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Received March 10, 1986
Revised version October 2, 1986 (2149)

STUDIA MATHEMATICA, T. LXXXVII. (1987)

Two weighted estimates for oscillating kernels I

by

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Abstract. In this paper we wish to determine those nonnegative weights w, v for which $\|Tf\|_{q,w} \leqslant c\|f\|_{p,v}$ where $\|\cdot\|_{s,u} = (\int|\cdot|^s u(t)\,dt)^{1/s}$. The operator $Tf(x)$ is a convolution transform with kernel

$$K_{a,b}(t) = (1+|t|^n)^{-b} e^{i|t|^a}, \quad a > 1.$$

Here, we study the cases where $b \leqslant 1-a/2$. Thus, we solve certain two weight problems for a wide class of transforms which includes the Fourier transform. The results agree with our earlier results on the Fourier transform.

§ 0. Introduction. In this paper we solve certain two weight problems for the kernels

$$K_{a,b}(t) = (1+|t|^n)^{-b} e^{i|t|^a}, \quad a > 1, \tag{0.1}$$

where n (a positive integer) coincides with the dimension of the variable t, i.e. $t = (t_1, t_2, \ldots, t_n)$, $|t| = (t_1^2+t_2^2+\ldots+t_n^2)^{1/2}$. Also, let $\|t\| = \max_{1\leqslant j\leqslant n}|t_j|$. We set

$$Tf(x) = \int K(x-t) f(t)\,dt$$

and we wish to determine those weights w, v for which

$$\|Tf\|_{q,w} \leqslant c\|f\|_{p,v}, \quad \text{where } \|g\|_{s,u} = \left(\int \|g\|^s u(t)\,dt\right)^{1/s}.$$

In this paper, we shall study the cases where $b \leqslant 1-a/2$. The arguments here work for a class of kernels more general than those defined through (0.1). This class is stated explicitly in Remark 1.6. The case where $a = 2$ and $b = 0$ in (0.1), which is identical to the Fourier kernel, is included among our results here (see e.g. Corollary 4.12). Hence this argument will furnish another way to solve a two weight problem for the Fourier transform, and agrees with our results in [4], but is general enough so that it works for a wider class of transforms.

Here, for the most part we shall just discuss the cases where $n = 1$ or 2.

We say a function $u(t)$ is *radial* if $u(t) = u(|t|)$. Furthermore, we say the radial function $u(t)$ is *essentially decreasing* over some region Ω if

$$u(t_1) \geqslant cu(t_2) \quad \text{where } |t_1| \leqslant |t_2|,\ t_1, t_2 \in \Omega,$$

and c is a positive constant independent of $t \in \Omega$. We denote this by $u(t) \searrow\searrow$ for $t \in \Omega$. Similarly we define $u(t) \nearrow\nearrow$ for $t \in \Omega$.

Now we shall define a class of functions, which in the applications will denote two classes of weights for which we solve the two weighted mapping problem.

DEFINITION 0.1. Let $u \geqslant 0$ be radial. In case $\beta > 0$ we say that $u \in F = F[\beta]$ if

(i) $u(t) \leqslant cu(x)$ if $|x|/2 \leqslant |t| \leqslant 2|x|$ for $t, x \in \mathbf{R}^n$,
(ii) $u(t) \nearrow\nearrow$ for $|t| \geqslant 1$,
(iii) $u(t) \searrow\searrow$ for $|t| \leqslant 1$, and
(iv) $u(t)/|t|^{\beta} \searrow\searrow$ for $|t| \geqslant 1$.

In case $\beta \leqslant 0$, conditions (i) and (ii) are dropped.

Notice that if $u \in F$ with $\beta > 0$ and $u > 0$ for $|t| \geqslant 1$, then it follows from (i) and (ii) that $u(t) \geqslant \alpha > 0$ for $|t| \geqslant 1$.

