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Generalized Nash–Moser smoothing operators and the structure of Fréchet spaces

by

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Abstract. In [3] E. Dubinsky related the Nash–Moser Inverse Function Theorem to the structure theory of Fréchet spaces via the smoothing operators of Nash–Moser type. Motivated by this, we introduce very general families of smoothing operators and show what implications their existence has on the structure of a Fréchet space.

Introduction. In recent times, some quite unexpected connections between two apparently unrelated topics in Functional Analysis, namely, the Inverse Function Theorem and the structure theory of Fréchet spaces, have begun to be noted (cf. [3]). The unexpectedness is due to the fact that, as everyone knows, the Inverse Function Theorem and linear analysis do not mix well. A crucial point of contact comes from the so-called Nash–Moser Theorem, which is an Inverse Function Theorem in Fréchet spaces based on a refinement of the old Newton's iteration method. (As is well known, the usual Banach space theorem does not go over to Fréchet spaces.) The technique, invented by J. Nash [12] in his solution of the isometric embedding problem for Riemannian manifolds, assumes the existence of an appropriate one-parameter family of smoothing operators on the space. The method was later fashioned by J. Moser [11] into an Inverse Function Theorem in Fréchet spaces which became known as the Nash–Moser Theorem, and wide applicability of the method and its subsequent generalizations (cf. e.g. [8]) was claimed by various authors over the years (see the survey article [5] by R. S. Hamilton). However, the impressive results of D. Vogt [20] (cf. also [4]) show that in the nuclear case, which is the most important in the applications, only a very small class of Fréchet spaces can support a family of smoothing operators of Nash–Moser type. In particular, the nuclear space $H(D)$ of analytic functions on the open unit disc D of the complex plane does not belong to such a class and in [3] smoothing operators supported by this space were found and, through their use, an Inverse Function Theorem valid

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in $H(D)$ was proved (but note that this theorem is quite different in character from the Nash–Moser one). Again, the smoothing operators introduced in [3] are tailored to $H(D)$ and hence are suitable only for a certain class of Fréchet spaces (see § 3).

The purpose of this paper, whose main idea was announced in [10], is to introduce the most general type of a family of smoothing operators and to show that, even in this extremely general setting, the existence of such a family has noteworthy implications for the structure of the Fréchet space under consideration. We wish to appeal to workers in Nonlinear Functional Analysis and it is for this that we repeat some definitions and facts that are well known to Fréchet space specialists. It remains, of course, a big open question whether our enlarged setting enables one to prove an Inverse Function Theorem valid in the more general class of spaces that will be produced here.

1. Preliminaries. Our notation and terminology are standard (cf. e.g. [6]). We also follow [2] for what concerns nuclear Fréchet spaces. Throughout this paper E will always stand for a *Fréchet space*, i.e. a locally convex space which is metrizable and complete. The topology of E is then generated by a sequence of seminorms $\|\ \|_k$, $k = 0, 1, 2, \ldots$ (which, in actual fact, will always be norms). Without loss of generality, we shall assume from now on that *the sequence of seminorms is increasing*, i.e. that, for every k,

(1) $$\|x\|_k \leqslant \|x\|_{k+1} \quad \text{for all } x \in E.$$

In this case, the sequence $(\|\ \|_k)$ is also called a *grading* and, once a choice of grading is made, the structure $(E, (\|\ \|_k))$ is said to be a *graded Fréchet space* (cf. [5], p. 133).

We shall also find it useful to set

$$E_k = (E, \|\ \|_k) \quad \text{and} \quad U_k = \{x \in E\colon \|x\|_k \leqslant 1\},$$

so that U_k is the unit ball of the seminormed space E_k and the sequence (U_k) forms a basis of neighbourhoods of 0 in E. Obviously (1) is equivalent to

(2) $$U_{k+1} \subset U_k \quad \text{for each } k = 0, 1, \ldots$$

The Fréchet space E is *Montel* if all of its bounded sets are relatively compact. Also, E will be called: (a) *Schwartz*, (b) *nuclear*, (c) *countably normed*, if the seminorms $\|\ \|_k$ may be chosen so that the canonical maps $E_{k+1} \to E_k$ are, respectively: (a) precompact, (b) nuclear, (c) such that the induced maps $\tilde{E}_{k+1} \to \tilde{E}_k$ are injective, where, for each k, $\tilde{E}_k$ is the completion of the normed space associated to E_k.

