarrow A\cap B = \emptyset$, and $c_B\in\mathbf{R}$, $B\in\mathscr{B}_i$, such that

$$e_i\,\chi_{E(e_i>\delta)} = \sum_{B\in\mathscr{B}} c_B\,\chi_B.$$

By the hypothesis on e_i, $\sup_{B\in\mathscr{B}_i} P(B)$ tends to zero as i tends to infinity. Therefore by the hypothesis on A^∞ which says that the union of small atoms is small, the claim is verified. So we can suppose (by taking a subsequence) that

$$P\Big(E(|e_j|^2>\delta)\setminus \bigcup_{i=j+1}^{\infty} E(|e_i|^2>\delta)\Big) > \tfrac{1}{2}P\big(E(|e_j|^2>\delta)\big).$$

We put everything together and estimate as in the proof of Theorem 1, part (b):

$$\begin{aligned}\big(\sum\lambda_i^2\big)^{1/2} &\geqslant C_1\int\big(\sum\lambda_i^2|e_i|^2\big)^{1/2}\\ &\geqslant C_2\int\big(\sum\lambda_i^2|e_i|^2\chi_{E(|e_i|>\delta)}\big)^{1/2} - C_2\big(\sum\lambda_i^2\big)^{1/2}\\ &\geqslant C_3\sum|\lambda_i| - C_4\,\delta\big(\sum\lambda_i^2\big)^{1/2};\end{aligned}$$

a contradiction.

3. Examples of badly complemented H_n^1 spaces in $H^1(\delta)$. In this section we construct isometric copies of H_n^1 in H^1. We isolate properties of embeddings $i_n\colon H_n^1 \to H^1$ which cause the norm of projections onto $i_n(H_n^1)$ to be large (cf. Theorem 3, part (a)).

These properties are in extreme contrast to those which cause a copy of H_n^1 to be "nicely" complemented (cf. Theorem 3, part (b)). Hence Theorem 3 sheds some light on the ideas behind the proofs in the previous sections.

Construction. Fix $n_0\in\mathbf{N}$. E_{ni} denotes a tree in $[0, 1]$ such that $|E_{ni}| = 2^{-n}$. C_{ni} denotes a collection of dyadic intervals such that

(a) $I, J\in C_{ni}$, $I\neq J$ implies $I\cap J = \emptyset$.

(b) $\bigcup_{I\in C_{ni}} I = E_{ni}$.

Define

$$\tilde{h}_{ni} = \sum_{I\in C_{ni}} h_I,$$

$$Y_{n_0} = \operatorname{span}\{\tilde{h}_{nj}\colon n < n_0,\ 0\leqslant j\leqslant 2^n-1\}.$$

It is easily seen that $H_{n_0}^1 \to H^1$, $h_{ni}\to\tilde{h}_{ni}$, is an isometry onto Y_{n_0}.

THEOREM 3. *Fix* $n_0\in\mathbf{N}$.

(a) *If for any* (m, j), (n, i), $I\in C_{mj}$, $J\in C_{ni}$, $I\subset J$ *implies* $m<n$, *then, for any projection* P_{n_0} *from* $H^1(\delta)$ *onto* Y_{n_0}, *we have* $\|P_{n_0}\| \geqslant \frac{1}{12}\sqrt{n_0}$.

(b) *If*

$$E_{n+1,2i} = E(\sum_{C_{ni}} h_I = 1) \quad \text{and} \quad E_{n+1,2i+1} = E(\sum_{C_{ni}} h_I = -1)$$

then there exists a projection P_{n_0} *from* $H^1(\delta)$ *onto* Y_{n_0} *such that* $\|P_{n_0}\| \leqslant 4$.

Remark. We prove Theorem 3, part (a), without using Bourgain's result on projections onto the image of order-inverting embeddings in H^1. The concrete (and specialized) situation above allows a different (and simple) proof which "lives" entirely in BMO.

Condition (b) connects the tree E_{ni} strongly with C_{ni} and is, in fact, the exact opposite of property (a).

