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Multilinear singular integrals involving a derivative of fractional order

by

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Abstract. In this paper, we obtain L^2 estimates for certain multilinear singular integrals, which are analogues of the Calderón commutators involving a derivative of fractional order. The estimates are obtained by an application of the tent space theory of Coifman, Meyer, and Stein.

1. Introduction. For $\lambda \in (0, 1]$, consider the derivative of fractional order λ, defined for tempered distributions $f \in \mathscr{S}'(\boldsymbol{R})$ by

$$(|D|^\lambda f)\hat{}(\xi) = |\xi|^\lambda \hat{f}(\xi). \tag{1.1}$$

Here, $\hat{}$ denotes the Fourier transform, defined according to the normalization

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx. \tag{1.2}$$

Let $A_1, \ldots, A_n$: $\boldsymbol{R} \to \boldsymbol{C}$ be locally integrable functions; let M_j, for $1 \leqslant j \leqslant n$, denote the operator of pointwise multiplication by A_j. If T is an operator, let $\delta_j(T) = [M_j, T] = M_j T - TM_j$; let Δ_n denote the iterated commutator $\delta_n \circ \delta_{n-1} \circ \delta_{n-2} \circ \ldots \circ \delta_1$. We consider the multilinear operators

$$C_{\lambda,n} = C_{\lambda,n}(A_1, \ldots, A_n) = \Delta_n(|D|^{\lambda n}), \tag{1.3}$$

$$\tilde{C}_{\lambda,n} = \tilde{C}_{\lambda,n}(A_1, \ldots, A_n) = \Delta_n(H|D|^{\lambda n}) \tag{1.4}$$

where H denotes the Hilbert transform, defined by

$$(Hf)\hat{}(\xi) = -i \operatorname{sgn} \xi \hat{f}(\xi). \tag{1.5}$$

It is easily seen that $C_{\lambda,n} = 0$ if λn is an even integer, and $\tilde{C}_{\lambda,n} = 0$ if λn is an odd integer. For all other positive integers n, it is easily seen that

$$C_{\lambda,n} f(x) = \gamma_n(\lambda) \text{ p.v.} \int K_n(x, y) f(y)\,dy, \tag{1.6}$$

$$\tilde{C}_{\lambda,n} f(x) = \tilde{\gamma}_n(\lambda) \text{ p.v.} \int \tilde{K}_n(x, y) f(y)\,dy \tag{1.7}$$

where $\gamma_n(\lambda)$, $\tilde{\gamma}_n(\lambda)$ are constants depending on n and λ, and

$$K_n(x, y) = |x-y|^{-n\lambda-1} \prod_{j=1}^{n} (A_j(x) - A_j(y)), \tag{1.8}$$

$$\tilde{K}_n(x, y) = \operatorname{sgn}(x-y) K_n(x, y) \tag{1.9}$$

(see [8], Chapter 3). These operators are generalizations of the so-called Calderón commutators, which arise when λ is taken equal to 1; these commutators have been extensively studied by Calderón, Coifman, McIntosh, Meyer, and others (see [1], [3], [4]). In particular, it is well known that $C_{1,n}$ is bounded on $L^2(\boldsymbol{R})$ if and only if $A_1 \in \mathrm{Lip}_1(\boldsymbol{R})$, i.e., $A_1' \in L^\infty(\boldsymbol{R})$ (see [1] and [6]). Coifman, McIntosh, and Meyer have shown ([3], Theorem III) that $C_{1,n}$ (for n odd) and $\tilde{C}_{1,n}$ (for n even) are bounded on $L^2(\boldsymbol{R})$ provided that $A_1, \ldots, A_n \in \mathrm{Lip}_1(\boldsymbol{R})$. L^2 estimates for Calderón commutators have also been obtained as a straightforward consequence of the amazing theorem of David and Journé ([5]).

Cohen, Gosselin, and others have asked whether it is possible to obtain L^2 estimates for the operators $C_{1,n}$ and $\tilde{C}_{1,n}$ under the assumption that the functions $A_1, \ldots, A_n$ all have differing degrees of smoothness. They found that the only way to obtain such estimates is to replace each occurrence of the quotient $(A_j(x)-A_j(y))(x-y)^{-1}$ with an appropriately adjusted Taylor series remainder of A_j. They were then able to estimate the L^2 norms of these modified operators in terms of the BMO norms of the higher derivatives of the A_j (see [2]).

The case of $\lambda < 1$ is fundamentally different. One might well expect that $A \in \mathrm{Lip}_\lambda(\boldsymbol{R})$ is a necessary and sufficient condition for the L^2 boundedness of $C_{\lambda,1}$ and $\tilde{C}_{\lambda,1}$. But the author has recently shown ([6]) that these operators are bounded on L^2 if and only if $\alpha_1 = |D|^\lambda A_1 \in \mathrm{BMO}(\boldsymbol{R})$; i.e., $A_1 \in I_\lambda(\mathrm{BMO})$, the BMO Sobolev space studied by Strichartz ([9]) which is properly contained in $\mathrm{Lip}_\lambda(\boldsymbol{R})$.

If we consider the restriction of the multilinear operators $C_{\lambda,n}$ and $\tilde{C}_{\lambda,n}$ to the diagonal $A_1 = A_2 = \ldots = A_n = A$, it is easy to obtain an estimate of the form

$$\|C_{\lambda,n}(A, \ldots, A)f\|_2,\ \|\tilde{C}_{\lambda,n}(A, \ldots, A)f\|_2 \leqslant C\|A\|_\lambda^{n-1}\|\alpha\|_*\|f\|_2 \tag{1.10}$$

where $\|\cdot\|_\lambda$ denotes the norm on Lip_λ, $\|\cdot\|_*$ denotes the BMO norm, $\alpha = |D|^\lambda A$, and C is a constant independent of A, f, and n. The author has shown (in [7], Chapter 2) that the estimate (1.10) is valid for $n = 2$; R. R. Coifman has pointed out that (1.10) for $n > 2$ is an immediate consequence, since $|K_n(x, y)|, |\tilde{K}_n(x, y)| \leqslant \|A\|_\lambda^{n-2} K_2(x, y)$ for $n > 2$ in the diagonal case.

It is natural to ask whether it is possible to obtain L^2 estimates for $C_{\lambda,n}$ and $\tilde{C}_{\lambda,n}$ when the functions $A_1, \ldots, A_n$ have differing degrees of smoothness. In this paper we answer the question affirmatively and prove the following result for $n = 2$ or 3:

MAIN THEOREM. *Suppose n is a positive integer, and let $\lambda_j \in (0, 1)$ for $1 \leqslant j \leqslant n$. Let $\lambda = n^{-1}(\lambda_1 + \ldots + \lambda_n)$ and suppose $A_j \in I_{\lambda_j}(\mathrm{BMO})$ with $\alpha_j = |D|^{\lambda_j} A_j$ for $1 \leqslant j \leqslant n$. Then*

$$\|C_{\lambda,n} f\|_2,\ \|\tilde{C}_{\lambda,n} f\|_2 \leqslant C\|f\|_2 \prod_{j=1}^{n} \|\alpha_j\|_* \tag{1.11}$$

where C is a constant independent of $A_1, \ldots, A_n, f$.

The proof of the Main Theorem may be extended to the case of arbitrary n, but in the interest of relative simplicity we give the proof in the case of $n = 2, 3$; the case $n = 1$ is the result already cited (see [6]). It should be noted that estimates of the form (1.11) cannot be obtained from the powerful theorem of David and Journé (see [5]).

We begin by showing that $C_{\lambda,n}$ and $\tilde{C}_{\lambda,n}$ may be expressed in terms of operators of the form

$$\int_0^\infty Y_{0,t}\Big\{\sum_{\sigma\in S_n} M_{a_{\sigma(1)}} Y_{1,t} M_{a_{\sigma(2)}} \ldots Y_{n-1,t} M_{a_{\sigma(n)}} Y_{n,t}\Big\}(t^{1-\lambda})^n \frac{dt}{t}. \tag{1.12}$$

Here, S_n denotes the symmetric group of degree n; for $1 \leqslant j \leqslant n$, M_{a_j} is the operator of pointwise multiplication by $a_j = A_j'$, and, for $0 \leqslant j \leqslant n$, $Y_{j,t} \in \{P_t, Q_t\}$, where $P_t = (I + t^2 D^2)^{-1}$, $Q_t = tDP_t$, and $D = -i(d/dx)$. Expressions of the form (1.12) are obtained by means of the symbolic calculus developed in [3]. Then, following [3], we show that the problem of estimating the operator norm of (1.11) may be reduced to certain estimates in the upper half-plane. The necessary quadratic estimates follow from certain remarkable identities involving the operators P_t and Q_t, together with the Tent Space techniques introduced by Coifman, Meyer, and Stein ([4]). These estimates are computed explicitly in the cases $n = 2, 3$; we then indicate how the proofs may be extended to the case of more general n.

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2. Integral representation formulas for the commutators. In this section, we use the techniques of Coifman, McIntosh, and Meyer to obtain integral representation formulas for the commutators. Following [3] and [6], we define, for $t \neq 0$, the operators $P_t = (I + t^2 D^2)^{-1}$ and $Q_t = tDP_t$. Then P_t and Q_t are the operators of convolution with p_t and q_t, respectively, where $\hat{p}_t(\xi) = \hat{p}(t\xi) = (1 + t^2\xi^2)^{-1}$, $\hat{q}_t(\xi) = \hat{q}(t\xi) = t\xi\hat{p}(t\xi)$, $p(x) = \frac{1}{2}e^{-|x|}$, and $q(x) = (i\,\mathrm{sgn}\,x)\,p(x)$. We also define $R_t = I - P_t$, which may be thought of as convolution with $\delta_0 - p_t$, where δ_0 is the Dirac measure concentrated at 0; and we set $L_t^+ = P_t + iQ_t$, $L_t^- = P_t - iQ_t$. We observe that

$$L_t^\pm = (I \mp itD)^{-1}, \tag{2.1}$$

$$P_t = \tfrac{1}{2}(I + itD)^{-1} + \tfrac{1}{2}(I - itD)^{-1} = \tfrac{1}{2}(L_t^- + L_t^+), \tag{2.2}$$

$$Q_t = \frac{i}{2}(I + itD)^{-1} - \frac{i}{2}(I - itD)^{-1} = \frac{i}{2}(L_t^- - L_t^+). \tag{2.3}$$

We obtain the following result:

LEMMA 2.1. *Let* $\nu\in(0,2)$ *and set* $\varrho_\nu=(2/\pi)\sin(\pi\nu/2)$. *Then*

(a) $$|D|^\nu=\varrho_\nu\int_0^\infty R_t\,t^{-\nu-1}\,dt,$$

(b) $$D\,|D|^{\nu-2}=iH\,|D|^{\nu-1}=\varrho_\nu\int_0^\infty Q_t\,t^{-\nu}\,dt.$$

Proof. Note that

(2.4) $$\int_0^\infty (t\,|\xi|)^2\{1+(t\,|\xi|)^2\}^{-1}t^{-\nu-1}\,dt=|\xi|^\nu\int_0^\infty s^{1-\nu}(1+s^2)^{-1}\,ds,$$

(2.5) $$\int_0^\infty t\xi\,\{1+(t\,|\xi|)^2\}^{-1}t^{-\nu}\,dt=\xi\,|\xi|^{\nu-2}\int_0^\infty s^{1-\nu}(1+s^2)^{-1}\,ds.$$

If we set $\varrho_\nu=\{\int_0^\infty s^{1-\nu}(1+s^2)^{-1}\,ds\}^{-1}$, then (2.4) and (2.5) yield

(2.6) $$|\xi|^\nu=\varrho_\nu\int_0^\infty\{1-\hat p_t(\xi)\}\,t^{-\nu-1}\,dt,$$

(2.7) $$\xi\,|\xi|^{\nu-2}=\varrho_\nu\int_0^\infty \hat q_t(\xi)\,t^{-\nu}\,dt.$$

A calculation using residues shows that $\varrho_\nu=(2/\pi)\sin(\pi\nu/2)$. The lemma now follows from the definition of R_t and Q_t. ∎

If h is any locally integrable function, we denote by M_h the operator of pointwise multiplication by h. If $h\in\mathcal{S}(\boldsymbol{R})$, then M_h is an element of $\mathcal{A}=\mathcal{L}(\mathcal{S}(\boldsymbol{R}))$, the algebra of continuous linear operators on the Schwartz class $\mathcal{S}(\boldsymbol{R})$.