1.1. Remark. (b) $\Rightarrow$ (a) $\Rightarrow$ Montel, while (c) implies that E *has a continuous norm* and is equivalent to the following (cf. [2], p. 168):

(*) *There is a k_0 with the property that, whenever $j > k \geqslant k_0$, if (x_n) is a sequence in E which is Cauchy in E_j and converges to 0 in E_k, then (x_n) converges to 0 also in E_j.*

We abbreviate "continuous linear map" to "operator" and denote by I_E the identity map of E. Then E is said to have:

(a) the *approximation property* (AP) if there is a net of finite-rank operators on E which converges to I_E uniformly on compact subsets of E;

(b) the *bounded approximation property* (BAP) if the net in (a) may be chosen so as to be equicontinuous;

(c) an *unconditional partition of the identity* (UPI) if there exists a sequence of finite-rank operators $T_n\colon E \to E$ such that

(3) $$x = \sum_n T_n x \quad \text{unconditionally for each } x \in E;$$

(d) a *finite-dimensional decomposition* (FDD) if the operators T_n in (c) are projections, i.e. such that $T_m T_n = \delta_{mn} T_n$;

(e) an *absolute UPI* if the series in (3) converges absolutely;

(f) a *basis* if the operators T_n in (d) are such that $\dim T_n(E) = 1$;

(g) a *Schauder decomposition* if in (d) the requirement that the projections T_n have finite rank is dropped.

1.2. Remarks. (i) We have the implications

$$\text{(f)} \Rightarrow \text{(d)} \Rightarrow \text{(b)} \Rightarrow \text{(a)}, \quad \text{(d)} \Rightarrow \text{(g)}, \quad \text{(c)} \Rightarrow \text{(b)}.$$

To see that both (c) and (d) imply (b) put $S_m = \sum_{n \leqslant m} T_n$, thereby obtaining a sequence (S_m) of finite-rank operators converging pointwise to I_E. But E is barrelled, hence the sequence (S_m) is equicontinuous and, therefore, it converges to I_E uniformly on compact subsets of E.

(ii) On an equicontinuous set of operators the topology of uniform convergence on compact subsets of E coincides with the topology of pointwise convergence on a dense subset D of E. The latter topology will be metrizable if E is separable and D is taken to be countable and hence, if E has also BAP, from the net in definition (b) we may extract a sequence of finite-rank operators converging to I_E pointwise. It follows that this happens in all Fréchet–Montel spaces, since the latter are separable.

(iii) Every nuclear space has AP. Also, in nuclear Fréchet spaces every basis is absolute, hence unconditional ([14], 10.1.2 and 10.2.1).

2. Nash–Moser smoothing operators. The smoothing operators introduced by Nash and Moser are a one-parameter family of operators $S_t\colon E \to E$ $(t \geqslant 1)$ on a graded Fréchet space E such that, for all $j \geqslant k \geqslant 0$ and all $x \in E$, we have

(4) $$\|S_t x\|_j \leqslant c t^{j-k} \|x\|_k \quad \text{and} \quad \|x - S_t x\|_k \leqslant d t^{k-j} \|x\|_j,$$

with c and d positive constants depending on k, j. A space E admitting a choice of grading for which there exists such a family (S_t) will be said to have the *Nash–Moser smoothing operator property* (NMSOP). We refer to [5], II,1.3, where many examples of such spaces are to be found. Unfortunately, although covering many usual function spaces (cf. [17] and [19]), the class of Fréchet spaces E with NMSOP is rather restricted, especially in the nuclear case, since we have

2.1. Theorem (D. Vogt [20]; E. Dubinsky and D. Vogt [4]). *If E is nuclear and has NMSOP, then E is isomorphic (via a "good" isomorphism) to a nuclear power series space of infinite type*

$$\Lambda_\infty(\alpha) = \{(\xi_n)\colon \|(\xi_n)\|_k = \sum_n e^{k\alpha_n}|\xi_n| < \infty \text{ for all } k\}.$$

In the general case we have the following weaker result, where we simply write $\otimes$ because the tensor products $\otimes_\pi$ and $\otimes_\varepsilon$ coincide in virtue of the nuclearity of $s = \Lambda_\infty(\log n)$.