Proof. Let P_{n_0} be a projection from $H^1_{n_0}$ onto Y_{n_0}. Arguing as in [1], p. 49, there exists a linear map ξ_{n_0}: $\mathrm{BMO}_{n_0} \to \mathrm{BMO}$ such that

$$(*) \qquad \|\xi_{n_0}\| \cdot \|\xi_{n_0}^{-1}|_{\xi_{n_0}(\mathrm{BMO}_{n_0})}\| \leqslant \sqrt{2}\,\|P_{n_0}\|,$$

$$(**) \qquad \xi_{n_0} h_{ni} \in \operatorname{span}\{h_I\colon I \in C_{ni}\}, \qquad n \leqslant n_0.$$

Let Q_{ni} denote $\{I \in C_{ni}\colon |\xi_{n_0} h_{ni}| > \delta\}$. For

$$\xi_{n_0} h_{ni} = \sum_{I \in C_{ni}} \alpha_I h_I$$

we get, by the special form of C_{ni},

$$Q_{ni} = \{I \in C_{ni}\colon |\alpha_I| > \delta\}.$$

Now define:

$$\mathscr{B} = \bigcup_{n=0}^{n_0} \bigcup_{i=0}^{2^n-1} Q_{ni}, \qquad h_n = \sum_{i=0}^{2^n-1} h_{ni},$$

$$R_n = \bigcup_{i=0}^{2^n-1} \bigcup_{I \in Q_{ni}} I, \qquad S_n = [0, 1] \setminus R_n, \qquad M = \|\xi_{n_0}^{-1}|_{\xi_{n_0}(\mathrm{BMO}_{n_0})}\|$$

and for δ take $1/(2M)$.

CLAIM 1.

$$\sup_{I \in \mathscr{B}} \frac{1}{|I|} \sum_{J \in \mathscr{B} \cap I} |J| \geqslant (1/M - \delta)^2 n_0.$$

Proof of Claim 1. Take any $\alpha_n \in \boldsymbol{R}$. Then

$$\frac{1}{M}(\sum \alpha_n^2)^{1/2} \leqslant \|\sum_{n=0}^{n_0} \alpha_n \xi_{n_0} h_n\|$$

$$\leqslant \|(\sum_{n=0}^{n_0} \alpha_n \xi_{n_0} h_n)\chi_{R_n}\| + \|(\sum_{n=0}^{n_0} \alpha_n \xi_{n_0} h_n)\chi_{S_n}\|$$

$$\leqslant \sup|\alpha_n| \cdot \sup\Big(\frac{1}{|I|} \sum_{J \in \mathscr{B} \cap I} |J|\Big)^{1/2} + \delta(\sum_{n=0}^{n_0} \alpha_n^2)^{1/2}.$$

Thus we obtain

$$(1/M - \delta)\frac{(\sum_{n=0}^{n_0} \alpha_n^2)^{1/2}}{\sup|\alpha_n|} \leqslant \sup_{I \in \mathscr{B}} \Big(\frac{1}{|I|} \sum_{J \in \mathscr{B} \cap I} |J|\Big)^{1/2}.$$

This last estimate proves Claim 1.

CLAIM 2. *There exists a sequence* $j(n)$, $0 \leqslant j(n) \leqslant 2^n - 1$, $n \leqslant n_0$, *such that*

$$E_{1,j(1)} \supset \ldots \supset E_{n,j(n)} \supset \ldots \supset E_{n_0,j(n_0)}$$

and such that for

$$\mathscr{A} = \bigcup_{n=1}^{n_0} Q_{n,j(n)}$$

we have

$$\sup_{I \in \mathscr{A}} \frac{1}{|I|} \sum_{J \in \mathscr{A} \cap I} |J| \geqslant (1/M - \delta)^2 n_0.$$

Proof of Claim 2. By the hypothesis on (C_{ni}) we may assume that there exist $j(n_0)$ and $I_0 \in C_{n_0,j(n_0)}$ such that

$$\frac{1}{|I_0|} \sum_{J \in \mathscr{B} \cap I_0} |J| \geqslant \tfrac{1}{2}(1/M - \delta)^2 n_0.$$

$(n_0, j(n_0))$ defines uniquely a sequence of nested dyadic intervals $(E_{n,j(n)})$ which contain $E_{n_0,j(n_0)}$. Again, by the hypothesis on C_{ni},

$$\frac{1}{|I_0|} \sum_{J \in \mathscr{B} \cap I_0} |J| = \frac{1}{|I_0|} \sum_{n=1}^{n_0} \sum_{J \in I_0 \cap Q_{n,j(n)}} |J|$$

and this proves Claim 2.