As in [5], we may assume without loss of generality that the functions $A_1,A_2,\ldots,A_n\in C_0^\infty(\boldsymbol{R})$. For $T\in\mathcal{A}$ and $1\leqslant j\leqslant n$, we define

(2.8) $$\delta_j(T)=[M_{A_j},T]=M_{A_j}T-TM_{A_j}.$$

It is easy to see that δ_j is a derivation of the complex algebra $\mathcal{A}$. We shall enumerate some of its most important properties (see also [3]). For ease of notation, let Δ_n denote the iterated commutator $\delta_1\circ\delta_2\circ\ldots\circ\delta_n$; for $1\leqslant j\leqslant n$, let $a_j=A'_j$. We then obtain the following (see [3]):

LEMMA 2.2. *Let* $\alpha,\beta\in\boldsymbol{C}$ *and* $S,T\in\mathcal{A}$. *Let n be a positive integer, let* $1\leqslant j,k\leqslant n$, *and let* $0\leqslant l\leqslant n-1$. *Then, with notation as above, we have*

(a) $\delta_j(\alpha S+\beta T)=\alpha\delta_j(S)+\beta\delta_j(T)$.

(b) $\delta_j(ST)=\delta_j(S)\,T+S\delta_j(T)$.

(c) $\delta_j\circ\delta_k=\delta_k\circ\delta_j$.

(d) *If S is invertible*, $\delta_j(S^{-1})=-S^{-1}\delta_j(S)\,S^{-1}$.

(e) $\delta_j(D)=iM_{a_j}$.

(f) $\delta_j(M_{A_k})=0$.

(g) $\Delta_n(D^l)=0$.

(h) *For* $t\neq0$, $\delta_j(L_t^\pm)=\mp tL_t^\pm M_{a_j}L_t^\pm$.

(i) *For* $t\neq0$, $\Delta_n(L_t^\pm)=(\mp t)^n\sum_{\sigma\in S_n}L_t^\pm M_{a_{\sigma(1)}}L_t^\pm M_{a_{\sigma(2)}}\cdot\ldots\cdot L_t^\pm M_{a_{\sigma(n)}}L_t^\pm$.

(j) *For* $t\neq0$, $$\Delta_n\big((tD)^k L_t^\pm\big)=(\mp i)^k\Delta_n(L_t^\pm)=(\mp1)^{k+n}i^k t^n\sum_{\sigma\in S_n}L_t^\pm M_{a_{\sigma(1)}}L_t^\pm M_{a_{\sigma(2)}}\cdot\ldots\cdot L_t^\pm M_{a_{\sigma(n)}}.$$

Proof. Properties (a)–(g) are elementary; (h) follows from (a), (d), (e), and (2.1). Property (i) follows from (h) by a simple induction argument. We shall indicate the proof of (j). Notice that

(2.9) $$[(tD)^k-i^k]\,L_t^-=i^k[(-itD)^k-I](I+itD)^{-1}=-i^k[I-(-itD)^k][I-(-itD)]^{-1}=-i^k\sum_{j=0}^{k-1}(-itD)^j.$$

Thus, by (a) and (g),

(2.10) $$\Delta_n\big((tD)^k L_t^-\big)=i^k\Delta_n(L_t^-)-i^k\sum_{j=0}^{k-1}(-it)^j\Delta_n(D^j)=i^k\Delta_n(L_t^-).$$

Similarly,

(2.11) $$[(tD)^k-(-i)^k]\,L_t^+=(-i)^k[(itD)^k-I](I-itD)^{-1}=-(-i)^k[I-(itD)^k](I-itD)^{-1}=-(-i)^k\sum_{j=0}^{k-1}(itD)^j$$

so that, by (a) and (g),

(2.12) $$\Delta_n\big((tD)^k L_t^+\big)=(-i)^k\Delta_n(L_t^+)-(-i)^k\sum_{j=0}^{k-1}(it)^j\Delta_n(D^j)=(-i)^k\Delta_n(L_t^+).$$

Then (j) follows from (i), (2.10), and (2.12). ∎

Combining Lemmas 2.1 and 2.2, we obtain

LEMMA 2.3. *Suppose* $\lambda\in(0,1)$. *Let* $[\cdot]$ *denote the greatest integer function, and set* $\mu=(n\lambda+1)-2[(n\lambda+1)/2]$, $\nu=n\lambda-2[n\lambda/2]$. *For any real number* x, *let* $\varrho_x = (2/\pi)\sin(\pi x/2)$. *Then*

(a) *If* $\nu \neq 0$ *and* $K(n, \lambda) = -\frac{1}{2}(-1)^{[n\lambda/2]}\varrho_\nu$, *then*

$$C_{\lambda,n} = K(n,\lambda)\int_0^\infty \sum_{\sigma\in S_n} [L_t^- M_{a_{\sigma(1)}} L_t^- M_{a_{\sigma(2)}} \cdot \ldots \cdot L_t^- M_{a_{\sigma(n)}} L_t^- + (-1)^n L_t^+ M_{a_{\sigma(1)}} L_t^+ M_{a_{\sigma(2)}} \cdot \ldots \cdot L_t^+ M_{a_{\sigma(n)}} L_t^+](t^{1-\lambda})^n \frac{dt}{t}. \tag{2.13}$$

(b) *If* $\mu \neq 0$ *and* $\tilde{K}(n, \lambda) = \frac{1}{2}(-1)^{[(n\lambda+1)/2]}\varrho_\mu$, *then*

$$\tilde{C}_{\lambda,n} = \tilde{K}(n,\lambda)\int_0^\infty \sum_{\sigma\in S_n} [L_t^- M_{a_{\sigma(1)}} L_t^- M_{a_{\sigma(2)}} \cdot \ldots \cdot L_t^- M_{a_{\sigma(n)}} L_t^- + (-1)^{n+1} L_t^+ M_{a_{\sigma(1)}} L_t^+ M_{a_{\sigma(2)}} \cdot \ldots \cdot L_t^+ M_{a_{\sigma(n)}} L_t^+](t^{1-\lambda})^n \frac{dt}{t}. \tag{2.14}$$

Proof. If $\nu \neq 0$, then $\nu \in (0, 2)$; since $n\lambda - \nu$ is even, we have

$$|D|^{n\lambda} = D^{n\lambda-\nu}|D|^\nu = \varrho_\nu \int_0^\infty (tD)^{n\lambda-\nu}(I-P_t)\, t^{-n\lambda-1}\, dt \tag{2.15}$$

by Lemma 2.1. Consequently,

$$\begin{aligned} C_{\lambda,n} = \Delta_n(|D|^{n\lambda}) &= -\varrho_\nu \int_0^\infty \Delta_n\big((tD)^{n\lambda-\nu}P_t\big)\, t^{-n\lambda-1}\, dt \\ &= -\tfrac{1}{2}\varrho_\nu \int_0^\infty \{\Delta_n\big((tD)^{n\lambda-\nu}L_t^-\big) + \Delta_n\big((tD)^{n\lambda-\nu}L_t^+\big)\}\, t^{-n\lambda}\frac{dt}{t}. \end{aligned} \tag{2.16}$$

Combining (2.16) and Lemma 2.2(j), we obtain (2.13).

If $\mu \neq 0$, then $\mu \in (0, 2)$; since $n\lambda + 1 - \mu$ is even, we have

$$H|D|^{n\lambda} = H|D|^\mu D^{n\lambda+1-\mu} = \varrho_\mu \int_0^\infty -iQ_t(tD)^{n\lambda-\mu+1}\, t^{-n\lambda-1}\, dt \tag{2.17}$$

by Lemma 2.1. Thus

$$\begin{aligned} \tilde{C}_{\lambda,n} = \Delta_n(H|D|^{n\lambda}) &= \varrho_\mu \int_0^\infty \Delta_n\big(-iQ_t(tD)^{n\lambda-\mu+1}\big)\, t^{-n\lambda-1}\, dt \\ &= \tfrac{1}{2}\varrho_\mu \int_0^\infty \{\Delta_n\big((tD)^{n\lambda-\mu+1}L_t^-\big) - \Delta_n\big((tD)^{n\lambda-\mu+1}L_t^+\big)\}\, t^{-n\lambda}\frac{dt}{t}. \end{aligned} \tag{2.18}$$

Combining (2.18) and Lemma 2.2(j) yields (2.14). ■

3. Reduction to estimates in the upper half-plane. By Lemma 2.3, $C_{\lambda,n}$ and $\tilde{C}_{\lambda,n}$ may be written as sums of symmetric multilinear operators of the form

$$S(a_1, \ldots, a_n) = \int_0^\infty (t^{1-\lambda})^n M_{n,t}(a_1, \ldots, a_n)\frac{dt}{t} \tag{3.1}$$

where, for $f \in L^2(\boldsymbol{R})$,

$$M_{n,t}(a_1, \ldots, a_n) f = \sum_{\sigma\in S_n} X_{1,t} M_{a_{\sigma(1)}} X_{2,t} M_{a_{\sigma(2)}} \ldots X_{n,t} M_{a_{\sigma(n)}} X_{n+1,t} f, \tag{3.2}$$

with $X_1, X_2, \ldots, X_{n+1} \in \{P, Q\}$. We aim to show that $S(a_1, \ldots, a_n) f$ satisfies the estimate

$$\|S(a_1, \ldots, a_n) f\|_2 \leqslant K(n)\Big(\prod_{j=1}^n \|\alpha_j\|_*\Big)\|f\|_2 \tag{3.3}$$

where $K(n)$ is a constant depending only upon n. Our Main Theorem is an immediate consequence of this.