2.2. Theorem. *If E has NMSOP, then E is isomorphic to a subspace of $l^\infty(I)\hat{\otimes}s$ and to a quotient of $l^1(I)\hat{\otimes}s$ for a suitable index set I.*

Proof. It is easily seen that E has D. Vogt's properties (DN) and (Ω) (cf. [15]) and hence the result follows from Lemmas 2.1 and 3.1 of [19].

3. Dubinsky's smoothing operators. The space $H(D)$ of analytic functions on the unit disc of the complex plane cannot have NMSOP. In fact, $H(D)$ is nuclear and isomorphic to the power series space of finite type

$$(5)\qquad \Lambda_1(n) = \{(\xi_n)\colon \|(\xi_n)\|_k = \sum_n e^{-n/k}|\xi_n| < \infty \text{ for all } k\}$$

and this space can never be isomorphic to a subspace of s, as shown in [21]. Thus, in order to prove an Inverse Function Theorem in $H(D)$, Dubinsky introduced a family (S_t) of smoothing operators satisfying, instead of (4), the following inequalities for all $j \geq k \geq 1$ and all $x \in E$:

$$(6)\qquad \|S_t x\|_j \leq ct^{1/k-1/j}\|x\|_k \quad \text{and} \quad \|x - S_t x\|_k \leq dt^{1/j-1/k}\|x\|_j,$$

with c and d positive constants depending on k, j. Of course, the estimates (6) are forced by the particular structure of $H(D)$ or, equivalently, of the space $\Lambda_1(n)$ in (5), and are obtained by considering the operators defined by

$$(7)\qquad S_t(\xi_n) = (\eta_n), \quad \text{with} \quad \eta_n = \begin{cases} \xi_n & \text{for } n \leq m, \\ 0 & \text{for } n > m \end{cases}$$

and $m = [\log t]$. Note that these operators have finite rank.

Unfortunately, as in the case of (4), the inequalities (6) are so strong that they too characterize a very special class of nuclear Fréchet spaces. Indeed, we have

3.1. Theorem. *The following assertions are equivalent:*

(i) *E is nuclear and admits a family (S_t) of operators satisfying (6) with respect to some grading.*

(ii) *E is isomorphic to a nuclear power series space of finite type*

$$\Lambda_1(\alpha) = \{(\xi_n)\colon \|(\xi_n)\|_k = \sum_n e^{-\alpha_n/k}|\xi_n| < \infty \text{ for all } k\}.$$

Proof. (i) $\Rightarrow$ (ii). Given any $k > 1$ we have, by (6),

$$\|x\|_k \leq \|S_t x\|_k + \|x - S_t x\|_k \leq ct^{1-1/(2k)}\|x\|_1 + dt^{-1/(2k)}\|x\|_{2k},$$

from which, computing the minimum on the right-hand side, we obtain

$$\|x\|_k \leq C\|x\|_1^{1/(2k)}\|x\|_{2k}^{1-1/(2k)}.$$

Thus E has property $(\underline{\mathrm{DN}})$ (cf. [16], Definition 1.2). Also, for each $k \geq 1$, if $r > 1$ and $l > 2k$ we have, for any $x \in U_{2k}$,

$$\|x - S_t x\|_k \leq dt^{-1/(2k)}\|x\|_{2k} \leq dt^{-1/(2k)},$$
$$\|S_t x\|_l \leq ct^{1/(2k)-1/l}\|x\|_{2k} \leq ct^{1/(2k)-1/l} \leq ct^{1/(2k)}$$

and hence, choosing $t = (dr)^{2k}$, we obtain (with $C = cd$)

$$U_{2k} \subset Cr\,U_l + \frac{1}{r}U_k.$$

Thus E has also property $(\bar{\Omega})$ (cf. [16], Definition 1.3) and (ii) follows from Satz 1.6 of [16].

