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On a theorem of Gleason, Kahane and Żelazko

by

RICHARD ARENS (Los Angeles, Cal.)

Abstract. Let A be a commutative Banach algebra with the sup norm. Let T and φ be continuous functionals where T is linear, and suppose F is some entire function for which $T(F(a)) = F(\varphi(a))$ for every a in A. Then either F is a polynomial of degree at most 1, or T is almost multiplicative, in the sense that $T(a^2)\,T(1) = T(a)^2$ for all a in A.

1. Introduction. A. M. Gleason [3] (see also J. P. Kahane and W. Żelazko [4]) characterized the maximal ideals of any commutative Banach algebra A. The characterization is based on the following

THEOREM. *Let A be a Banach algebra with unit. Let T be a continuous linear functional defined on A such that*

(H_{inv}) *T of any inverse is an inverse.*

Then for any element f of A one must have

(h) $$T(f^2)\,T(1) = T(f)^2.$$

Gleason proceeds by pointing out that (H_{inv}) implies (H_{exp}), which is to say, *T of any exponential is an exponential*, and from this he deduces (h).

One might entertain the conjecture that (H_F), where (H_F) is the condition

For each f in A, $T(F(f))$ is one of the values of F,

(F being some fixed, nontrivial entire function, used in place of Gleason's exponential function), might also imply (h). But such a conjecture would be idle, as (H_F) is fulfilled by any T and any A if $F(z)$ is z^2. One has only to take $\varphi(f)$ to be a square root of $T(f^2)$, and of course $T(F(f)) = F(\varphi(f))$.

Observing that this φ would have to be discontinuous, we amend the hypothesis for the conjecture. F, as before, is a fixed entire function.

($H_{F,\text{cont.}}$) *There is a continuous complex-valued function φ on A such that*
$$T(F(f)) = F(\varphi(f)) \quad \textit{for each } f \textit{ in } A.$$

We conjecture that ($H_{F,\text{cont.}}$) implies (h) above.

2. A variation on Gleason's theorem. We have good evidence for the conjecture.

THEOREM 1. *Let A be a Banach algebra with unit, and let F be an entire function for which $F''(z)$ is not identically 0.*

Suppose that $(H_{F,\mathrm{cont}})$ *holds. Suppose moreover that either*

(p) F *is a polynomial*

or

(s) *A is an algebra of functions on a set Ω and has the norm $\|f\| = \sup|f(\omega)|$.*

Then (h) *holds.*

We begin our proof with some lemmas.

LEMMA L. *Suppose there holds a condition*

(Hzw) $$T(F(z+wf)) = F(\varphi(z, w))$$

where φ is a linear function. Then

(h) $$T(1)\,T(f^2) = T(f)^2.$$

Proof. Suppose $\varphi(z, w) = a+bz+cw$. Then $F(a+bz+cw) = \sum_n F^{(n)}(a)(bz+cw)^n/n!$. By Hzw, this is equal to

$$\sum_n F^{(n)}(0)\,T[(z+wf)^n]/n!.$$

Hence

$$F^{(n)}(0)\,T[(z+wf)^n] = F^{(n)}(a)(bz+cw)^n$$

for each n. Since F is not linear, there must exist an n, at least 2, such that $F^{(n)}(0)$ is not zero. Hence $T(f^k) = K_n b^{n-k} c^k$. The desired (h) follows at once.

LEMMA AC. *Let F, ψ, and F_1 be entire, and let M, respectively M_1, be the maxima of their moduli on the circle of radius R around the origin. Suppose there are positive constants a, b, c such that*

$$F_1(z) = F(\psi(z)) + \text{constant } K, \qquad M_1(R) \leqslant aM(bR+c).$$

Then either F is constant or ψ is a polynomial of degree 1 at most.

Proof. We consider two cases.

Case 1. There is an R_0 such that whenever z_1 and z_2 are given with $|z_2| = 2R$ and $|z_1| = R > R_0$, then there is a w with $|w| < R^{2/3}$ and such that $\psi(w)$ is either z_1 or z_2. Select $R > R_0$, and select z_1 such that $|F(z_1)| = M(R)$, and z_2 such that $|F(z_2)| = M(2R)$. If $\psi(w)$ is z_1, then $F(z_1) = F(\psi(w)) = F_1(w) - K$, whence $M(R) \leqslant M_1(R^{2/3}) + |K|$. If on the other hand $\psi(w)$ is z_2, then we obtain $M(2R) \leqslant M_1(R^{2/3}) + |K|$. We may thus conclude that $M(R) \leqslant M_1(R^{2/3}) + |K|$, and thus $M(R) \leqslant aM(bR^{2/3}+c) + |K|$, or $M(R) \leqslant aM(bR^h+c)+k$, where $0 < h < 1$. From this we can deduce that $M(R)$ is bounded.