Let $\boldsymbol{R}_+^2 = \boldsymbol{R} \times (0, \infty)$, and let $\|\cdot\|_2^+$ denote the norm on $L^2(\boldsymbol{R}_+^2, dx\, dt/t)$. In this section, we contend that (3.3), and hence the Main Theorem, are consequences of the following:

MAIN LEMMA. *With notation as above, and under the hypotheses of the Main Theorem, there exists a constant K independent of $A_1, \ldots, A_n, f$, such that, for $X \in \{P, Q\}$,*

$$\|(t^{1-\lambda})^n X_t M_{a_n} M_{n-1,t}(a_1, \ldots, a_{n-1}) f\|_2^+ \leqslant K\Big(\prod_{j=1}^n \|\alpha_j\|_*\Big)\|f\|_2. \tag{3.4}$$

(We convene that, for $n = 1$, "$M_{n-1,t}(a_1, \ldots, a_{n-1})$" is simply P_t or Q_t).

In the interest of simplicity we restrict our attention to the case $n = 2$; the general case is similar. For notational ease we write $a_1 = a$, $a_2 = b$, $X_1 = X$, $X_2 = Y$, $X_3 = Z$. Abusing notation in the usual way, we do not distinguish between a and M_a, b and M_b. We are interested in estimating the L^2 norm of

$$S(a, b) f = \int_0^\infty (X_t a Y_t b Z_t + X_t b Y_t a Z_t) f (t^{1-\lambda})^2 \frac{dt}{t}. \tag{3.5}$$

We claim, first of all, that an expression of the form (3.5) can be written as a sum of expressions of the following types:

$$L(a, b) f = \int_0^\infty (Q_t X_t a Y_t b Z_t + Q_t X_t b Y_t a Z_t) f (t^{1-\lambda})^2 \frac{dt}{t}, \tag{3.6}$$

$$R(a, b) f = \int_0^\infty (X_t a Y_t b Z_t Q_t + X_t b Y_t a Z_t Q_t) f (t^{1-\lambda})^2 \frac{dt}{t}, \tag{3.7}$$

$$I(a, b) f = \int_0^\infty (X_t a W_t Y_t b Z_t + X_t b W_t Y_t a Z_t) f (t^{1-\lambda})^2 \frac{dt}{t} \tag{3.8}$$

where $W, X, Y, Z \in \{P, Q\}$, and the X, Y, Z occurring in (3.6)–(3.8) need not be the same as those occurring in (3.5).

We can easily compute the L^2 norms of (3.6)–(3.8) by duality. If f, g are complex-valued functions in $L^2(\boldsymbol{R})$, let us define the (real) inner product of f and g by setting

$$\langle f \mid g\rangle = \int_{\boldsymbol{R}} f(x)\, g(x)\, dx. \tag{3.9}$$

With respect to this inner product, $P_t^* = P_t$, $Q_t^* = -Q_t$, and multiplication operators are selfadjoint. Let us compute the norm of the operator $L(a, b)$ by duality; it is equal to

$$\sup_{\|f\|_2 = \|g\|_2 = 1} |\langle L(a, b) f \mid g\rangle|. \tag{3.10}$$

Now note that

$$\begin{aligned} |\langle L(a, b) f \mid g\rangle| &= \left|\int_0^\infty \langle Q_t(X_t aY_t bZ_t f + X_t bY_t aZ_t f) \mid g\rangle (t^{1-\lambda})^2 \frac{dt}{t}\right| \\ &= \left|\int_0^\infty \langle X_t aY_t bZ_t f + X_t bY_t aZ_t f \mid Q_t g\rangle (t^{1-\lambda})^2 \frac{dt}{t}\right| \\ &\leqslant \|(t^{1-\lambda})^2 (X_t aY_t bZ_t f + X_t bY_t aZ_t f)\|_2^+ \, \|Q_t g\|_2^+ \\ &\leqslant \|Q_t g\|_2^+ \, \{\|(t^{1-\lambda})^2 X_t aY_t bZ_t f\|_2^+ \\ &\quad + \|(t^{1-\lambda})^2 X_t bY_t aZ_t f\|_2^+\} \end{aligned} \tag{3.11}$$

where we have used the fact that $Q_t^* = -Q_t$, together with the Schwarz inequality and the triangle inequality. An application of the Plancherel theorem shows that

$$\|Q_t g\|_2^+ \leqslant \frac{1}{\sqrt{2}} \|g\|_2 \tag{3.12}$$

(see [3], Proposition 4). Thus estimating the operator norm of $L(a, b)$ is reduced to estimating

$$\|(t^{1-\lambda})^2 X_t aY_t bZ_t f\|_2^+ \quad \text{and} \quad \|(t^{1-\lambda})^2 X_t bY_t aZ_t f\|_2^+. \tag{3.13}$$

The problem of estimating the operator norm of $R(a, b)$ is completely analogous, in view of the fact that $R(a, b)$ and $L(a, b)$ are "essentially" adjoint to one another.

It remains to estimate the operator norm of $I(a, b)$. We have

$$\begin{aligned} |\langle I(a, b) f \mid g\rangle| &= \left|\int_0^\infty \langle X_t aW_t Y_t bZ_t f + X_t bW_t Y_t aZ_t f \mid g\rangle (t^{1-\lambda})^2 \frac{dt}{t}\right| \\ &= \left|\int_0^\infty \{\langle X_t aW_t Y_t bZ_t f \mid g\rangle + \langle X_t bW_t Y_t aZ_t f \mid g\rangle\} (t^{1-\lambda})^2 \frac{dt}{t}\right| \\ &= \left|\int_0^\infty \{\langle Y_t bZ_t f \mid W_t aX_t g\rangle + \langle Y_t aZ_t f \mid W_t bX_t g\rangle\} (t^{1-\lambda})^2 \frac{dt}{t}\right| \\ &\leqslant \left|\int_0^\infty \langle Y_t bZ_t f \mid W_t aX_t g\rangle (t^{1-\lambda})^2 \frac{dt}{t}\right| \\ &\quad + \left|\int_0^\infty \langle Y_t aZ_t f \mid W_t bX_t g\rangle (t^{1-\lambda})^2 \frac{dt}{t}\right|. \end{aligned} \tag{3.14}$$

If we let $\lambda_1 = \delta$, $\lambda_2 = \varepsilon$, then $(t^{1-\lambda})^2 = t^{1-\delta} t^{1-\varepsilon}$, and, by the Schwarz inequality, we obtain

$$\begin{aligned} |\langle I(a, b) f \mid g\rangle| \leqslant{}& \|(t^{1-\varepsilon}) Y_t bZ_t f\|_2^+ \, \|(t^{1-\delta}) W_t aX_t g\|_2^+ \\ &+ \|(t^{1-\delta}) Y_t aZ_t f\|_2^+ \, \|(t^{1-\varepsilon}) W_t bX_t g\|_2^+. \end{aligned} \tag{3.15}$$

Hence the problem of estimating the operator norm of $I(a, b)$ is reduced to that of estimating expressions such as

$$\|(t^{1-\delta}) Y_t aZ_t f\|_2^+ \quad \text{and} \quad \|(t^{1-\varepsilon}) Y_t bZ_t f\|_2^+ \tag{3.16}$$

where $Y, Z \in \{P, Q\}$.

Thus it remains for us to establish our claim that any expression of the form (3.5) may be written as a sum of expressions of the form (3.6)–(3.8). We shall make use of the following identities:

LEMMA 3.1. (a) $P_t = P_t^2 + Q_t^2$.

(b) $t \dfrac{\partial}{\partial t} P_t = -2Q_t^2$.

(c) $t \dfrac{\partial}{\partial t} Q_t = 2P_t Q_t - Q_t$.

Proof. Identities (b) and (c) are given in Proposition 2 of [3]. To prove (a), note that the symbol of $P_t^2 + Q_t^2$ is given by

$$(1+t^2\xi^2)^{-2} + t^2\xi^2(1+t^2\xi^2)^{-2} = (1+t^2\xi^2)^{-1} \tag{3.17}$$

which is the symbol of P_t. ∎

To prove our claim, we consider various cases, corresponding to the various possible values of X, Y, and Z in (3.5):

Case 1: $Y = P$. In this case,

$$S(a, b) f = \int_0^\infty (X_t aP_t bZ_t + X_t bP_t aZ_t) f (t^{1-\lambda})^2 \frac{dt}{t}. \tag{3.18}$$

We may use Lemma 3.1(a) to write $S(a, b)f$ as the sum of $I_0(a, b)f + I_1(a, b)f$, where, for $j = 0$ or 1,

$$I_j(a, b)f = \int_0^\infty (X_t a Y_{j,t}^2 b Z_t + X_t b Y_{j,t}^2 a Z_t) f (t^{1-\lambda})^2 \frac{dt}{t} \tag{3.19}$$

with $Y_0 = P$, $Y_1 = Q$. $I_0(a, b)$ and $I_1(a, b)$ have the same structure as $I(a, b)$.

Case 2: $X = Y = Q$, $Z = P$. In this case,

$$S(a, b)f = \int_0^\infty (Q_t a Q_t b P_t + Q_t b Q_t a P_t) f (t^{1-\lambda})^2 \frac{dt}{t}; \tag{3.20}$$

we integrate by parts, using Lemma 3.1(b), (c). Let $du = t^{1-2\lambda}\,dt$ and $v = (Q_t a Q_t b P_t + Q_t b Q_t a P_t) f$; we obtain

$$S(a, b)f = \frac{1}{1-\lambda} S(a, b)f - \frac{1}{1-\lambda} L_2(a, b)f - \frac{1}{1-\lambda} I_2(a, b)f + \frac{1}{1-\lambda} R_2(a, b)f \tag{3.21}$$

where

$$L_2(a, b)f = \int_0^\infty (Q_t P_t a Q_t b P_t + Q_t P_t b Q_t a P_t) f (t^{1-\lambda})^2 \frac{dt}{t}, \tag{3.22}$$

$$I_2(a, b)f = \int_0^\infty (Q_t a Q_t P_t b P_t + Q_t b Q_t P_t a P_t) f (t^{1-\lambda})^2 \frac{dt}{t}, \tag{3.23}$$

$$R_2(a, b)f = \int_0^\infty (Q_t a Q_t b Q_t^2 + Q_t b Q_t a Q_t^2) f (t^{1-\lambda})^2 \frac{dt}{t}. \tag{3.24}$$

Consequently, we have

$$S(a, b)f = \frac{1}{\lambda} L_2(a, b)f + \frac{1}{\lambda} I_2(a, b)f - \frac{1}{\lambda} R_2(a, b)f, \tag{3.25}$$

where $L_2(a, b)$, $I_2(a, b)$, and $R_2(a, b)$ have the same structure as $L(a, b)$, $I(a, b)$, and $R(a, b)$ respectively.

Case 3: $X = P$, $Y = Z = Q$. This case is essentially adjoint to Case 2. An analogous integration by parts shows that $S(a, b)$ is again expressible as a sum of operators of the form $L(a, b)$, $I(a, b)$, and $R(a, b)$.