(ii) $\Rightarrow$ (i). It suffices to give E the grading of $\Lambda_1(\alpha)$ and to define S_t as in (7), where now m is such that $e^{\alpha_m} \leq t < e^{\alpha_{m+1}}$.

4. Generalized smoothing operators. In §§ 2 and 3 we have seen that the existence on a Fréchet space of a family of smoothing operators with certain properties forces the space to belong to a well determined class. This is so because such a family imposes, by its nature, rather strong conditions on the structure of the space. Also, inequalities (4) and (6) depend on the choice of grading. It is in order to obtain a family of smoothing operators that can be supported by as many spaces as possible that we give the following very general definition, which is also "grading free" in a fairly general sense.

We say that a Fréchet space E has the *smoothing operator property* (SOP) if it admits a family of operators $S_t\colon E \to E$ $(t \geq 1)$ such that, with respect to some grading $(\|\ \|_k)$ on E,

(8) $S_t\colon E_k \to E_j$ is continuous for all $k, j \geq 0$;

(9) $\|x - S_t x\|_k \leq r_{kj}(t)\|x\|_j$ for all $0 \leq k < j$ and all $x \in E$,

where the functions r_{kj} are such that $r_{kj}(t) \to 0$ as $t \to +\infty$.

We now investigate what kind of structure spaces with SOP must have. First of all, note that (8) is equivalent to

(10) $$\|S_t x\|_j \leqslant s_{jk}(t)\|x\|_x \quad \text{for all } 0 \leqslant k \leqslant j \text{ and all } x \in E,$$

for some positive functions $s_{jk}(t)$.

Clearly (9) and (10) continue to hold if we pass to an equivalent sequence of norms.

4.1. LEMMA. *If E has SOP then it admits a continuous norm.*

Proof. It suffices to show that $\|\ \|_0$ is a norm. Supposing the contrary, we can find an $x \in E$ with $\|x\|_0 = 0$ and $\|x\|_k \neq 0$ for some $k > 0$. We apply (10). For any $t \geqslant 1$ we have

$$\|S_t x\|_k \leqslant s_{k0}(t)\|x\|_0 = 0$$

and hence $\|S_t x\|_k = 0$. It follows that

$$\|x\|_k \leqslant \|x - S_t x\|_k \leqslant r_{k,k+1}(t)\|x\|_{k+1} \to 0 \quad \text{as } t \to \infty,$$

hence $\|x\|_k = 0$ and we have a contradiction.

4.2. LEMMA. *If E has SOP, then the following assertions are equivalent*:

(i) *E is normable* (*hence Banach*).

(ii) *There is an equivalent sequence of norms on E with respect to which* (10) *holds with*

$$\sup_{t \geqslant 1} s_{k+1,k}(t) = c_k < \infty \quad \textit{for all } k.$$

Proof. (i) $\Rightarrow$ (ii). Take $S_t = I_E$ (= the identity of E) for all $t \geqslant 1$.

(ii) $\Rightarrow$ (i). For every k and for every $x \in E$ we have

$$\|x\|_{k+1} \leqslant \|x - S_t x\|_{k+1} + \|S_t x\|_{k+1} \leqslant r_{k+1,k+2}(t)\|x\|_{k+2} + c_k\|x\|_k \leqslant 2c_k\|x\|_k$$

if t is chosen sufficiently large. Thus all norms are equivalent and (i) follows.

