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On the M-structure of the operator space $L(CK)$

by

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Abstract. Let K be a compact Hausdorff space. We determine the centralizer of the space of bounded operators on a complex $C(K)$-space, $L(CK)$, and give a new characterization of the M-ideals of $L(CK)$ which does not resort to higher duals of this operator space.

Introduction. In this note we completely describe the M-structure of $L(CK)$, the space of bounded operators on a complex $C(K)$-space.

In the first section we determine the centralizer of $L(CK)$, which yields a characterization of the duals among the $L(CK)$-spaces as well as a Banach–Stone type theorem. The second section contains a characterization of the M-ideals of $L(CK)$, based on the earlier paper [8]. Our description has the advantage of avoiding higher duals of operator spaces and yields also some further information on the special nature of these spaces.

Our notation and terminology is standard, special objects of M-structure are explained at the beginning of the respective section. B_X denotes the closed unit ball of a Banach space X.

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1. The centralizer. Let us recall that an operator T on a complex Banach space X is said to belong to the algebra Mult X whenever $T'(p) = a_T(p)\,p$ for all $p \in \operatorname{ex} B_{X'}$ and some $a_T(p) \in C$. Belonging to Mult X is known to be equivalent to the following condition [2, p. 57]:

(A) For all $x \in X$, $\|y - \lambda x\| \leqslant r$ for all λ with $|\lambda| \leqslant \|T\|$ and some r implies $\|y - Tx\| \leqslant r$.

The *centralizer* $\mathscr{Z}(X)$ consists of those T in Mult X for which there is an operator $T^* \in$ Mult X such that $a_{T^*}(p) = \overline{a_T(p)}$ for all $p \in \operatorname{ex} B_{X'}$.

(When e.g. A is a unital C^*-algebra, $\mathscr{Z}(A)$ may be canonically identified with the centre of A.)

Let us write $\mathscr{Z}^{\mathbf{R}}(X)$ for the set of all those elements of $\mathscr{Z}(X)$ for which all the eigenvalues $a_T(p)$ are real. It is not difficult to see that

(B) $\quad \mathscr{Z}(X) = \mathscr{Z}^{\mathbf{R}}(X) \oplus i\,\mathscr{Z}^{\mathbf{R}}(X) \quad$ and $\quad \mathscr{Z}^{\mathbf{R}}(X) = \operatorname{Mult} X \cap \mathscr{H}(X)$, where $\mathscr{H}(X)$ denotes the collection of Hermitian operators on X. (An operator T is called *Hermitian* iff $x'Tx$ is real for all $(x', x) \in \Pi(X) := \{(x', x) \in X' \times X\colon x'(x) = \|x\| = \|x'\| = 1\}$, cf. [4].)

1.1. Lemma. *Let $T \in \operatorname{Mult} X$ and put $L_T(S) := TS$ for all $S \in L(X)$. Then L_T belongs to $\operatorname{Mult} L(X)$ and, in addition, $L_T \in \mathscr{Z}^{\mathbf{R}}(L(X))$ whenever $T \in \mathscr{Z}^{\mathbf{R}}(X)$.*

Proof. It is a straightforward calculation involving condition (A) that $L_T \in \operatorname{Mult} L(X)$. $T \in \mathscr{Z}^{\mathbf{R}}(X)$ is also contained in $\mathscr{H}(X)$ and thus L_T is contained in $\mathscr{H}(L(X))$ (use [4, p. 46 and 84]). Now (B) yields $L_T \in \mathscr{Z}^{\mathbf{R}}(L(X))$.

1.2. Theorem. $\mathscr{Z}(L(CK)) = \{L_T\colon T \in \mathscr{Z}(C(K))\}$.

Remark. $\mathscr{Z}(C(K))$ consists of all the operators $\pi(f), f \in C(K)$, defined by $\pi(f)(g) := fg$.

Proof. Define mappings $I_1\colon \mathscr{Z}^{\mathbf{R}}(C(K)) \to \mathscr{Z}^{\mathbf{R}}(L(CK))$ and $I_2\colon \mathscr{Z}^{\mathbf{R}}(L(CK)) \to L(CK)$ by $I_1(T) := L_T$ and $I_2(\tilde{T}) := \tilde{T}(\mathrm{Id})$, respectively, where Id denotes the unit in $L(CK)$. The fact that I_1 is well defined is contained in the above lemma. We show that I_2 is injective: To this end, note first that for $T \in \operatorname{Mult} X$ the kernel and image of T are M-orthogonal, which means that for all $x \in \operatorname{Ker} T$ and $y \in X$

$$\|x + Ty\| = \max\{\|x\|, \|Ty\|\}.$$

Now suppose that $\tilde{T}(\mathrm{Id}) = 0$ for some $\tilde{T} \in \operatorname{Mult} L(CK)$ and let $S \in B_{L(CK)} \cap \operatorname{Im} \tilde{T}$. Then by the above,

$$\|\mathrm{Id} - S\| = \|\mathrm{Id} + S\| = 1,$$

which yields $S = 0$ and $\tilde{T} = 0$ since the identity operator is an extreme point of the unit ball.