Case 4: $X = Y = Z = Q$. In this case

$$S(a, b)f = \int_0^\infty (Q_t a Q_t b Q_t + Q_t b Q_t a Q_t) f (t^{1-\lambda})^2 \frac{dt}{t}. \tag{3.26}$$

Once again, we integrate by parts, using Lemma 3.1(c), letting $du = t^{1-2\lambda}\,dt$, $v = (Q_t a Q_t b Q_t + Q_t b Q_t a Q_t) f$. In this manner we obtain

$$S(a, b)f = \frac{3}{2-2\lambda} S(a, b)f - \frac{1}{1-\lambda} L_3(a, b)f - \frac{1}{1-\lambda} I_3(a, b)f - \frac{1}{1-\lambda} R_3(a, b)f \tag{3.27}$$

where

$$L_3(a, b)f = \int_0^\infty (Q_t P_t a Q_t b Q_t + Q_t P_t b Q_t a Q_t) f (t^{1-\lambda})^2 \frac{dt}{t}, \tag{3.28}$$

$$I_3(a, b)f = \int_0^\infty (Q_t a Q_t P_t b Q_t + Q_t b Q_t P_t a Q_t) f (t^{1-\lambda})^2 \frac{dt}{t}, \tag{3.29}$$

$$R_3(a, b)f = \int_0^\infty (Q_t a Q_t b P_t Q_t + Q_t b Q_t a P_t Q_t) f (t^{1-\lambda})^2 \frac{dt}{t}. \tag{3.30}$$

Consequently, provided $\lambda \neq \frac{1}{2}$, we have

$$S(a, b)f = \frac{2}{2\lambda-1} L_3(a, b)f + \frac{2}{2\lambda-1} I_3(a, b)f + \frac{2}{2\lambda-1} R_3(a, b)f. \tag{3.31}$$

Note that, if $\lambda = \frac{1}{2}$, the operator $\tilde{C}_{\lambda,2} = 0$; moreover, consideration of the formula (2.13) shows that, regardless of the value of λ, the operator (3.26) does not arise in the expansion of $C_{\lambda,2}$. Thus, whenever the operator (3.26) arises, it can be expressed as a sum of operators having the same form as $L(a, b)$, $I(a, b)$, and $R(a, b)$.

This establishes our claim, and thereby shows that, for $n = 2$, the proof of the Main Theorem can be reduced to proving the Main Lemma.

We make a few remarks concerning the case of more general n. Analogous arguments, making use of Lemma 3.1, can be used to show that, in general, any operator of the form (3.1) arising in the expansion of $C_{\lambda,n}$ or $\tilde{C}_{\lambda,n}$ may be expressed as the sum of operators of the form

$$I(a_1, \ldots, a_n) = \int_0^\infty \sum_{\sigma \in S_n} Y_{1,t} M_{a_{\sigma(1)}} \ldots Y_{n,t} M_{a_{\sigma(n)}} Y_{n+1,t} (t^{1-\lambda})^n \frac{dt}{t}, \tag{3.32}$$

in which, for some $j \in \{2, 3, \ldots, n\}$, $Y_{j,t} \in \{P_t^2, Q_t^2, P_t Q_t\}$, and for all other values of j, $Y_j \in \{P, Q\}$; and operators of the form

$$\int_0^\infty (t^{1-\lambda})^n Q_t M_{n,t}(a_1, \ldots, a_n) \frac{dt}{t} \quad \text{or} \quad \int_0^\infty (t^{1-\lambda})^n M_{n,t}(a_1, \ldots, a_n) Q_t \frac{dt}{t} \tag{3.33}$$

with $M_{n,t}$ defined as in (3.2). Duality arguments may then be used to show that the Main Theorem follows from the Main Lemma in the case of $n \geqslant 3$.

4. The tent space. In order to prove the Main Lemma, we will make use of certain ideas from the theory of tent spaces of Coifman, Meyer, and Stein ([4]) together with facts from Hardy space theory. We begin with some definitions.

DEFINITION 4.1. Let $f\colon \boldsymbol{R}_+^2 \to \boldsymbol{C}$ be a measurable function with respect to the measure $dx\,dt$.

(a) The *square area function* of f, $S(f)$, is given by

$$S(f)(x) = \Big[\iint_{|x-y|\leqslant t} |f(y,t)|^2 t^{-2}\,dy\,dt\Big]^{1/2}. \tag{4.1}$$

(b) We say that f is an element of the *tent space* $T_{2,1}$ if and only if $S(f)\in L^1(\boldsymbol{R})$. We define

$$\|f\|_{T_{2,1}} = \|S(f)\|_1. \tag{4.2}$$

(c) We say that f is an *atom* for $T_{2,1}$ if and only if there is a finite interval $I\subseteq \boldsymbol{R}$ such that f is supported in $\hat{I} = \{(x,t)\in \boldsymbol{R}^2_+:\ [x-t,x+t]\subseteq I\}$ and

$$\Big[\iint_{\hat{I}} |f(y,t)|^2 t^{-1}\,dy\,dt\Big]^{1/2} \leqslant |I|^{-1/2}. \tag{4.3}$$

The set $\hat{I}$ is called the *tent based on* I.

Coifman, Meyer, and Stein have obtained the following useful characterization of $T_{2,1}$ (see [4], Lemmas 1 and 2 and Theorem 2):

PROPOSITION 4.1. *Let $f\colon \boldsymbol{R}^2_+\to \boldsymbol{C}$ be a measurable function with respect to $dx\,dt$.*

(a) $\|S(f)\|_2 = \sqrt{2}\,\|f\|_2^+$; *moreover, if f is a $T_{2,1}$-atom, then $f\in T_{2,1}$ and* $\|f\|_{T_{2,1}}\leqslant \sqrt{2}$.

(b) *f is an element of $T_{2,1}$ if and only if there is a sequence $\langle a_k\rangle$ of $T_{2,1}$-atoms and a sequence $\langle \lambda_k\rangle$ of complex coefficients such that*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k, \tag{4.4}$$

$$\sum_{k=1}^{\infty} |\lambda_k| < +\infty. \tag{4.5}$$

Moreover, the $T_{2,1}$ norm of f is equivalent to the infimum over all representations (4.4) *of the sums* (4.5).

There is an intimate relation between the space $T_{2,1}$ and the Hardy space H^1, defined in terms of atoms. We shall recall a few theorems and definitions from Hardy space theory (see [10], section 2).

DEFINITION 4.2. Suppose $q>1$, s is a nonnegative integer, $\varepsilon>\max\{s,0\}$, $\omega = \varepsilon+(1-1/q)$, and $x_0\in\boldsymbol{R}$. Let f be a locally integrable function on $\boldsymbol{R}$, and let $f^{\omega}(x) = f(x)|x-x_0|^{\omega}$.

(a) f is called a $(1,q,s)$-*atom centered at* x_0 if and only if f is supported in a finite interval I centered at x_0, and

$$\|f\|_q \leqslant |I|^{1/q-1}, \tag{4.6}$$

$$\int f(x)x^j\,dx = 0 \quad \text{for all nonnegative integers } j\leqslant s. \tag{4.7}$$

The atomic space $H^{1,q,s}$ is the set of all locally integrable functions f such that

$$f = \sum \lambda_k a_k \tag{4.8}$$

where $\lambda_k\in\boldsymbol{C}$, a_k is a $(1,q,s)$-atom, and $\sum|\lambda_k| < +\infty$.

(b) f is called a $(1,q,s,\varepsilon)$-*molecule centered at* x_0 if and only if $f, f^{\omega}\in L^q(\boldsymbol{R})$, f satisfies (4.7), and

$$\|f\|_q^{\varepsilon/\omega}\,\|f^{\omega}\|_q^{1-\varepsilon/\omega} = \Omega(f) < +\infty. \tag{4.9}$$

It has been shown that, for all $q>1$ and for all nonnegative integers s, $H^{1,q,s} = H^1$, the atomic Hardy space whose dual is BMO (see [10]). Moreover, the quantity

$$\inf\Big\{\sum|\lambda_k|:\ f = \sum\lambda_k a_k,\ a_k\ (1,q,s)\text{-atoms},\ \sum|\lambda_k| < +\infty\Big\} \tag{4.10}$$

is equivalent to all other norms on H^1. Moreover, the $(1,q,s,\varepsilon)$-molecules belong to H^1 and are fundamental building blocks for the space, in the following sense:

PROPOSITION 4.2. *Let $q>1$, and suppose s is a nonnegative integer and $\varepsilon>\max\{s,0\}$. There is a constant C depending on q, s, ε such that*

(a) *If f is a $(1,q,s,\varepsilon)$-molecule, then $f\in H^{1,q,s}$ and*

$$\|f\|_{H^1}\leqslant C\Omega(f). \tag{4.11}$$

(b) *If f is a $(1,q,s)$-atom, then it is also a $(1,q,s,\varepsilon)$-molecule and*

$$\Omega(f)\leqslant C. \tag{4.12}$$

Proof. This is the content of Proposition 2.3 and Theorem 2.9 of [10]. ∎

We now examine the relationship between H^1 and $T_{2,1}$. For convenience, we begin with the following definition.

DEFINITION 4.3. Let $f\in L^1(\boldsymbol{R})$ and let $B,\beta>0$. We say that f is a (B,β)-*psi function* if and only if

$$\hat{f}(0) = 0; \tag{4.13}$$

$$\text{if } |x|\geqslant 1, \text{ then } |f(x)|\leqslant B|x|^{-1-\beta}; \tag{4.14}$$

$$\int_0^{\infty} |\hat{f}(\pm\xi)|\frac{d\xi}{\xi}\leqslant B^2. \tag{4.15}$$

We obtain the following generalization of Theorem 3 of [4]:

PROPOSITION 4.3. *Let ψ be a (B,β)-psi function, $\psi_t(x) = t^{-1}\psi(xt^{-1})$, $f_t(x) = f(x,t)\in T_{2,1}$, and set*

$$g = \int_0^{\infty}\psi_t * f_t\frac{dt}{t}. \tag{4.16}$$

Suppose, moreover, that $0<\varepsilon<\beta$.

(a) *If* f_t *is a* $T_{2,1}$*-atom, then* g *is a* $(1, 2, 0, \varepsilon)$*-molecule, and*

$$\Omega(g) \leqslant C(\varepsilon, \beta) B \tag{4.17}$$

where $C(\varepsilon, \beta)$ *is a constant depending only upon* ε, β.

(b) *If* f_t *is any* $T_{2,1}$ *function, then* $g \in H^1$, *and*

$$\|g\|_{H^1} \leqslant C(\varepsilon, \beta) B \|f_t\|_{T_{2,1}} \tag{4.18}$$

where $C(\varepsilon, \beta)$ *depends only upon* ε, β.

Proof. Note first that (b) is immediate from (a) by Proposition 4.1 and Proposition 4.2(a). Thus it suffices to prove (a).

We begin by observing that, so long as g is integrable, we must have

$$\int_{\boldsymbol{R}} g(x)\, dx = 0 \tag{4.19}$$

because $\hat{\psi}(0) = 0$. The integrability of g will follow from our estimate of $\Omega(g)$.