Now we come back to the proof of Theorem 3(a). Take $I_0 \subset E_{n_0,j(n_0)}$ and $(n, j(n))$, $n \leqslant n_0$, as obtained in Claim 2. We will now estimate $\|\sum \xi h_{n,j(n)}\|_{\mathrm{BMO}}$, $\xi = \xi_{n_0}$, from below.

$$\|\sum \xi h_{n,j(n)}\|^2_{\mathrm{BMO}} \geqslant \frac{1}{|I_0|} \int_{I_0} \big(\sum_n \xi h_{n,j(n)} - (\sum_n \xi h_{n,j(n)})_{I_0}\big)^2$$

$$= \frac{1}{|I_0|} \int_{I_0} (\sum_n \xi h_{n,j(n)})^2 = \frac{1}{|I_0|} \int_{I_0} \sum_n (\xi h_{n,j(n)})^2$$

$$= \frac{1}{|I_0|} \int_{I_0} \sum_{m=1}^{n_0} \sum_{I \subset C_{n,j(n)}} h_I^2 \alpha_I^2$$

$$\geqslant \frac{1}{|I_0|} \sum_n \sum_{\substack{I \in Q_{n,j(n)} \\ I \subset I_0}} |I| |\alpha_I|^2$$

$$\geqslant \delta^2 \frac{1}{|I_0|} \sum_{\substack{I \in Q_{n,j(n)} \\ I \subset I_0}} |I| \geqslant \delta^2 (1/M - \delta)^2 n_0.$$

On the other hand we get:

$$\Big\| \sum_{n=1}^{n_0} h_{n,j(n)} \Big\|_{\mathrm{BMO}} \leqslant 4.$$

Hence

$$\|P_{n_0}\| \geqslant \|\xi\| \, \|\xi^{-1}\| \geqslant \frac{1}{\delta} 4 \frac{\|\sum_n \xi h_{n,j(n)}\|}{\|\sum h_{n,j(n)}\|} \geqslant \frac{1}{\delta} \frac{\delta^2}{4} n_0^{1/2}.$$

Using the fact that i_{n_0} is an isometry and (*) we obtain the estimate $\delta \geqslant \frac{1}{3}$ and consequently $\|P_{n_0}\| \geqslant \frac{1}{12} n_0^{1/2}$.

Part (b) is a special case of Theorem 0. The estimate $\|P_{n_0}\| \leqslant 4$ follows from the calculations in this special case.

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Continuous factorizations of covariance operators and Gaussian processes

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Abstract. A bounded linear operator $Q \in L(E', E)$, defined on the dual E' of a Banach space E with values in E, is called a *covariance operator* if Q is positive, symmetric and compact. If E is separable, such an operator Q is always of the form $Q = T \circ T^*$ where T is a bounded linear operator from the Hilbert space l^2 into E. The following theorem is proved. Let $P_c(E)$ denote the set of all covariance operators. Then there is a universal map T from $P_c(E)$ into $L(l^2, E)$ such that $Q = T(Q) \circ T(Q)^*$ for all $Q \in P_c(E)$ and such that T is continuous, if $P_c(E)$ and $L(l^2, E)$ are equipped e.g. with the norm topology. Roughly speaking, it is always possible to make a continuous choice of "square roots" for a given continuous family of covariance operators. This pure functional analytic theorem has the following application to probability theory. If $(\varrho_s)_{s \in S}$ is a continuously indexed family of Gaussian measures on a separable Banach space E (continuous relative to the topology of weak convergence of probability measures), then there is always a Gaussian process $(X_s)_{s \in S}$ associated with the family $(\varrho_s)_{s \in S}$ which is e.g. mean square continuous.

1. Introduction. A (centered) Gaussian measure ϱ on a real separable Banach space E is usually defined as a probability measure on E such that all one-dimensional projections of ϱ are normal distributions with mean zero. It follows that the Fourier transform $\hat{\varrho}$: $E' \to \mathbf{C}$, defined on the dual E' of E, is given by

$$\hat{\varrho}(f) = \exp\Big(-\tfrac{1}{2} \int_E \langle x, f \rangle^2 \, \varrho(dx)\Big)$$

for all $f \in E'$. Hence ϱ is uniquely determined by the bilinear form $\int_E x \otimes x \, \varrho(dx)$ on $E' \times E'$, defined by

$$\Big(\int_E x \otimes x \, \varrho(dx)\Big)(f, g) = \int_E \langle x, f \rangle \langle x, g \rangle \, \varrho(dx)$$

for all $f, g \in E'$. Since for a Gaussian measure we always have $\int \|x\|^2 \varrho(dx) < \infty$, it follows that the bilinear form $\int x \otimes x \, \varrho(dx)$ is given by a continuous linear operator Q: $E' \to E$, where

$$\langle Qf, g \rangle = \int \langle x, f \rangle \langle x, g \rangle \, \varrho(dx) \qquad (f, g \in E').$$