Case 2: the denial of case 1. Now for every R_0 one has an $R > R_0$ and z_1, z_2 with moduli R and $2R$ such that for every w with $|w| \leqslant R^{2/3}$, $\psi(w)$ is neither z_1 nor z_2. Let $g(t) = [\psi(R^{2/3}t) - z_1]/[z_1 - z_2]$. Then $g(t)$ is neither 0 nor 1 for $|t| \leqslant 1$.

The value $g(0)$ is independent of R. Thus we may apply Schottky's theorem [6], and conclude that $|g(t)| \leqslant C$ where C is independent of R, for $|t| \leqslant 1/2$. From this one can deduce that $|\psi(R^{2/3}t)| \leqslant RC_1$ where C_1 is independent of R, for $|t| \leqslant 1/2$. This shows that $\psi''(t) = 0$.

LEMMA FA. *Suppose $T(F(f)) = F(\varphi(f))$ where φ is continuous on the Banach algebra A (with unit), F is entire, and T is a continuous linear functional. Let z and w be complex variables, and let*

$$G(z, w) = T(F(z+wf)).$$

Suppose (s) *holds, namely that A is a function algebra. Then*

(*) $$|G(z, w)| \leqslant \|T\|\, M(|z| + |w|\,\|f\|),$$

where M is the maximum-modulus function for F, and

$$G(z, w) = F(\psi(z, w))$$

where ψ is holomorphic.

The inequality (*) follows at once from the sup norm assumption (s). The function ψ is holomorphic because it is a continuous solution of a holomorphic relation.

Proof of Theorem 1 with assumption (s). Because of (*), we can apply Lemma AC. We deduce that $\psi(z, w)$ is linear in z for each w, and linear in w for each z. We can therefore apply Lemma L and deduce (h).

I am very grateful to Prof. L. Carleson for suggesting the use of Schottky's theorem for a proof of the case where A is just C^2.

Proof of Theorem 1 with assumption (p). As already noted, if $T(F(f)) = F(\varphi(f))$ where φ is continuous on the Banach algebra A, then we get $T(F(z+wf)) = F(\varphi(z, w))$, where φ is holomorphic. But if F is a polynomial, then φ can be at most of degree 1. Then Lemma L applies.

A remark about $T(1)$. For some F, $T(1)$ is arbitrary, but not for all. For example, when $F(z) = z^2$, and α is any complex number, then a T can be easily exhibited for which $T(1) = \alpha$. On the other hand, when $F(z) = z^2+1$, one always has $T(1) = 1$, unless T is the 0 functional.

3. Remarks about the case in which T maps not into C but into another commutative Banach algebra B. Suppose $T: A \to B$ is a bounded linear transformation such that $(H_{F,\mathrm{cont}})$ holds, where now φ is a continuous map $A \to B$. If now either (p) or (s) holds, we can deduce that

(h_0) $$T(1)\,T(f^2) - T(f)^2 \text{ lies in the radical } R \text{ of } B.$$

This we do by applying the theorem to the pair $\beta \circ T$ and $\beta \circ \varphi$, for each complex-valued homomorphism β of B. We want to discuss the type of T satisfying (h_0), regardless of how it was obtained. We assume $T(1) = 1$. As Gleason observes, this makes T into a homomorphism, of course, modulo the radical. Such a T can be obtained in the following way. Let U be a genuine Banach-algebra homomorphism of A into B, with $U(1) = 1$. Let V be a linear bounded map of A into R. Then $T = U + V$ satisfies (h_0).

We want to show by an example that in general such a decomposition is not possible.

Let $\mathfrak{A}$ be the algebra of C. Feldman [2], in the notation of [5, p. 297]. This algebra has a radical spanned by an element q, and the quotient algebra is isomorphic to a certain $l_2(\Lambda)$. The latter shall be our A. A is spanned by certain elements u_k. Feldman's algebra shall be our B, and our T shall be defined by $T(u_k) = [u_k, q]$, again in Rickart's notation. It is easy to verify that $T(u_k u_m) - T(u_k) T(u_m) = q$.

One can strengthen Feldman's argument (cf. [1]) to show that for any homomorphism U of $l_2(\Lambda)$ into $\mathfrak{A}$, one must have $U(u_k) = 0$ for almost all k. This makes a representation of T as $U + V$ where V maps A into the radical, impossible.

References

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA
405 Hilgard Avenue, Los Angeles, California 90024, U.S.A.

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