Next, we make the observation that the image of I_2 is contained in $\mathscr{Z}^{\mathbf{R}}(CK)$: This is implied by the fact that the range of I_2 is contained in $\mathscr{H}(CK)$ (this is true since $(x' \otimes x, \mathrm{Id}) \in \Pi(L(X))$ for all $(x', x) \in \Pi(X)$), where the latter set is known to be equal to $\mathscr{Z}^{\mathbf{R}}(CK)$ [5, p. 92].

Combining the above results, we see that $I_1 I_2\colon \mathscr{Z}^{\mathbf{R}}(L(CK)) \to \mathscr{Z}^{\mathbf{R}}(L(CK))$ is an injective projection and hence is surjective. In particular, I_1 is surjective, and this is enough to prove our claim.

Remarks. 1. The statement together with its proof that I_2 is injective represents a simplified version of a more general result which is due to G. Wodinski, who has shown that for arbitrary unital Banach algebras I_2 is even isometric [11].

2. The above proof applies for Banach spaces X for which $\mathscr{H}(X)$ is contained in $\mathscr{Z}(X)$, a condition which is satisfied e.g. when X belongs to the class of L^1-preduals (combine [4, p. 85], [5, p. 92], and the fact that $T'' \in \mathscr{Z}(X'')$ implies $T \in \mathscr{Z}(X)$ for a proof).

1.3. Corollary. *$L(CK_1)$ and $L(CK_2)$ are isometrically isomorphic if and only if K_1 and K_2 are homeomorphic.*

Proof. If $L(CK_1) \cong L(CK_2)$ then

$$C(K_1) \cong \mathscr{Z}(L(CK_1)) \cong \mathscr{Z}(L(CK_2)) \cong C(K_2),$$

and with the aid of the classical Banach–Stone theorem the more difficult implication follows.

1.4. Corollary. *$L(CK)$ is a dual precisely when CK is.*

Proof. For dual spaces X (with predual X_*) we always have $L(X) \cong (X \hat{\otimes}_\pi X_*)'$.

On the other hand, assume that $L(CK)$ is isometric to the dual of some Banach space. Using $\mathscr{Z}(L(CK)) \cong C(K)$ we infer that $C(K)$ is a dual, too (i.e. K is hyperstonean) since $\mathscr{Z}(X)$ is known to be a dual space whenever X is ([2, p. 114], [3, p. 25]).

2. M-ideals. This section is devoted to the description of the M-ideals of $L(CK)$. Recall [1, 2] that an *M-ideal* J of a Banach space X is defined to be a closed subspace the polar J° of which is an L-summand of X', i.e. there is a projection P onto J° satisfying the norm condition $\|x'\| = \|Px'\| + \|x' - Px'\|$ for all $x' \in X'$.

Let $D \subset K$. Define a closed subspace $J_{(D)}$ of $L(CK)$ by

$$J_{(D)} = \{T\colon \lim_{t \to k} \|T'(\delta_t)\| = 0 \text{ for all } k \in D\}.$$

We shall show that exactly the subspaces $J_{(D)}$ with D closed are the M-ideals of $L(CK)$. It will be convenient to introduce some more notation. For $T \in L(CK)$ put

$$\nu_T(k) = \|T'(\delta_k)\|, \qquad |T|(k) = \limsup_{t \to k} \nu_T(t)$$

so that $J_{(D)} = \{T\colon |T||_D = 0\}$. Moreover, let $J_D = \{f \in C(K)\colon f|_D = 0\}$. It is well known that the spaces J_D, $D \subset K$ closed, constitute the M-ideals of $C(K)$ [2, p. 40].

2.1. Proposition. *Let $D \subset K$ be closed. Then*

$$J_{(D)} = \{\pi(h) \circ S\colon h \in J_D,\ S \in L(CK)\}.$$

Proof. Let $T \in J_{(D)}$ with $\|T\| = 1$. According to a result of Tong's [9, p. 27] there is $h \in C(K)$ such that

$$|T|^{1/2} \leqslant h \leqslant 1_{K \setminus D}.$$

(Note that $|T|$ is upper semicontinuous by definition.) In particular, $h \in J_D$. It follows easily that

$$|(Tf)(k)| \leqslant h^2(k)\|f\|$$

for $f \in C(K)$ and $k \in K$. Define a function g on K by $g(k) = 1/h(k)$ if $h(k) \neq 0$, $g(k) = 0$ otherwise, and put $Sf = g \cdot Tf$ for $f \in C(K)$. Then $S \in L(CK)$ and $T = \pi(h) \circ S$. The other inclusion is obvious.