The above lemma shows that SOP is uninteresting for normable spaces, hence from now on we shall assume that E is not normable or, equivalently, that

(11) $$s_{jk}(t) \to +\infty \quad \text{as} \quad t \to +\infty \quad \text{whenever} \quad k < j, \quad \text{and} \quad s_{jj}(t) = \text{const}.$$

We now strengthen Lemma 4.1 by proving

4.3. THEOREM. *If E has SOP, then it is countably normed.*

Proof. We use the equivalent formulation (*) of § 1. Let then $j > k \geqslant 0$ and suppose that (x_n) is a sequence in E which is Cauchy for $\|\ \|_j$ and converges to 0 for $\|\ \|_k$. Let $\varepsilon > 0$ be given and let n_ε be such that

$$\|x_n - x_{n_\varepsilon}\|_j < \varepsilon \quad \text{for } n \geqslant n_\varepsilon.$$

Next take a sequence $t_n \nearrow +\infty$ for which $s_{jk}(t_n)\|x_n\|_k \to 0$, so that there will be an m_ε with

$$s_{jk}(t_n)\|x_n\|_k < \varepsilon \quad \text{for } n \geqslant m_\varepsilon.$$

Now choose $p_\varepsilon \geqslant \max(n_\varepsilon, m_\varepsilon)$ such that

$$r_{j,j+1}(t_n)\|x_{n_\varepsilon}\|_{j+1} < \varepsilon \quad \text{for } n \geqslant p_\varepsilon.$$

Then for $n \geqslant p_\varepsilon$ we have

$$\begin{aligned}\|x_n\|_j &\leqslant \|x_n - S_{t_n}x_n\|_j + \|S_{t_n}x_n\|_j \\ &\leqslant \|x_n - x_{n_\varepsilon}\|_j + \|x_{n_\varepsilon} - S_{t_n}x_{n_\varepsilon}\|_j + \|S_{t_n}(x_{n_\varepsilon} - x_n)\|_j + \|S_{t_n}x_n\|_j \\ &\leqslant (1 + s_{jj})\|x_n - x_{n_\varepsilon}\|_j + r_{j,j+1}(t_n)\|x_{n_\varepsilon}\|_{j+1} + s_{jk}(t_n)\|x_n\|_k \\ &< (3 + s_{jj})\varepsilon.\end{aligned}$$

Thus $\|x_n\|_j \to 0$ and E is countably normed.

Lemma 4.1 and Theorem 4.3 already enable us to give the following examples of Fréchet spaces that cannot have SOP: spaces of continuous or C^∞ functions on open sets, strict projective limits of Fréchet spaces with continuous norms, in particular, countable products of Banach spaces, quojections and the twisted spaces constructed in [9] (because for all these spaces Lemma 4.1 fails), as well as Dubinsky's [2] and Vogt's [18] counter-examples to BAP (because BAP $\Rightarrow$ countably normed by [2], Proposition (3.1.6), p. 169).

4.4. PROPOSITION. *If E has SOP and F is a complemented subspace of E, then F has SOP.*

Proof. If P: $E \to F$ is a continuous projection, then (PS_t) is a family of smoothing operators for F. In fact, if $\|Px\|_k \leqslant c_k\|x\|_{\sigma(k)}$ $(x \in E)$ with $\sigma(k)$ a suitable function, then for each k and each $x = Px \in F$ we have

$$\|x - PS_t x\|_k \leqslant c_k\|x - S_t x\|_{\sigma(k)} \leqslant c_k r_{\sigma(k),\sigma(k)+1}(t)\|x\|_{\sigma(k)+1}$$

so that from the sequence $(\|\ \|_k)$ of norms we may extract a subsequence $(\|\ \|_{k_n})$, with $k_{n+1} = \sigma(k_n) + 1$, with respect to which the estimates (9) hold. Then the inequalities in (10) are automatically true:

$$\|PS_t x\|_{k_n} \leqslant c_{k_n}\|S_t x\|_{\sigma(k_n)} \leqslant c_{k_n}s_{\sigma(k_n),k_m}(t)\|x\|_{k_m} \quad \text{for } m < n.$$

4.5. LEMMA. *If E has SOP and $S_t(E)$ is closed* (*in particular, if S_t is a projection*), *then $S_t(E)$ is Banach.*

Proof. Let $E_t = E/S_t^{-1}(0)$, let φ be the quotient map and let R_t: $E_t \to E$ be the induced map, i.e. $R_t \circ \varphi = S_t$. By the Open Mapping Theorem R_t is an isomorphism onto $S_t(E)$, so that there must be a k such that