Since the Fourier transform is included among our results, the weights satisfy certain additional integral conditions, as was shown in [4]. This leads us to the following requirements depending on p, q, ε, δ:

$$\text{(0.2)}\quad \begin{aligned} &\text{(i)} && \sup_{0<s\leqslant 1}\Big(\int\limits_{1\leqslant |t|\leqslant s^{1/(1-a)}} \frac{w(t)}{|t|^{\delta}}\,dt\Big)^{1/q}\Big(\int\limits_{s^{1/(1-a)}\leqslant |t|} \frac{|t|^{\varepsilon}\,dt}{(w(t))^{p/[q(p-1)]}}\Big)^{1/p'} < \infty,\\ &\text{(ii)} && \sup_{s\geqslant 1}\Big(\int\limits_{|t|\leqslant s^{1-a}} w(t)\,dt\Big)^{1/q}\Big(\int\limits_{s^{1-a}\leqslant |t|\leqslant 1} \frac{dt}{|t|^{\varepsilon/(b-1)}(w(t))^{p/[q(p-1)]}}\Big)^{1/p'} < \infty. \end{aligned}$$

We define (using χ for characteristic functions)

$$\text{(0.3)}\quad \begin{aligned} &\text{(i)} && v_0(t) = |t|^{\alpha(1-p)}\max\big((w(t))^{p/q}, (w(|t|^{1-a}))^{p/q}, 1\big)\chi(|t| \geqslant 1)\\ & && \qquad + |t|^{\gamma}(w(|t|^{1/(1-a)}))^{p/q}\chi(|t| \leqslant 1),\\ &\text{(ii)} && T_{\lambda} f(x) = \int \frac{e^{i|t|^a} f(x-t)}{(1+|t|^n)^{b(1-\lambda)}}\,dt. \end{aligned}$$

In case $1/p+1/q \leqslant 1$, we shall state our major result. We state the dual result in Theorem 4.10 in case $\beta > 0$ and in Theorem 4.11 in case $\beta \leqslant 0$; in fact, we just state the sufficiency there since the necessity is done as before.

COROLLARY 0.2. *Let $1 < p \leqslant q < \infty$, $1/p+1/q \leqslant 1$, $b \leqslant 1-a/2$, $a > 1$, and $n = 1$ or 2. Suppose*

(i) *$w \in F$ with $\beta = n(b(pq-q+p)/p+a-2)$,*
(ii) *w satisfies (0.2) with*

$$\delta = n\frac{b}{p}(pq-q+p), \qquad \varepsilon = \frac{n(pq-q+p)}{q(1-p)}(1-b).$$

Then, for $1/q = 1/p-\lambda$, $0 \leqslant \lambda \leqslant 1$,

$$\textit{If}\quad v(t) \geqslant c_2 v_0(t), \quad \textit{then}\quad \|T_{\lambda} f\|_{q,w} \leqslant c_1 \|f\|_{p,v}$$

where $T_{\lambda} f$ and v_0 are defined in (0.3), α is defined in Lemma 4.5, and γ is defined in Lemma 4.6. Furthermore, if v satisfies Definition 0.1(i) and is radial, then

$$\|T_{\lambda} f\|_{q,w} \leqslant c_1 \|f\|_{p,v} \quad \textit{implies that}\quad v(t) \geqslant c_2 v_0(t).$$

We shall next discuss the power weights $w(t)$ that satisfy Corollary 0.2 (note the restrictions on a, b, p, q and n there). In other words, if

$$\text{(0.4)}\qquad w(t) = |t|^{nr_1}\chi(|t| \geqslant 1) + |t|^{nr_2}\chi(|t| \leqslant 1)$$

then we wish to determine r_1, r_2, p, q such that

$$\|T_{\lambda} f\|_{q,w} \leqslant c_1 \|f\|_{p,v} \quad \text{with} \quad v(t) \geqslant c_2 v_0(t).$$

We see for example from Proposition 5.3 that $r_1 \leqslant bq(1-1/p+1/q)+a-2$. Notice that if r_1, a, b are known this inequality places a restriction on p and q.