2.2. THEOREM. *A closed subspace J of $L(CK)$ is an M-ideal if and only if there is a closed subset D of K such that*

$$J = J_{(D)} = \{T\colon \lim_{t \to k} \|T'(\delta_t)\| = 0 \text{ for all } k \in D\}.$$

Proof. To prove that $J_{(D)}$ is an M-ideal for closed D, we make use of Lima's characterization [10, Th. 6.17]. Hence it suffices to show:

Given $\varepsilon > 0$ and $T_1, T_2, T_3 \in J_{(D)}$, $T \in L(CK)$ with $\|T_i\| \leqslant 1$, $\|T\| \leqslant 1$, there is $S \in J_{(D)}$ such that $\|T_i + T - S\| \leqslant 1 + \varepsilon$ for $i = 1, 2, 3$.

In fact, consider the open set $V = \bigcap_{i \leqslant 3} \{k\colon |T_i|(k) < \varepsilon\}$ and choose a continuous function h with $0 \leqslant h \leqslant 1$, $h(k) = 0$ for $k \in D$, $h(k) = 1$ for $k \notin V$. It is easily verified that $S = \pi(h) T$ has the required properties.

Conversely, assume J is an M-ideal in $L(CK)$. It has been shown in [8] that there is a closed subset D of K such that $J = \{T\colon \pi''(1_D) \cdot T = 0\}$. Here we understand 1_D to be an element of $C(K)''$, and we identify T with its canonical image in $L(CK)''$. Finally, the multiplication of the two elements of the second dual of the Banach algebra $L(CK)$ is the first Arens multiplication as defined e.g. in [4, p. 106].

$J \subset J_{(D)}$: Consider the directed set $\mathscr{V} = \{V \subset K\colon V$ is an open neighbourhood of $D\}$. Choose continuous functions $f_V\colon K \to [0, 1]$ indexed by $V \in \mathscr{V}$ such that $f_V(t) = 0$ for $t \notin V$, $f_V(t) = 1$ for $t \in D$. Then the net $(f_V)_{V \in \mathscr{V}}$ converges to 1_D w.r.t. the topology $\sigma(C(K)'', C(K)')$; this is ensured by the regularity of the measures $\mu \in C(K)'$. It follows that

$$\pi''(f_V) = \pi(f_V) \underset{V \in \mathscr{V}}{\to} \pi''(1_D)$$

w.r.t. $\sigma\big((L(CK))'', (L(CK))'\big)$.

Now let $T \in J$. A continuity property of the Arens multiplication, which is easily derived from the definition, allows us to conclude

$$\pi(f_V) \cdot T \underset{V \in \mathscr{V}}{\to} \pi''(1_D) \cdot T = 0$$

w.r.t. $\sigma\big(L(CK), (L(CK))'\big)$. Note that $h_V := 1 - f_V \in J_D$ and thus

$$\pi(h_V) \cdot T \to T$$

weakly. Therefore T is in the weak closure of $J_{(D)}$, which equals $J_{(D)}$, by Proposition 2.1.

$J_{(D)} \subset J$: Stick to the above notation. Then for $f \in J_D$, $S \in L(CK)$:

$$\begin{aligned}\pi''(1_D) \cdot \pi(f) \cdot S &= \text{weak*-limit}\big(\pi(f_V) \cdot \pi(f) \cdot S\big)\\ &= \text{weak limit}\big(\pi(f_V \cdot f) \cdot S\big)\\ &= \pi(\text{weak limit } f_V f) \cdot S\\ &= 0,\end{aligned}$$

and again the result follows from Proposition 2.1.

Theorem 2.2 shows why the M-ideal $J_{(D)}$ differs from the space of J_D-valued operators $L(CK, J_D)$ in general. T belongs to the latter space if the numerical function ν_T defined above vanishes on D, whereas in order that T belong to $J_{(D)}$ it is necessary that not only ν_T but also its upper semicontinuous regularization $|T|$ vanishes on D.

We are going to examine the relationship between $J_{(D)}$ and $L(CK, J_D)$ a bit closer.

2.3. PROPOSITION. (a) *A compact J_D-valued operator belongs to $J_{(D)}$.*

(b) *Every operator in $J_{(D)}$ is compact iff $K \setminus D$ is discrete.*

Proof. Let $T \in K(CK, J_D)$ (i.e. T is compact and J_D-valued). Then ν_T is a continuous function on K [7, p. 490] so that $|T|$ coincides with ν_T. Hence assertion (a) follows from the above remark.