$\|R_t^{-1}y\|_0 \leqslant c\|y\|_k$ for all $y \in S_t(E)$. Put $\hat{x} = R_t^{-1}y$ and choose $x \in \hat{x}$ such that $\|x\|_0 \leqslant 2\|\hat{x}\|_0$. Then if $j > k$ we have, for $y \in S_t(E)$,

$$\|y\|_k \leqslant \|y\|_j = \|R_t\hat{x}\|_j = \|S_t x\|_j \leqslant s_{j0}(t)\|x\|_0$$
$$\leqslant 2s_{j0}(t)\|R_t^{-1}y\|_0 \leqslant 2cs_{j0}(t)\|y\|_k.$$

Thus all norms $\| \ \|_j$ with $j \geqslant k$ are equivalent on $S_t(E)$ and the result follows.

4.6. THEOREM. *Suppose that E has SOP and that the family (S_t) contains a sequence (S_n) such that each map $S_n - S_{n-1}$ $(S_0 = 0)$ has closed range. Then E is isomorphic to a complemented subspace of a space with a Schauder decomposition.*

Proof. Put $T_1 = S_1$ and $T_n = S_n - S_{n-1}$ for $n > 1$. A proof similar to that of Lemma 4.5 shows that each subspace $T_n(E)$ is closed in E; hence is a Banach space. Now note that, by (9), $x = \sum_n T_n x$ for all $x \in E$. Then it is standard that on E the norms

$$|x|_k = \sup_n \Big\|\sum_{i=1}^{n} T_i x\Big\|_k$$

form a sequence equivalent to the sequence $(\| \ \|_k)$. Define

$$F = \{(y_n):\ y_n \in T_n(E) \text{ and } \sum_n y_n \text{ converges in } E\},$$

with topology given by the sequence of norms

$$|(y_n)|_k = \sup_n \Big\|\sum_{i=1}^{n} y_i\Big\|_k.$$

It is then easy to see that F is a Fréchet space in which the coordinate projections form a Schauder decomposition, that the map $x \to (T_n x)$ is an isomorphism of E into F and that the map $(y_n) \to \big(T_n(\sum_i y_i)\big)$ is a continuous projection.

Note that the spaces considered by R. S. Hamilton in [5] form a particular case of Theorem 4.6 and of the following

4.7. PROPOSITION. *Let B be a Banach space and let E be a Fréchet–Schwartz space with an absolute basis. Then the space $E \hat{\otimes}_\pi B$ has SOP and the corresponding family (S_t) of smoothing operators may be chosen to consist of projections.*

Proof. By assumption, E is isomorphic to a Köthe space

$$\lambda(P) = \{(\xi_n):\ \|(\xi_n)\|_k = \sum_n a_{kn}|\xi_n| < \infty \text{ for all } k\},$$

where the matrix $P = (a_{kn})$ may be chosen to satisfy

$$\lim_n \frac{a_{kn}}{a_{k+1,n}} = 0 \quad \text{for each } k.$$

Since $\lambda(P)$ is perfect and B is complete, by a result of A. Pietsch [13] (cf. also [7], § 41.7.(5)) $E \hat{\otimes}_\pi B$ is isomorphic to the space

$$\lambda(P)(B) = \{(x_n):\ x_n \in B,\ \|(x_n)\|_k = \sum_n a_{kn}\|x_n\|_B < \infty \text{ for all } k\}$$

and the latter is easily seen to have SOP with smoothing operators S_t given by

$$S_t(x_n) = (y_n), \quad \text{where} \quad y_n = \begin{cases} x_n & \text{if } n \leqslant t, \\ 0 & \text{if } n > t. \end{cases}$$

It is clear that such maps S_t are projections.

The above proof also shows that the assertion holds for spaces of the type

$$\lambda(P)(B_n) = \{(x_n):\ x_n \in B_n,\ \|(x_n)\|_k = \sum_n a_{kn}\|x_n\|_{B_n} < \infty \text{ for all } k\},$$

where the B_n are Banach spaces and $\lambda(P)$ is Fréchet–Schwartz.