R e m a r k 0.3. The power weights $w(t)$ in (0.4) that satisfy Corollary 0.2, as explained above, are as follows. The conditions throughout are:

$$-1 < r_2 \leqslant 0, \quad r_1 \leqslant bq\Big(1-\frac{1}{p}+\frac{1}{q}\Big)+a-2, \quad r_1 > bq\Big(1-\frac{1}{p}+\frac{1}{q}\Big)-1.$$

Case 1: *$1 < a \leqslant 2$ and $b > 0$.* Then either

$$bq\Big(1-\frac{1}{p}+\frac{1}{q}\Big)-1 < r_1 \quad \text{and} \quad \frac{1}{q} \leqslant \frac{b}{1-b}\Big(1-\frac{1}{p}\Big), \quad \text{or}$$

$$0 \leqslant r_1 \quad \text{and} \quad \frac{b}{1-b}\Big(1-\frac{1}{p}\Big) \leqslant \frac{1}{q} \leqslant 1-\frac{1}{p}.$$

Case 2: *$1 < a < 2$ and $b \leqslant 0$.* Then $2b-1 < r_1 \leqslant 2b+a-2$ and $1/p+1/q = 1$.

Case 3: *$a = 2$ and $b \leqslant 0$.* Then $r_1 \leqslant 0$ and $1/p+1/q \leqslant 1$.

Case 4: *$a > 2$.* Then either $r_1 = 0$ and $1/p+1/q = 1$, or $r_1 < 0$ and $1/p+1/q \leqslant 1$.

By duality we get a corresponding theorem in case $1/p+1/q \geqslant 1$. And although we would obtain sufficient conditions for the kernels in (0.1) for $0 < a < 1$ by this approach, we would not obtain definitive results (see also [2] for these cases). This explains why we have not stated our results for this case.

Note that in the Fourier transform case, i.e. $a = 2$ and $b = 0$ here, we get $\beta = 0$ and so (i) and (ii) of Corollary 0.2 reduce to $w(t) \searrow\searrow$, while $\delta = 0$,

$\gamma = (n/q)(pq-q-p)$, and so (0.2) coincides with (5.1) of [4], where $w(t) = W(\gamma_n |t|^n)$. Hence for $n = 1$ or 2, Corollary 0.2 coincides with Theorem 2 of [4] in case $a = 2$ and $b = 0$ (see also Corollary 4.12).

The significant development here is that we are able to solve a two weight problem for a general class of kernels (even more general than those defined in (0.1)) which when $a = 2$ and $b = 0$ agrees with our Fourier transform result in [4].

Our conventions are that circles or squares in two dimensions will stand for intervals in one dimension; and so in two dimensions $I(u) = [-u, u] \times [-u, u]$, $R(u_1, u_2) = I(u_2) - I(u_1)$ with $u_2 > u_1$, and $R(u) = I(2u) - I(u)$. And we let R stand for a rectangle with sides parallel to the coordinate axes or annuli as described above.

We let the letters $c_1, c_2, \ldots$ denote positive constants, independent of the specified variable quantities, and we use the letter c generically. They are not necessarily the same at each occurrence.

§ 1. **Fundamental L^2 estimates.** In this section, we determine power weighted L^2 estimates for our operators, which are the foundations of all our results. Along with the kernels $K_{a,b}$ defined in (0.1), we also define the following kernels:

$$\Omega_\tau(t) = (1+|t|^n)^{-i\tau} K_{a,b} \quad \text{for } \tau \text{ real}, \tag{1.0}$$
$$\Omega_{\tau,a}(t) = \Omega_\tau(t) \quad \text{with } b = 1 - a/2.$$

Following [7] and the notation in § 0, we define

$$S(R) = S(R, x, \tau) = \Big|\int_R \Omega_\tau(t) e^{-it\cdot x} dt\Big|.$$

The next result can be found in [5] for $n = 1$, and in [11] for $n = 2$.

LEMMA A. *Let $a > 1$, $b = 1 - a/2$ and $n = 1$ or 2. Then*

$$S(R) \leqslant c(1+|\tau|^n)$$

with c independent of R, x, τ.