Suppose that f_t is a $T_{2,1}$-atom supported in a tent $\hat{I}$, and let x_0 be the center of I. We shall show that g is a $(1, 2, 0, \varepsilon)$-molecule centered at x_0. If $q=2$ in Definition 4.2, then $\omega=\varepsilon+1/2$, $\varepsilon/\omega = 2\varepsilon(2\varepsilon+1)^{-1}$, $1-\varepsilon/\omega = (2\varepsilon+1)^{-1}$. Thus

$$\Omega(g) = \|g\|_2^{2\varepsilon(2\varepsilon+1)^{-1}} \|g^{\omega}\|_2^{(2\varepsilon+1)^{-1}} \tag{4.20}$$

where

$$\|g^{\omega}\|_2^{(2\varepsilon+1)^{-1}} = \Big[\int_{\boldsymbol{R}} |g(x)|^2 |x-x_0|^{2\varepsilon+1}\, dx\Big]^{1/(4\varepsilon+2)}. \tag{4.21}$$

We compute $\|g\|_2$ by duality. For $h \in L^2(\boldsymbol{R})$, we have

$$\begin{aligned} |\langle h \,|\, g\rangle| &= \left|\int_0^\infty \langle h \,|\, \psi_t * f_t\rangle \frac{dt}{t}\right| \\ &= \left|\int_0^\infty \langle \tilde{\psi}_t * h \,|\, f_t\rangle \frac{dt}{t}\right| \\ &= (2\pi)^{-1} \left|\iint_{\boldsymbol{R}_+^2} \hat{\psi}(-t\xi)\, \hat{h}(\xi)\, \hat{f}_t(-\xi) \frac{dt\, d\xi}{t}\right| \\ &\leqslant (2\pi)^{-1} \left[\iint_{\boldsymbol{R}_+^2} |\hat{\psi}(-t\xi)\, \hat{h}(\xi)|^2 \frac{d\xi\, dt}{t}\right]^{1/2} \left[\iint_{\boldsymbol{R}_+^2} |\hat{f}_t(-\xi)|^2 \frac{d\xi\, dt}{t}\right]^{1/2} \\ &\leqslant B \|h\|_2 \|f_t\|_2^{\ddagger} \leqslant B \|h\|_2 |I|^{-1/2}, \end{aligned} \tag{4.22}$$

where we have used Plancherel's Theorem and (4.3). Thus

$$\|g\|_2^{2\varepsilon(2\varepsilon+1)^{-1}} \leqslant B^{2\varepsilon(2\varepsilon+1)^{-1}} |I|^{-\varepsilon(2\varepsilon+1)^{-1}}. \tag{4.23}$$

Moreover,

$$\int_{\boldsymbol{R}} |g(x)|^2 |x-x_0|^{2\varepsilon+1}\, dx = I_1 + I_2 \tag{4.24}$$

where, letting $|I|$ denote the Lebesgue measure of I,

$$\begin{aligned} I_1 &= \int_{|x-x_0| \leqslant 10|I|} |g(x)|^2 |x-x_0|^{2\varepsilon+1}\, dx \\ &\leqslant 10^{2\varepsilon+1} |I|^{2\varepsilon+1} \|g\|_2^2 \leqslant 10^{2\varepsilon+1} B^2 |I|^{2\varepsilon}, \end{aligned} \tag{4.25}$$

$$I_2 = \int_{|x-x_0| \geqslant 10|I|} |g(x)|^2 |x-x_0|^{2\varepsilon+1}\, dx. \tag{4.26}$$

For $|x-x_0| \geqslant 10|I|$, we have

$$\begin{aligned} |g(x)| &= \left|\int_0^{|I|/2} \int_I t^{-1} \psi\left(\frac{x-y}{t}\right) f(y, t)\, dy \frac{dt}{t}\right| \\ &\leqslant B \int_0^{|I|/2} \int_I t^{\beta} |x-y|^{-1-\beta} |f(y, t)| \frac{dy\, dt}{t} \\ &\leqslant C(\beta) B |x-x_0|^{-1-\beta} \int_0^{|I|/2} \int_I t^{\beta} |f(y, t)| \frac{dy\, dt}{t} \end{aligned} \tag{4.27}$$

where $C(\beta)$ is a constant depending on β. Now

$$\begin{aligned} &\int_0^{|I|/2} \int_I t^{\beta} |f(y, t)| \frac{dy\, dt}{t} \\ &\leqslant \Big[\int_0^{|I|/2} \int_I t^{2\beta-1}\, dy\, dt\Big]^{1/2} \left(\int_0^{|I|/2} \int_I |f(y, t)|^2 \frac{dy\, dt}{t}\right)^{1/2} \\ &\leqslant (2\beta)^{-1/2} 2^{-\beta} |I|^{\beta+1/2} |I|^{-1/2} = (2\beta)^{-1/2} 2^{-\beta} |I|^{\beta}. \end{aligned} \tag{4.28}$$

Thus

$$|g(x)| \leqslant BC(\beta) |I|^{\beta} |x-x_0|^{-1-\beta} \tag{4.29}$$

where $C(\beta)$ is a constant depending on β. Moreover,

$$\int_{|x-x_0| \geqslant 10|I|} |x-x_0|^{2\varepsilon-2\beta-1}\, dx = (2\varepsilon-2\beta)^{-1} (10|I|)^{2\varepsilon-2\beta} \tag{4.30}$$

whence, by (4.26), (4.29), and (4.30),

$$I_2 \leqslant C(\varepsilon, \beta) B^2 |I|^{2\varepsilon}. \tag{4.31}$$

Thus, by (4.21), (4.24), (4.25), and (4.31),

$$\|g^{\omega}\|_2^{(2\varepsilon+1)^{-1}} \leqslant C(\varepsilon, \beta) B^{(2\varepsilon+1)^{-1}} |I|^{\varepsilon(2\varepsilon+1)^{-1}} \tag{4.32}$$

so that, by (4.20), (4.23), and (4.32),

$$\Omega(g) \leqslant C(\varepsilon, \beta) B. \tag{4.33}$$

This completes the proof of (a). ■

We are now in a position to prove the following useful generalization of Lemma 9 of [4]:

Proposition 4.4. *There is a constant K such that, if*

(a) *φ_1, φ_2 are two functions such that, setting $\varphi_{j,t} = t^{-1}\varphi_j(\cdot\, t^{-1})$, we have*

$$|\varphi_j(x)| \leqslant C_j(1+|x|^2)^{-1}, \tag{4.34}$$

where C_j is independent of $x \in \boldsymbol{R}$ for $j = 1, 2$;

(b) *$g(x, t) = g_t(x)$ satisfies $\gamma = \sup_{t>0} |g_t| \in L^2(\boldsymbol{R})$;*

(c) *$f(x, t) = f_t(x)$ satisfies $f_t \in L^2(\boldsymbol{R}_+^2, dx\,dt/t)$;*

(d) *$M(x, t) = M_t(x)$ satisfies $M_* = \sup_{t>0} \|M_t\|_\infty < \infty$,*

then $F(x, t) = (\varphi_{1,t} * f_t)(x)(\varphi_{2,t} * g_t)(x) M_t(x)$ defines a function in $T_{2,1}$ with norm dominated by $KC_1 C_2 M_* \|f\|_2^+ \|\gamma\|_2$.

Proof. Let φ_t denote the Poisson kernel on $\boldsymbol{R}$; i.e.,

$$\varphi_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}. \tag{4.35}$$

Then, for $j = 1, 2$ and $x \in \boldsymbol{R}$,

$$|\varphi_{j,t}(x)| \leqslant \pi C_j |\varphi_t(x)|. \tag{4.36}$$

Thus

$$|F(y, t)| \leqslant \pi^2 C_1 C_2 M_* \cdot (\varphi_t * |f_t|) \cdot (\varphi_t * |g_t|)(y). \tag{4.37}$$

If $|x-y| \leqslant t$, it is easily seen that

$$(\varphi_t * |f_t|)(y) \leqslant 5 f_t^*(x), \qquad (\varphi_t * |g_t|)(y) \leqslant 5 g_t^*(x) \tag{4.38}$$

where $*$ denotes the Hardy–Littlewood maximal function. Thus the square area function satisfies

$$\begin{aligned} S(F)(x) &\leqslant 25\pi^2 C_1 C_2 M_* \Big[\iint_{|x-y|\leqslant t} |f_t^*(x) g_t^*(x)|^2 t^{-2}\, dy\, dt\Big]^{1/2} \\ &\leqslant 25\sqrt{2}\pi^2 C_1 C_2 M_* \gamma^*(x) \Big(\int_0^\infty |f_t^*(x)|^2 \frac{dt}{t}\Big)^{1/2} \end{aligned} \tag{4.39}$$

whence

$$\|S(F)\|_1 \leqslant 25\sqrt{2}\pi^2 C_1 C_2 M_* \|\gamma^*\|_2 \|f_t^*\|_2^+ \tag{4.40}$$

by the Schwarz inequality. Since the Hardy–Littlewood maximal operator is bounded on $L^2(\boldsymbol{R})$, we obtain, by Fubini's theorem,

$$\|S(F)\|_1 \leqslant KC_1 C_2 M_* \|\gamma\|_2 \|f_t\|_2^+ . \;■ \tag{4.41}$$

The following generalization of Theorem 4 of [4] will be crucial to our proof of the Main Lemma:

Proposition 4.5. *Let $\beta \in (0, 1)$, and suppose that*

(a) *For $j = 1, 2$, φ_j is an L^1 function and C_j, T_j are constants such that $\operatorname{supp}\hat{\varphi}_j \subseteq [-T_j, T_j]$, $\hat{\varphi}_j \in C^2(\boldsymbol{R})$, and, for all $\xi \in \boldsymbol{R}$,*

$$|\hat{\varphi}_j^{(s)}(\xi)| \leqslant C_j T_j^{-s} \tag{4.42}$$

where the superscript denotes the derivative of order s, and $s \in \{0, 1, 2\}$;

(b) *$M(x, t) = M_t(x)$ is a function and $T_0 \geqslant 1$ a constant such that $\operatorname{supp}\hat{M}_t \subseteq [-t^{-1}T_0, t^{-1}T_0]$ and*

$$M_* = \sup_{t>0} \|M_t\|_\infty < +\infty; \tag{4.43}$$

(c) *$g(x, t) = g_t(x)$ satisfies $\gamma = \sup_{t>0} |g_t| \in L^2(\boldsymbol{R})$;*

(d) *$f(x, t) = f_t(x)$ satisfies $f \in L^2(\boldsymbol{R}_+^2, dx\,dt/t)$.*

Furthermore, set $\varphi_{j,t} = t^{-1}\varphi_j(\cdot\, t^{-1})$ for $j = 1, 2$ and define

$$G = H|D|^\beta \int_0^\infty M_t \cdot (\varphi_{1,t} * f_t) \cdot (\varphi_{2,t} * g_t)\, t^{\beta-1}\, dt. \tag{4.44}$$

Then $G \in H^1(\boldsymbol{R})$, and

$$\|G\|_{H^1} \leqslant K_\beta C_1 C_2 (T_0+T_1+T_2)^\beta M_* \|\gamma\|_2 \|f_t\|_2^+ \tag{4.45}$$

where K_β is a constant depending only upon β.