4.8. Remark. By Proposition 4.4, complemented subspaces of the spaces in Proposition 4.7 also have SOP.

4.9. Remark. The proof of Proposition 4.7 also shows that Köthe spaces $\lambda(P)$ of type Fréchet–Schwartz have the stronger property FSOP discussed in the next section (the same being true, of course, of their complemented subspaces).

5. A stronger property. Often in the applications the smoothing operators S_t have finite rank, as shown by the examples given in § 3. This leads us to introduce the following stronger property: a Fréchet space E will be said to have the *finite-dimensional smoothing operator property* (FSOP) if it has SOP and for a subsequence $(S_n) \subset (S_t)$ the operators S_n have finite rank.

Clearly, also FSOP is invariant for complemented subspaces.

Our central result is the following

5.1. THEOREM. *For a Fréchet space E the following assertions are equivalent:*

(i) *E is Montel and has AP and SOP.*

(ii) *E has FSOP.*

(iii) *E is Schwartz and has BAP and a continuous norm.*

Proof. (i) $\Rightarrow$ (ii). By (10), $S_t(U_0) \subset s_{j0}(t)U_j$ for all $j \geqslant 0$ and all $t \geqslant 1$ and so each S_t is strongly bounded (i.e. maps a neighbourhood of 0 onto a bounded set). Because E is Montel, $S_t(U_0)$ is relatively compact for $t \geqslant 1$. Put $K_n = \overline{S_n(U_0)}$. Because E has AP, for each n there exists an operator T_n: $E \to E$ such that $\dim T_n(E) < \infty$ and

$$\sup_{y \in K_n} \|y - T_n y\|_n \leqslant r_{0n}(n). \tag{12}$$

Given k, j with $k<j$, for all $n \geqslant j$ we have from (12)

$$\|x-T_n S_n x\|_k \leqslant \|x-S_n x\|_k + \|S_n x - T_n S_n x\|_n$$
$$\leqslant r_{kj}(n)\|x\|_j + r_{0n}(n)\|x\|_0 \leqslant 2r_{kj}(n)\|x\|_j \quad (n \text{ large}).$$

Thus putting $R_t = T_n S_n$ for $n \leqslant t < n+1$, we see that (R_t) is a family as in the definition of FSOP for E with the same rate of decreasing $r_{kj}(t)$ as the original family (S_t).

Note that we also have

$$\|R_t x\|_j \leqslant \|T_n S_n x - S_n x\|_n + \|S_n x\|_j \leqslant r_{0n}(n)\|x\|_0 + s_{jk}(n)\|x\|_k$$
$$\leqslant 2s_{jk}(n)\|x\|_k.$$

(ii) $\Rightarrow$ (iii). Pick $m > \dim S_1(E)$ and then choose a sequence $(t_n\colon n \geqslant m)$ such that $t_n \to \infty$ and $\dim S_{t_n}(E) < n$ (allowing repetition of the t_n's if necessary). If k is fixed, then for every $x \in U_{k+1}$ we have

$$x = (x - S_{t_n} x) + S_{t_n} x \in r_{k,k+1}(t_n)\, U_k + S_{t_n}(E)$$

so that for the diameters of U_{k+1} with respect to U_k (cf. [14]) we have

$$d_n(U_{k+1}, U_k) \leqslant r_{k,k+1}(t_n) \to 0 \quad \text{as } n \to \infty.$$

Thus U_{k+1} is precompact in E_k and, since this holds for every k, E is Schwartz. Also, we have $S_{t_n} x \to x$, so that E has BAP. Finally, the existence of a continuous norm on E follows from Lemma 4.1.