LEMMA B. *Let $a > 1$, $u \geqslant 1$, $n = 1$ or 2. Set $Q = R(c_1 u^{a-1}, c_2 u^{a-1})$ and $J = \boldsymbol{R}^n - Q$. Then for suitable c_1, c_2*

$$S(R(u)) \leqslant c(1+|\tau|^n) u^{n(1-b-a/2)} \{\chi_Q(x) + u^{-n\gamma a} \chi_J(x)\}$$

where $\gamma = 1$ for $n = 1$ and $\gamma = \frac{1}{4}$ for $n = 2$.

The proof of Lemma B can be found in Lemma 3.1 of [2] in case $n = 1$, and in Lemma 2.1 of [11] in case $n = 2$. It is these lemmas that have restricted the dimension in our results. It should be pointed out that Lemma B works as long as u is bounded away from 0, i.e. if $u \geqslant \alpha > 0$; however, this just gives us other constants c, c_1, c_2 in the lemma.

In obtaining our estimates it is considerably simpler to work over rectangles R than to do estimates over circles or ellipses. Even for these kernels, the curvature of the domain creates all kinds of complications. In fact, Lemma A and Lemma B is not even true if we replace the rectangles by spheres in dimension $n \geqslant 3$, for the kernels in (0.1).

Therefore we define

$$\varphi(x) = \begin{cases} 0 & \text{if } \|x\| < 1, \\ 2^m & \text{if } 2^m \leqslant \|x\| < 2^{m+1}. \end{cases}$$

That is, think of $\varphi(x)$ as a replacement for $|x|$.

We begin with our first result.

THEOREM 1.1. *Let $a > 1$, $u \geqslant 1$, and $n = 1$ or 2. Then*

$$\begin{aligned} \mathrm{I} + \mathrm{II} = & \int_{u \leqslant \varphi(x) < 2u} \Big| \int_{\varphi(t) \leqslant \varphi(x)/4} \Omega_\tau(x-t) f(t)\, dt\Big|^2 dx \\ & + \int_{u \leqslant |x| \leqslant 2u} \Big| \int_{|t| \leqslant u/2} \Omega_\tau(x-t) f(t)\, dt\Big|^2 dx \\ \leqslant\ & c(1+|\tau|^n)^2 u^{n(2-2b-a)} \Big\{\int_Q |\hat{f}|^2 dt \\ & + \int_{c_1 u \leqslant \varphi(t) \leqslant c_2 u} |f|^2 dt + u^{-2n\gamma a} \int |f|^2 dt\Big\}, \end{aligned}$$

where $Q = Q(u)$ and γ are both defined in Lemma B.

Proof. The proof of II is similar to the proof of I so we will just give the argument for I, where we may assume that u is a power of 2.

For such $u \geqslant 1$, but otherwise fixed, we can view the support of the kernel $\Omega_\tau(x-t)$, in the integral I, as contained in the set $R(c_1 u, c_2 u)$ where c_1, c_2 are positive constants that depend only on the dimension. We set

$$\Omega_\tau^u(t) = \Omega_\tau(t) \chi(t \in R(c_1 u, c_2 u)).$$

Using the fact that

$$f(t) = f(t)\chi(\varphi(t) < \varphi(x)/4) + f(t)\chi(\varphi(t) \geqslant \varphi(x)/4)$$

we get

$$\mathrm{I} \leqslant c\Big\{\int_{u \leqslant \varphi(x) < 2u} |\Omega_\tau^u * f|^2 dx + \int_{u \leqslant \varphi(x) < 2u} \Big|\int_{\varphi(t) \geqslant \varphi(x)/4} \Omega_\tau^u(x-t) f(t)\, dt\Big|^2 dx\Big\}.$$