Proof. Let $S = T_0+T_1+T_2$. For $j = 0, 1, 2$, let $\Phi_j = \varphi_{j,S}$; i.e.,

$$\Phi_j(x) = S^{-1}\varphi_j(xS^{-1}), \qquad \hat{\Phi}_j(\xi) = \hat{\varphi}_j(S\xi). \tag{4.46}$$

Note that $\operatorname{supp}\hat{\Phi}_j \subseteq [-1, 1]$. For $t > 0$, we let $\Phi_{j,t} = t^{-1}\Phi_j(\cdot\, t^{-1})$.

Let $\eta \in C_0^\infty(\boldsymbol{R})$ be an even nonnegative function, supported in $[-2, 2]$ and identically one on $[-1, 1]$. Define Ψ to be the function for which

$$\hat{\Psi}(\xi) = -i \operatorname{sgn}\xi\, |\xi|^\beta \eta(\xi). \tag{4.47}$$

It is not difficult to show that there is a constant B depending upon β for which Ψ is a (B, β)-psi function. It is easily seen that

$$\begin{aligned} G(x) &= S^\beta \int_0^\infty \Psi_{t/S} * \{M_t \cdot (\varphi_{1,t} * f_t) \cdot (\varphi_{2,t} * g_t)\}(x) \frac{dt}{t} \\ &= S^\beta \int_0^\infty \Psi_t * \{M_{St} \cdot (\eta_t * \Phi_{1,t} * f_{St}) \cdot (\eta_t * \Phi_{2,t} * g_{St})\}(x) \frac{dt}{t}. \end{aligned} \tag{4.48}$$

Now note that

$$(4.49)\qquad \|\Phi_{1,t} * f_{St}\|_2^+ = (2\pi)^{1/2}\left[\iint_{\mathbf{R}_+^2} |\hat{\Phi}(t\xi)\hat{f}_{St}(\xi)|^2 \frac{d\xi\, dt}{t}\right]^{1/2} \leqslant C_1 \|f_t\|_2^+ .$$

Moreover,

$$(4.50)\qquad \big\|\sup_{t>0}\{|\Phi_{2,t} * g_{St}|\}\big\|_2 \leqslant \big\|\sup_{t>0}\{|\varphi_{2,t}| * \gamma\}\big\|_2 \leqslant 4C_2\|\gamma^*\|_2$$

by an application of Theorem 2, Chapter 3 of [8].

It is easily seen that there is a constant $C_\eta > 0$ for which

$$(4.51)\qquad |\eta(x)| \leqslant C_\eta(1+|x|^2)^{-1}.$$

Thus we may apply Proposition 4.4 to obtain

$$(4.52)\ \|M_{St}\cdot(\eta_t * \Phi_{1,t} * f_{St})\cdot(\eta_t * \Phi_{2,t} * g_{St})\|_{T_{2,1}} \leqslant KC_\eta^2 C_1 C_2 M_* \|f_t\|_2^+ \|\gamma\|_2,$$

where K is a purely geometric constant. The estimate (4.45) then follows from (4.48), (4.52), and Proposition 4.3. ■

5. H^1 estimates. We now turn to the proof of the Main Lemma, which involves an estimate in $L^2(\mathbf{R}_+^2, dx\,dt/t)$ which we can obtain by duality. Note that

$$(5.1)\qquad \|(t^{1-\lambda})^n X_t M_{a_n} M_{n-1,t}(a_1, \ldots, a_{n-1}) f\|_2^+$$
$$= \sup_{\|h_t\|_2^+ = 1} \left|\int_0^\infty (t^{1-\lambda})^n \langle X_t M_{a_n} M_{n-1,t}(a_1, \ldots, a_{n-1}) f \mid h_t\rangle \frac{dt}{t}\right|$$
$$= \sup_{\|h_t\|_2^+ = 1} \left|\int_0^\infty (t^{1-\lambda})^n \langle M_{a_n} M_{n-1,t}(a_1, \ldots, a_{n-1}) f \mid X_t h_t\rangle \frac{dt}{t}\right|$$
$$= \sup_{\|h_t\|_2^+ = 1} |\langle \alpha_n | G_{n-1}(a_1, \ldots, a_{n-1}) f\rangle|$$

where

$$(5.2)\qquad (G_{n-1}(a_1, \ldots, a_{n-1}) f)(x)$$
$$= H|D|^{1-\lambda_n}\int_0^\infty [X_t h_t](x)\{M_{n-1,t}(a_1, \ldots, a_{n-1}) f\}(x)(t^{1-\lambda})^n \frac{dt}{t}.$$

Thus the Main Lemma is proved once we have estimates of the form

$$(5.3)\qquad \left\|H|D|^{1-\lambda}\int_0^\infty [X_t h_t](x)[Y_t f](x)(t^{1-\lambda})\frac{dt}{t}\right\|_{H^1} \leqslant C\|h_t\|_2^+\|f\|_2,$$

$$(5.4)\qquad \|G_{n-1}(a_1, \ldots, a_{n-1}) f\|_{H^1} \leqslant K\Big(\prod_{j=1}^{n-1}\|\alpha_j\|_*\Big)\|h_t\|_2^+\|f\|_2, \qquad n \geqslant 2,$$

where C is a constant depending only upon λ, K is independent of $A_1, \ldots, A_{n-1}$, h_t, and f, and Y is equal to P or Q. In this section, we will use

results from the previous section, together with identities involving P and Q, to establish (5.3), and (5.4) for $n = 2, 3$. Finally, we will indicate how (5.4) can be obtained for $n \geqslant 4$.

We proceed via a series of lemmas.

LEMMA 5.1. *Suppose that* $\delta \in (0, 1)$, $A \in \mathscr{S}(\mathbf{R}) \cap \mathrm{Lip}_\delta(\mathbf{R})$, $a = A'$, *and* $X \in \{P, Q\}$. *Then, for all* $t > 0$,

$$(5.5)\qquad \|t^{1-\delta}[X_t a]\|_\infty \leqslant \Gamma(1+\delta)\|A\|_\delta$$

where $\|\cdot\|_\delta$ *denotes the norm in* Lip_δ.

Proof. Since $tDP_t = Q_t$ and $tDQ_t = R_t = I - P_t$, we have

$$(5.6)\qquad t^{1-\delta}[P_t a] = it^{-\delta}(q_t * A),$$

$$(5.7)\qquad t^{1-\delta}[Q_t a] = it^{-\delta}(A - p_t * A).$$

Now, $\hat{q}_t(0) = 0$, $\hat{p}_t(0) = 1$, and $|q_t(z)| = p_t(z) = \frac{1}{2}e^{-|z|}$, so

$$(5.8)\qquad |t^{1-\delta}P_t a(x)| = t^{-1-\delta}\left|\int_{\mathbf{R}} q\left[\frac{x-y}{t}\right]\{A(x)-A(y)\}\,dy\right|$$
$$\leqslant t^{-1-\delta}\int_{\mathbf{R}} p\left[\frac{x-y}{t}\right]\|A\|_\delta |x-y|^\delta\,dy$$
$$= \tfrac{1}{2}\|A\|_\delta \int_{\mathbf{R}} e^{-|z|}|z|^\delta\,dz = \Gamma(1+\delta)\|A\|_\delta;$$

$$(5.9)\qquad |t^{1-\delta}Q_t a(x)| = t^{-1-\delta}\left|\int_{\mathbf{R}} p\left[\frac{x-y}{t}\right]\{A(x)-A(y)\}\,dy\right|$$
$$\leqslant t^{1-\delta}\int_{\mathbf{R}} p\left[\frac{x-y}{t}\right]\|A\|_\delta |x-y|^\delta\,dy = \Gamma(1+\delta)\|A\|_\delta. \ ■$$

LEMMA 5.2. *Let* $\delta \in (0, 1)$. *Then* $I_\delta(\mathrm{BMO})$ *is properly contained in* $\mathrm{Lip}_\delta(\mathbf{R})$. *Moreover, there is a constant* C_δ *such that if* $A \in I_\delta(\mathrm{BMO})$ *and* $\alpha = |D|^\delta A$, *then*

$$(5.10)\qquad \|A\|_\delta \leqslant C_\delta\|\alpha\|_*.$$

Proof. See the proof of Theorem 3.4 of [9]. ■

LEMMA 5.3. *Let* $p(x) = \frac{1}{2}e^{-|x|}$ *and* $q(x) = i\,\mathrm{sgn}\,p(x)$, *as above. Then there are sequences* $\langle p_k\rangle_{k=0}^\infty$, $\langle q_k\rangle_{k=0}^\infty \subseteq \mathscr{S}(\mathbf{R})$ *having the following properties:*

(a) $\hat{p}_k$ *and* $\hat{q}_k$ *are supported in* A_k, *where* $A_0 = [-1, 1]$ *and* $A_k = \{x\colon 2^{k-2} < |x| < 2^k\}$ *for* $k \geqslant 1$.

(b) $\hat{p} = \sum_{k=0}^\infty \hat{p}_k$ *and* $\hat{q} = \sum_{k=0}^\infty \hat{q}_k$.

(c) *There is a constant* C *such that for all nonnegative integers* k *and for* $j = 0, 1, 2,$ *we have*

$$|\hat{p}_k^{(j)}(\xi)| \leqslant C2^{-(2+j)k}, \tag{5.11}$$

$$|\hat{q}_k^{(j)}(\xi)| \leqslant C2^{-(1+j)k}, \tag{5.12}$$

$$|p_k(x)| \leqslant 2C \inf\{2^{-k}, 8^{-k}|x|^{-2}\}, \tag{5.13}$$

$$|q_k(x)| \leqslant 2C \inf\{1, 4^{-k}|x|^{-2}\}, \tag{5.14}$$

$$\|p_k\|_1 \leqslant 4C\cdot 4^{-k}, \tag{5.15}$$

$$\|q_k\|_1 \leqslant 4C\cdot 2^{-k}, \tag{5.16}$$

where, in (5.11) *and* (5.12), *the superscript denotes the derivative of order* j.

Proof. The functions $\hat{p}_k$ and $\hat{q}_k$ are defined and discussed in [6], Section 3; properties (a) and (b) and inequalities (5.11) and (5.12) follow from that discussion. Inequalities (5.13) through (5.16) can be shown via direct computation using the inverse Fourier transform. ∎

In what follows, let $p_{k,t} = t^{-1}p_k(\cdot t^{-1})$ and $q_{k,t} = t^{-1}q_k(\cdot t^{-1})$ for each nonnegative integer k.