To prove (b) assume first that $K \setminus D$ is discrete. Let $T = \pi(f) \cdot S \in J_{(D)}$. It is enough to show that $\pi(f)$ is compact, i.e. $t \mapsto \delta_t \circ \pi(f) = f(t) \cdot \delta_t$ is norm-continuous (cf. [7] again). By assumption we only have to consider this property on D, where in fact the function in question is norm-continuous since $f \in J_D$.

On the other hand, let $t_0 \in K \setminus D$ be a cluster point of $K \setminus D$. Choose $h \in J_D$ with $h(t_0) = 1$. It is easy to see that the operator $\pi(h)$ is a noncompact member of $J_{(D)}$.

Remarks. 1) $W(CK, J_D) \subset J_{(D)}$ is false in general (W denoting the operator ideal of weakly compact operators). In fact, let $K = \beta N$ and $D = \beta N \setminus N$. Then by Proposition 2.3, $J_{(D)} = K(CK, J_D) = K(l^\infty, c_0)$. The Josefson–Nissenzweig theorem tells us that $K(X, c_0) \neq L(X, c_0)$ for a Banach space X of infinite dimension, finally the Grothendieck property of l^∞ is equivalent to $L(l^\infty, c_0) = W(l^\infty, c_0)$ so that $J_{(D)}$ is a proper subspace of $W(CK, J_D)$ in this case. (The reader who is unfamiliar with the notions and results employed above is referred to [6] for relevant information on these topics.)

2) Letting $K = \beta N$ and $D = \beta N \setminus N$ we obtain the (well-known) result that $K(l^\infty, c_0)$ is an M-ideal in $L(l^\infty, l^\infty)$.

3) Let us point out that the M-structure of the space $K(CK)$ is far easier to determine. Since $K(CK)$ is isometrically isomorphic to $C(K, M(K))$, it follows from [2, p. 168] that the M-ideals of $K(CK)$ are in one-to-one correspondence with the subspaces $K(CK, J_D)$, $D \subset K$ closed.

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Multilinear singular integrals involving a derivative of fractional order

by

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Abstract. In this paper, we obtain L^2 estimates for certain multilinear singular integrals, which are analogues of the Calderón commutators involving a derivative of fractional order. The estimates are obtained by an application of the tent space theory of Coifman, Meyer, and Stein.

1. Introduction. For $\lambda \in (0, 1]$, consider the derivative of fractional order λ, defined for tempered distributions $f \in \mathscr{S}'(\boldsymbol{R})$ by

$$(|D|^\lambda f)\hat{}(\xi) = |\xi|^\lambda \hat{f}(\xi). \tag{1.1}$$

Here, $\hat{}$ denotes the Fourier transform, defined according to the normalization

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx. \tag{1.2}$$

Let $A_1, \ldots, A_n$: $\boldsymbol{R} \to \boldsymbol{C}$ be locally integrable functions; let M_j, for $1 \leqslant j \leqslant n$, denote the operator of pointwise multiplication by A_j. If T is an operator, let $\delta_j(T) = [M_j, T] = M_j T - TM_j$; let Δ_n denote the iterated commutator $\delta_n \circ \delta_{n-1} \circ \delta_{n-2} \circ \ldots \circ \delta_1$. We consider the multilinear operators

$$C_{\lambda,n} = C_{\lambda,n}(A_1, \ldots, A_n) = \Delta_n(|D|^{\lambda n}), \tag{1.3}$$

$$\tilde{C}_{\lambda,n} = \tilde{C}_{\lambda,n}(A_1, \ldots, A_n) = \Delta_n(H|D|^{\lambda n}) \tag{1.4}$$

where H denotes the Hilbert transform, defined by

$$(Hf)\hat{}(\xi) = -i \operatorname{sgn} \xi \hat{f}(\xi). \tag{1.5}$$

It is easily seen that $C_{\lambda,n} = 0$ if λn is an even integer, and $\tilde{C}_{\lambda,n} = 0$ if λn is an odd integer. For all other positive integers n, it is easily seen that

$$C_{\lambda,n} f(x) = \gamma_n(\lambda) \text{ p.v.} \int K_n(x, y) f(y)\,dy, \tag{1.6}$$

$$\tilde{C}_{\lambda,n} f(x) = \tilde{\gamma}_n(\lambda) \text{ p.v.} \int \tilde{K}_n(x, y) f(y)\,dy \tag{1.7}$$

where $\gamma_n(\lambda)$, $\tilde{\gamma}_n(\lambda)$ are constants depending on n and λ, and

$$K_n(x, y) = |x-y|^{-n\lambda-1} \prod_{j=1}^{n} (A_j(x) - A_j(y)), \tag{1.8}$$

$$\tilde{K}_n(x, y) = \operatorname{sgn}(x-y) K_n(x, y) \tag{1.9}$$