Since $\Omega_\tau^u(x-t)$ is zero if $x - t \notin R(c_1 u, c_2 u)$, there is a c_3 so that for $\varphi(t) \geqslant c_3 \varphi(x)$ the integrand in the second term is zero. Set $f^u(t) = f(t)\chi(\varphi(x)/4 \leqslant \varphi(t) \leqslant c_3 \varphi(x))$. Then we get

$$\begin{aligned} \mathrm{I} &\leqslant c\{\int |(\Omega_\tau^u)^\wedge|^2 |\hat{f}|^2\,dt + \int |(\Omega_\tau^u)^\wedge|^2 |(f^u)^\wedge|^2\,dt\} \\ &\leqslant c(1+|\tau|^n)^2 u^{n(2-2b-a)} \{\int_Q |\hat{f}|^2\,dt \\ &\quad + \int |(f^u)^\wedge|^2\,dt + u^{-2n\gamma a} \int_J |\hat{f}|^2\,dt\}, \end{aligned}$$

where we employed Lemma B to obtain the last inequality, and our result follows.

As an immediate consequence of Theorem 1.1 we get

PROPOSITION 1.2. *Let $a > 1$ and $n = 1$ or 2. Then*

$$(1.1)\qquad \int_{\varphi(x)\geqslant 1} \Big|\int_{\varphi(t)\leqslant\varphi(x)/4} \Omega_\tau(x-t) f(t)\,dt\Big|^2 |x|^{n(2b+a-2)}\,dx \leqslant c(1+|\tau|^n)^2 \int |f|^2\,dt,$$

$$(1.2)\qquad \int_{\varphi(x)\geqslant 1} \Big|\int_{\varphi(t)\geqslant 4\varphi(x)} \Omega_\tau(x-t) f(t)\,dt\Big|^2\,dx \leqslant c(1+|\tau|^n)^2 \int_{\|t\|\geqslant 1} |f|^2 |t|^{n(2-2b-a)}\,dt,$$

$$(1.3)\qquad \int_{|x|\geqslant 1} \Big|\int_{|t|\leqslant 1} \Omega_\tau(x-t) f(t)\,dt\Big|^2 |x|^{n(2b+a-2)}\,dx \leqslant c(1+|\tau|^n)^2 \int_{|t|\leqslant 1} |f|^2\,dt,$$

$$(1.4)\qquad \int_{|x|\leqslant 1} \Big|\int_{|t|\geqslant 1} \Omega_\tau(x-t) f(t)\,dt\Big|^2\,dx \leqslant c(1+|\tau|^n)^2 \int_{|t|\geqslant 1} |f|^2 |t|^{n(2-2b-a)}\,dt.$$

Proof. We first note that (1.1) follows from Theorem 1.1 by summing over the outer integral, and (1.2) follows from the dual of (1.1), by restricting the support of f.

We now prove (1.3). Using the estimate for II in Theorem 1.1 and then restricting the support of f to $|t| \leqslant \frac{1}{2}$ and summing over the outer integral we get

$$(1.3')\qquad \int_{1\leqslant|x|} \Big|\int_{|t|\leqslant 1/2} \Omega_\tau(x-t) f(t)\,dt\Big|^2 |x|^{n(2b+a-2)}\,dx \leqslant c(1+|\tau|^n)^2 \int_{|t|\leqslant 1/2} |f|^2\,dt.$$

Next notice that

$$\begin{aligned} \int_{|x|\geqslant 1} \Big|\int_{1/2\leqslant|t|\leqslant 1} \Omega_\tau(x-t) f(t)\,dt\Big|^2 |x|^{n(2b+a-2)}\,dx &\leqslant c \int_{1\leqslant|x|\leqslant 2} |\ldots|^2 |x|^{n(2b+a-2)}\,dx \\ &\quad + c\int_{|x|\geqslant 2} |\ldots|^2 |x|^{n(2b+a-2)}\,dx = \mathrm{I}+\mathrm{II} \leqslant c(1+|\tau|^n)^2 \int_{|t|\leqslant 1} |f|^2\,dt, \end{aligned}$$

where the estimate for II follows just like in (1.3′), while for I, $\Omega_\tau(x-t)\in L^1$ when $|x| \leqslant 2$ and $|t| \leqslant 1$. Finally, (1.4) is the dual of (1.3).