(iii) $\Rightarrow$ (i). The hypotheses imply that E is Montel and has AP. To show that it has also SOP we proceed as follows. Let (S_n) be a sequence of finite-rank operators such that $S_n x \to x$ for all $x \in E$. Clearly (8) is satisfied. Also, because the set (S_n) is equicontinuous and E is Schwartz, we may choose a sequence $(\|\ \|_k)$ of norms such that each U_{k+1} is precompact in E_k and

$$\bigcup_n (I_E - S_n)(U_{k+1}) \subset U_k \quad \text{for all } k \geqslant 0.$$

Now $I_E - S_n \to 0$ (as $n \to \infty$) pointwise in E_n, hence also uniformly on each precompact subset of E_k and, in particular, on each U_j with $j > k$. This means that inequalities (11) must hold.

5.2. Corollary. *If E satisfies one of the equivalent assertions of Theorem 5.1, then E has an absolute UPI and is a complemented subspace of a Fréchet–Schwartz space with an FDD and a continuous norm.*

Proof. Apply the results of [1].

Since every nuclear space has AP, from Theorem 5.1 we obtain, in particular,

5.3. Corollary. *In a nuclear Fréchet space SOP is equivalent to FSOP.*

Hence in most of the usual spaces of analysis (which are nuclear) we can always choose smoothing operators with finite rank.

A further case of equivalence is afforded by the

5.4. Proposition. *Suppose that E has SOP and that the family (S_t) contains a sequence of maps S_n with closed ranges. Then $\dim S_n(E) < \infty$ (and hence E has FSOP) if E is Montel or if there is a sequence $(\|\ \|_k)$ of norms for which all the canonical maps $E_j \to E_k$ $(j > k)$ are strictly singular.*

Proof. Apply Lemma 4.5.

5.5. Remark. Finally, note that, by virtue of Theorem 5.1, the spaces $E \hat{\otimes}_\pi B$ of Proposition 4.7 exhibit examples of Fréchet spaces with SOP but without FSOP if $\dim B = \infty$.

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On the M-structure of the operator space $L(CK)$

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Abstract. Let K be a compact Hausdorff space. We determine the centralizer of the space of bounded operators on a complex $C(K)$-space, $L(CK)$, and give a new characterization of the M-ideals of $L(CK)$ which does not resort to higher duals of this operator space.

Introduction. In this note we completely describe the M-structure of $L(CK)$, the space of bounded operators on a complex $C(K)$-space.

In the first section we determine the centralizer of $L(CK)$, which yields a characterization of the duals among the $L(CK)$-spaces as well as a Banach–Stone type theorem. The second section contains a characterization of the M-ideals of $L(CK)$, based on the earlier paper [8]. Our description has the advantage of avoiding higher duals of operator spaces and yields also some further information on the special nature of these spaces.

Our notation and terminology is standard, special objects of M-structure are explained at the beginning of the respective section. B_X denotes the closed unit ball of a Banach space X.

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1. The centralizer. Let us recall that an operator T on a complex Banach space X is said to belong to the algebra Mult X whenever $T'(p) = a_T(p)\,p$ for all $p \in \operatorname{ex} B_{X'}$ and some $a_T(p) \in C$. Belonging to Mult X is known to be equivalent to the following condition [2, p. 57]:

(A) For all $x \in X$, $\|y - \lambda x\| \leqslant r$ for all λ with $|\lambda| \leqslant \|T\|$ and some r implies $\|y - Tx\| \leqslant r$.

The *centralizer* $\mathscr{Z}(X)$ consists of those T in Mult X for which there is an operator $T^* \in$ Mult X such that $a_{T^*}(p) = \overline{a_T(p)}$ for all $p \in \operatorname{ex} B_{X'}$.

(When e.g. A is a unital C^*-algebra, $\mathscr{Z}(A)$ may be canonically identified with the centre of A.)

Let us write $\mathscr{Z}^{\mathbf{R}}(X)$ for the set of all those elements of $\mathscr{Z}(X)$ for which all the eigenvalues $a_T(p)$ are real. It is not difficult to see that

(B) $\mathscr{Z}(X) = \mathscr{Z}^{\mathbf{R}}(X) \oplus i\,\mathscr{Z}^{\mathbf{R}}(X)$ and $\mathscr{Z}^{\mathbf{R}}(X) = \operatorname{Mult} X \cap \mathscr{H}(X)$,