Lemma 5.4. *For* $\delta \in (0, 1)$ *there is a constant* C_δ *such that for all* $A \in \mathrm{Lip}_\delta(\boldsymbol{R})$, *and for all* $t > 0$,

$$t^{-\delta}\|A - p_{0,t} * A\|_\infty \leqslant C_\delta \|A\|_\delta, \tag{5.17}$$

$$t^{-\delta}\|p_{k,t} * A\|_\infty \leqslant C_\delta 2^{-(2+\delta)k}\|A\|_\delta \quad \text{for } k > 0, \tag{5.18}$$

$$t^{-\delta}\|q_{k,t} * A\|_\infty \leqslant C_\delta 2^{-(1+\delta)k}\|A\|_\delta \quad \text{for } k \geqslant 0. \tag{5.19}$$

Proof. Note that, since $\hat{p}_0(0) = 1$, we have

$$\begin{aligned} |A(x) - p_{0,t} * A(x)| &= \left| t^{-1}\int_{\boldsymbol{R}} p_0\left[\frac{x-y}{t}\right]\{A(x) - A(y)\}\,dy\right| \\ &\leqslant t^\delta \|A\|_\delta \int_{\boldsymbol{R}} |p_0(z)|\,|z|^\delta\,dz \\ &\leqslant t^\delta \|A\|_\delta \cdot 4C\left\{\int_0^1 z^\delta\,dz + \int_1^\infty z^{\delta-2}\,dz\right\} \\ &= t^\delta \|A\|_\delta \cdot \frac{4C}{1-\delta^2} \end{aligned} \tag{5.20}$$

where the second inequality follows from (5.13). This establishes (5.17). If $k \geqslant 1$, we have $\hat{p}_k(0) = 0$, so that

$$\begin{aligned} |p_{k,t} * A(x)| &= \left| t^{-1}\int_{\boldsymbol{R}} p_k\left[\frac{x-y}{t}\right]\{A(x) - A(y)\}\,dy\right| \\ &\leqslant t^\delta \|A\|_\delta \int_{\boldsymbol{R}} |p_k(z)|\,|z|^\delta\,dz \\ &\leqslant t^\delta \|A\|_\delta \cdot 4C\left\{2^{-k}\int_0^{2^{-k}} z^\delta\,dz + 8^{-k}\int_{2^{-k}}^\infty z^{\delta-2}\,dz\right\} \\ &\leqslant t^\delta \|A\|_\delta \cdot 2^{-(2+\delta)k}\cdot\frac{4C}{1-\delta^2}, \end{aligned} \tag{5.21}$$

where again we have made use of (5.13). This establishes (5.18); an analogous argument, using $\hat{q}_k(0) = 0$ and (5.14), establishes (5.19). ∎

Lemma 5.5. *There is a constant* C *such that if*

(a) $\lambda_1, \ldots, \lambda_n \in (0, 1)$, *with* $\lambda_1 + \lambda_2 + \ldots + \lambda_n = n\lambda$;

(b) *For each* $j \in \{1, \ldots, n\}$, $B_j \in \mathscr{S}(\boldsymbol{R}) \cap \mathrm{Lip}_{\lambda_j}(\boldsymbol{R})$ *and* $b_j = B_j'$;

(c) *For each* $j \in \{1, \ldots, n\}$, $X_j \in \{P, Q\}$, *while* $X_t \in \{P_t, Q_t, R_t\}$;

(d) $g_t(x) = g(x, t)$ *satisfies* $\gamma = \sup_{t>0}|g_t| \in L^2(\boldsymbol{R})$,

then $w_t = (t^{1-\lambda})^n [X_{1,t} b_1]\cdot[X_{2,t} b_2] \ldots [X_{n,t} b_n]\cdot[X_t g_t]$ *satisfies* $\omega = \sup_{t>0}|w_t| \in L^2(\boldsymbol{R})$, *with*

$$\|\omega\|_2 \leqslant C\left(\prod_{j=1}^n \Gamma(1+\lambda_j)\|B_j\|_{\lambda_j}\right)\|\gamma\|_2. \tag{5.22}$$

Proof. By Lemma 5.1,

$$\|\omega\|_2 \leqslant \left(\prod_{j=1}^n \Gamma(1+\lambda_j)\|B_j\|_{\lambda_j}\right)\left\|\sup_{t>0}[X_t g_t]\right\|_2. \tag{5.23}$$

Now note that

$$\sup_{t>0}|[R_t g_t](x)| \leqslant \sup_{t>0}\{|g_t(x)| + |[P_t g_t](x)|\} \leqslant \gamma(x) + \sup_{t>0}|[P_t g_t](x)|. \tag{5.24}$$

Moreover, if $X \in \{P, Q\}$, it is easily seen, by Theorem 2, p. 62 of [8], that

$$\begin{aligned} \sup_{t>0}|[X_t g_t](x)| &\leqslant \left\{\int_{\boldsymbol{R}} \sup_{|y| \geqslant |z|} |p(y)|\,dz\right\}\gamma^*(x) \\ &= \left\{\tfrac{1}{2}\int_{\boldsymbol{R}} e^{-|z|}\,dz\right\}\gamma^*(x) = \gamma^*(x) \end{aligned} \tag{5.25}$$

where * denotes the Hardy–Littlewood maximal function. The result follows from the L^2 boundedness of the Hardy–Littlewood maximal operator. ∎

Lemma 5.6. *Let* n *be an integer greater than* 1, *let* $Y \in \{P, Q\}$, *and let* $h_t \in L^2(\boldsymbol{R}^2_+, dx\,dt/t)$. *Under the hypotheses of Lemma* 5.5, *the functions* G *and* A

defined by

$$G = H|D|^{1-\lambda_n}\int_0^\infty \Big(\prod_{j=1}^{n-1}[X_{j,t}b_j]\Big)\cdot[X_t h_t]\cdot[Y_t g_t](t^{1-\lambda})^n \frac{dt}{t}, \tag{5.26}$$

$$\Lambda = H|D|^{1-\lambda}\int_0^\infty [X_t h_t]\cdot[Y_t g_t](t^{1-\lambda})\frac{dt}{t} \tag{5.27}$$

are in H^1, *and satisfy the estimates*

$$\|G\|_{H^1} \leqslant C\Big(\prod_{j=1}^{n-1}\|B_j\|_{\lambda_j}\Big)\|\gamma\|_2\,\|h_t\|_2^+, \tag{5.28}$$

$$\|\Lambda\|_{H^1} \leqslant K\|\gamma\|_2\,\|h_t\|_2^+ \tag{5.29}$$

where K *is a constant depending on* λ *and* C *is a constant depending on* n *and* $\lambda_1, \ldots, \lambda_n$.

Proof. The proof is an application of Proposition 4.5. Notice that, for each $j \in \{1, 2, \ldots, n-1\}$,

$$(t^{1-\lambda_j})[X_{j,t}b_j] = \sum_{l=0}^{\infty}(t^{1-\lambda_j})\,x_{j,l_j,t}\,b_j \tag{5.30}$$

where $x_{j,l_j,t} = p_{l_j,t}$ or $q_{l_j,t}$ according as $X_j = P$ or Q. In turn, we have

$$(t^{1-\lambda_j})\,x_{j,l_j,t} * b_j = \begin{cases} t^{-\lambda_j}(A - p_{0,t} * A) & \text{if } X_j = Q \text{ and } l_j = 0,\\ t^{-\lambda_j}(p_{l_j,t} * A) & \text{if } X_j = Q \text{ and } l_j \geqslant 1,\\ t^{-\lambda_j}(q_{l_j,t} * A) & \text{if } X_j = P \text{ and } l_j \geqslant 0.\end{cases} \tag{5.31}$$

By Lemma 5.4, we have

$$\sup_{t>0}\|(t^{1-\lambda_j})\,x_{j,l_j,t} * b_j\|_\infty \leqslant C_{\lambda_j}2^{-(1+\lambda_j)l_j}\|B_j\|_{\lambda_j}. \tag{5.32}$$

If we set

$$M_t(l_1, \ldots, l_{n-1}) = \prod_{j=1}^{n-1}(t^{1-\lambda_j})\,x_{j,l_j,t} * b_j, \tag{5.33}$$

$$M_*(l_1, \ldots, l_{n-1}) = \sup_{t>0}\|M_t(l_1, \ldots, l_{n-1})\|_\infty \tag{5.34}$$

then we have

(5.35)

$$\operatorname{supp} M_t(l_1, \ldots, l_{n-1}) \subseteq [-t^{-1}4(2^{l_1}+\ldots+2^{l_{n-1}}),\ t^{-1}4(2^{l_1}+\ldots+2^{l_{n-1}})],$$

$$M_*(l_1, \ldots, l_{n-1}) \leqslant \prod_{j=1}^{n-1}C_{\lambda_j}2^{-(1+\lambda_j)l_j}\|B_j\|_{\lambda_j}. \tag{5.36}$$

Moreover, we have

$$G = \sum_{l_0=0}^{\infty}\sum_{l_1=0}^{\infty}\ldots\sum_{l_n=0}^{\infty}G(l_0, l_1, \ldots, l_n) \tag{5.37}$$

where

$$G(l_0, l_1, \ldots, l_n) = H|D|^{1-\lambda_n}\int_0^\infty (x_{l_0,t} * h_t)\cdot M_t(l_1, \ldots, l_{n-1})\cdot(y_{l_n,t} * g_t)\,t^{-\lambda_n}\,dt \tag{5.38}$$

and $x_{l_0,t} = p_{l_0,t}$ or $q_{l_0,t}$ according as $X = P$ or Q, with $y_{l_n,t}$ defined analogously.

Using Lemma 5.3(a), (c), we may apply Proposition 4.5 with $\beta = 1-\lambda_n$, to see that $G(l_0, l_1, \ldots, l_n) \in H^1(\boldsymbol{R})$, and

$$\begin{aligned}\|G(l_0, l_1, \ldots, l_n)\|_{H^1} &\leqslant K_{\lambda_n}\Big[\sum_{j=0}^{n}2^{l_j}\Big]^{1-\lambda_n}\cdot 2^{-l_0}\cdot 2^{-l_n}\cdot M_*(l_1, \ldots, l_{n-1})\|\gamma\|_2\,\|h_t\|_2^+\\ &\leqslant K_{\lambda_1,\ldots,\lambda_n}\Big[\sum_{j=0}^{n}2^{l_j}\Big]^{1-\lambda_n}\cdot 2^{-(l_0+l_1+\ldots+l_n)}\cdot\Big(\prod_{j=1}^{n-1}\|B_j\|_{\lambda_j}\Big)\|\gamma\|_2\,\|h_t\|_2^+.\end{aligned} \tag{5.39}$$

The estimate (5.28) now follows on combining (5.37) and (5.38).

The estimate (5.29) is still easier. Note that we may write

$$\Lambda = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\Lambda(l, m) \tag{5.40}$$

where

$$\Lambda(l, m) = H|D|^{1-\lambda}\int_0^\infty (x_{l,t} * h_t)\cdot(y_{m,t} * g_t)\,t^{-\lambda}\,dt. \tag{5.41}$$

By a simplification of the argument in Proposition 4.5, we obtain the estimate

$$\|\Lambda(l, m)\|_{H^1} \leqslant K_\lambda(2^l+2^m)^{1-\lambda}2^{-l}2^{-m}\|\gamma\|_2\,\|h_t\|_2^+; \tag{5.42}$$

combining (5.40) and (5.42) yields (5.29). ∎

To complete the proof of the Main Lemma, we show how the estimate (5.4) can be obtained from Lemma 5.6. To do this, we need the following identities involving P and Q:

Lemma 5.7. *Let* f, g *be functions in* $\mathscr{S}(\boldsymbol{R})$, *possibly depending upon* t. *Let* $D_t = tD$. *We have*

(a) $P_t(fg) = [P_t f]\cdot g - Q_t([P_t f]\cdot[D_t g]) - P_t([Q_t f]\cdot[D_t g])$,

(b) $Q_t(fg) = [Q_t f]\cdot g + P_t([P_t f]\cdot[D_t g]) - Q_t([Q_t f]\cdot[D_t g])$,

(c) $$P_t fQ_t g = [P_t f]\cdot[Q_t g]-Q_t([P_t f]\cdot[R_t g])-P_t([Q_t f]\cdot[R_t g]),$$

(d) $$P_t fP_t g = [P_t f]\cdot[P_t g]-Q_t([P_t f]\cdot[Q_t g])-P_t([Q_t f]\cdot[Q_t g]),$$

(e) $$Q_t fQ_t g = [Q_t f]\cdot[Q_t g]+P_t([P_t f]\cdot[R_t g])-Q_t([Q_t f]\cdot[R_t g]),$$

(f) $$Q_t fP_t g = [Q_t f]\cdot[P_t g]+P_t([P_t f]\cdot[Q_t g])-Q_t([Q_t f]\cdot[Q_t g]).$$

Proof. It is not difficult to show that

(5.43) $$L_t^{\pm}(fg) = [L_t^{\pm} f]\cdot g \pm it\, L_t^{\pm}([L_t^{\pm} f]\cdot[D_t g])$$

where, as before, $L^{\pm} = P_t \pm iQ_t = (I \mp iD_t)^{-1}$ (see Section 6 of [3]). From this it is easy to establish (a) and (b). Identities (c) and (e) follow from (a) and (b) by letting $[Q_t g]$ play the role of g; likewise, (d) and (f) follow from (a) and (b) by letting $[P_t g]$ play the role of g. ■

LEMMA 5.8. *Let a, b be functions in $\mathcal{S}(\boldsymbol{R})$, possibly depending upon t. We have*

(a) $$Q_t aQ_t b+Q_t bQ_t a = R_t([Q_t a]\cdot[Q_t b])+R_t([P_t a]\cdot[P_t b]),$$

(b) $$P_t aQ_t b+P_t bQ_t a = Q_t([Q_t a]\cdot[Q_t b])+Q_t([P_t a]\cdot[P_t b]).$$

Proof. Since $Q_t = D_t P_t$ and $R_t = D_t Q_t$, (a) will follow from (b). By Lemma 5.7(c), letting $f = a$ and $g = b$, we have

(5.44) $$P_t aQ_t b = [P_t a]\cdot[Q_t b]-Q_t([P_t a]\cdot[R_t b])-P_t([Q_t a]\cdot[R_t b]).$$

Moreover, since $R_t = I-P_t$, we have

(5.45) $$-Q_t([P_t a]\cdot[R_t b]) = -Q_t bP_t a+Q_t([P_t a]\cdot[P_t b]),$$

(5.46) $$-P_t([Q_t a]\cdot[R_t b]) = -P_t bQ_t a+P_t([Q_t a]\cdot[P_t b])$$

whence, substituting into (5.44), we obtain

(5.47) $$P_t aQ_t b+P_t bQ_t a = [P_t a]\cdot[Q_t b]-Q_t bP_t a +Q_t([P_t a]\cdot[P_t b])+P_t([Q_t a]\cdot[P_t b]).$$

By Lemma 5.7(f), we have

(5.48) $$[P_t a]\cdot[Q_t b]-Q_t bP_t a+P_t([Q_t a]\cdot[P_t b]) = Q_t([Q_t a]\cdot[Q_t b]).$$

Combining (5.47) and (5.48), we obtain (b). ■

We now use Lemmas 5.7 and 5.8 to show how the estimate (5.4) may be obtained in the cases $n = 2$ and $n = 3$. We begin with the case $n = 2$, in which we are concerned with the function

(5.49) $$G_1(a_1)f = H|D|^{1-\lambda_2}\int_0^{\infty}[X_t h_t]\cdot\{X_{1,t}a_1 X_{2,t} f\}(t^{1-\lambda})^2\frac{dt}{t}$$

where X, X_1, $X_2 \in \{P, Q\}$. By Lemma 5.7(c)–(f), $G_1(a_1)f$ may be rewritten as the sum of

(5.50) $$G = H|D|^{1-\lambda_2}\int_0^{\infty}[X_t h_t]\cdot[X_{1,t}a_1]\cdot[X_{2,t} f](t^{1-\lambda})^2\frac{dt}{t}$$

and two functions of the form

(5.51) $$\Lambda = \pm H|D|^{1-\lambda_2}\int_0^{\infty}[X_t h_t]\cdot W_t([Y_t a_1]\cdot[Z_t f])(t^{1-\lambda})^2\frac{dt}{t}$$

where W, $Y \in \{P, Q\}$ and $Z \in \{P, Q, R\}$. We claim that G and Λ are in H^1, with norm bounded by $C\|A_1\|_{\lambda_1}\|f\|_2\|h_t\|_2^{+}$, where C depends only on λ_1 and λ_2. In the case of the function G, this is immediate from Lemma 5.6. For Λ, we use Lemma 5.5 to see that

$$w_t = (t^{1-\lambda_1})[Y_t a_1]\cdot[Z_t f]$$

satisfies $\omega = \sup_{t>0}|w_t| \in L^2(\boldsymbol{R})$, with $\|\omega\|_2 \leqslant C_{\lambda_1}\|A_1\|_{\lambda_1}\|f\|_2$. The desired estimate for Λ is then a consequence of Lemma 5.6. We obtain (5.4) in the case of $n = 2$ by applying Lemma 5.2.

In the case $n = 3$, we consider the function

(5.52) $$G_2(a_1, a_2)f = H|D|^{1-\lambda_3}\int_0^{\infty}[X_t h_t]\cdot M_{2,t}(a_1, a_2)f(t^{1-\lambda})^3\frac{dt}{t}$$

where

(5.53) $$M_{2,t}(a_1, a_2)f = X_{1,t}a_1 X_{2,t}a_2 X_{3,t} f+X_{1,t}a_2 X_{2,t}a_1 X_{3,t} f.$$

We make repeated use of Lemma 5.7(c)–(f), beginning with the expression $X_{2,t}a_j X_{3,t} f$ for $j = 1, 2$. In consequence we see that

(5.54) $$M_{2,t}(a_1, a_2)f = S_{2,t}(a_1, a_2)f+E_{2,t}(a_1, a_2)f$$

where

(5.55) $$S_{2,t}(a_1, a_2)f = X_{1,t}(a_1\cdot[X_{2,t}a_2]\cdot[X_{3,t} f])+X_{1,t}(a_2\cdot[X_{2,t}a_1]\cdot[X_{3,t} f])$$

and $E_{2,t}(a_1, a_2)f$ is a sum of functions of the form

(5.56) $$W_t([Y_{0,t}a_j]\cdot Z_{0,t}([Y_{1,t}a_k]\cdot[Z_{1,t} f]))$$

where $W_t \in \{I, P_t, Q_t\}$, Y_0, $Y_1 \in \{P, Q\}$, and Z_0, $Z_1 \in \{P, Q, R\}$. Defining

(5.57) $$E_2(a_1, a_2)f = H|D|^{1-\lambda_3}\int_0^{\infty}[X_t h_t]\cdot E_{2,t}(a_1, a_2)f(t^{1-\lambda})^3\frac{dt}{t},$$

we easily obtain an estimate of the form

(5.58) $$\|E_2(a_1, a_2)f\|_{H^1} \leqslant C_{\lambda_1,\lambda_2,\lambda_3}\|A_1\|_{\lambda_1}\|A_2\|_{\lambda_2}\|h_t\|_2^{+}\|f\|_2$$

by Lemmas 5.5 and 5.6. It therefore remains to estimate

$$S_2(a_1, a_2)f = H|D|^{1-\lambda_3}\int_0^\infty [X_t h_t]\cdot S_{2,t}(a_1, a_2) f (t^{1-\lambda})^3 \frac{dt}{t}. \tag{5.59}$$

It is in estimating this operator that we make crucial use of the fact that $S_{2,t}(a_1, a_2)$ is symmetric in a_1 and a_2, in an application of Lemma 5.8.

By Lemma 5.7(c)–(f), $S_{2,t}(a_1, a_2)f$ may be written as a sum of expressions of the form

$$W_t([Y_t a_1 X_{2,t} a_2 + Y_t a_2 X_{2,t} a_1]\cdot[Z_t f]) \tag{5.60}$$

where $W_t \in \{I, P_t, Q_t\}$, $Y_t \in \{P_t, Q_t\}$, and $Z_t \in \{P_t, Q_t, R_t\}$. If $X_{2,t} = P_t$, Lemma 5.7(d),(f) shows that $[Y_t a_1 X_{2,t} a_2 + Y_t a_2 X_{2,t} a_1]$ may be expressed as a sum of functions of the form

$$W_{1,t}([Y_{1,t} a_1][Y_{2,t} a_2]), \tag{5.61}$$

with $W_{1,t} \in \{I, P_t, Q_t\}$, $Y_{j,t} \in \{P_t, Q_t\}$. If $X_{2,t} = Q_t$, then $[Y_t a_1 X_{2,t} a_2 + Y_t a_2 X_{2,t} a_1]$ has a similar expression; but in this case the symmetry is crucial and Lemma 5.8 must be used. In any case, Lemmas 5.7 and 5.8 enable us to write $S_2(a_1, a_2)f$ as the sum of H^1 functions which can be estimated by Lemma 5.6.

In the case of $n = 2, 3$, we have actually proved the following:

LEMMA 5.9. *Under the hypotheses of the Main Theorem, we have, for* $n \geqslant 2$,

$$\|G_{n-1}(a_1, \ldots, a_{n-1})f\|_{H^1} \leqslant K\Big(\prod_{j=1}^{n-1} \|A_j\|_{\lambda_j}\Big)\|h_t\|_2^{+}\|f\|_2 \tag{5.62}$$

where K *is independent of* $A_1, \ldots, A_{n-1}, h_t$, *and* f.

To prove Lemma 5.9 for $n \geqslant 4$, we must make repeated use of Lemmas 5.7 and 5.8 to express $G_{n-1}(a_1, \ldots, a_{n-1})f$ in terms of H^1 functions which can be estimated by Lemma 5.6. At every stage there will be at least one term for which symmetry in $a_1, \ldots, a_{n-1}$ is crucial. To treat these terms, analogues to Lemma 5.8 may be developed which show, for example, that the expression

(5.63)
$$\sum_{\sigma\in S_{n-1}} [Q_t a_{\sigma(1)} Q_t a_{\sigma(2)} \ldots Q_t a_{\sigma(n-1)}] - [Q_t a_1][Q_t a_2][Q_t a_3] \ldots [Q_t a_{n-1}]$$

can be written as a sum of functions built up from P_t, Q_t, $[P_t a_j]$, and $[Q_t a_j]$, where $1 \leqslant j \leqslant n-1$. We omit the details.

The Main Lemma, and hence the Main Theorem, now follow from Lemmas 5.9 and 5.2.

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