We refer to $\Omega_\tau(x-t)\ \chi(\varphi(t) \leqslant \varphi(x)/4)$ and $\Omega_\tau(x-t)\chi(\varphi(t) \geqslant 4\varphi(x))$ as the off diagonal pieces of $\Omega_\tau(x-t)$, and $\Omega_\tau(x-t)\chi(\varphi(x)/4 < \varphi(t) < 4\varphi(x))$ as the middle piece.

The next result is the one that forces the restriction $b \leqslant 1-a/2$.

LEMMA 1.3. *Let $a > 1$, $u \geqslant 1$, $b \leqslant 1-a/2$ and $n = 1$ or 2. Then*

$$\mathrm{I} = \Big|\int_{\varphi(t)\leqslant u} \Omega_\tau(t) e^{-ix\cdot t}\,dt\Big| \leqslant c(1+|\tau|^n) u^{n(1-b-a/2)}.$$

Proof. When $b = 1-a/2$, the result follows from Lemma A. In case $b < 1-a/2$, we sum the inequality of Lemma B, where the terms between the braces are replaced by 1. This completes the proof.

In the next result we obtain estimates for the middle kernel.

LEMMA 1.4. *Let $a > 1$, $u \geqslant 1$, $b \leqslant 1-a/2$ and $n = 1$ or 2. Then*

$$\text{(a)}\qquad \mathrm{I} = \int_{u\leqslant\varphi(x)<2u} \Big|\int_{\varphi(x)/4<\varphi(t)<4\varphi(x)} \Omega_\tau(x-t) f(t)\,dt\Big|^2\,dx \leqslant c(1+|\tau|^n)^2 \int_{u/4\leqslant\varphi(t)\leqslant 8u} |t|^{n(2-2b-a)} |f|^2\,dt,$$

$$\text{(b)}\qquad \int_{u\leqslant\varphi(x)<2u} |x|^{n(2b+a-2)} \Big|\int_{\varphi(x)/4<\varphi(t)<4\varphi(x)} \Omega_\tau(x-t) f(t)\,dt\Big|^2\,dx \leqslant c(1+|\tau|^n)^2 \int_{u/4<\varphi(t)\leqslant 8u} |f|^2\,dt.$$

Proof. Set ${}_u\Omega(t) = \Omega_\tau(t)\chi(\varphi(t) \leqslant cu)$, c some fixed absolute constant. We just need to prove (a) since (b) is an immediate consequence of (a).

Using the notation in the proof of Theorem 1.1 (assuming u to be a power of 2), we get

$$\begin{aligned} \mathrm{I} &= \int_{u\leqslant\varphi(x)<2u} |{}_u\Omega * f^u|^2\,dx \leqslant \int |{}_u\Omega * f^u|^2\,dx \\ &= \int |({}_u\Omega)^\wedge|^2 |(f^u)^\wedge|^2\,dt \leqslant c(1+|\tau|^n)^2 u^{n(2-2b-a)} \int |(f^u)^\wedge|^2\,dt \end{aligned}$$

where the last inequality follows from Lemma 1.3. Hence we obtain our result.

An immediate consequence of Lemma 1.4 is

PROPOSITION 1.5. *Let $a > 1$, $b \leqslant 1-a/2$ and $n = 1$ or 2. Then*

$$(1.5)\qquad \begin{aligned} &\text{(a)}\quad \int_{\varphi(x)\geqslant 1} \Big|\int_{\varphi(x)/4<\varphi(t)<4\varphi(x)} \Omega_\tau(x-t) f(t)\,dt\Big|^2\,dx \leqslant c(1+|\tau|^n)^2 \int_{\|t\|\geqslant 1} |f|^2 |t|^{n(2-2b-a)}\,dt, \\ &\text{(b)}\quad \int_{\varphi(x)\geqslant 1} |x|^{n(2b+a-2)} \Big|\int_{\varphi(x)/4<\varphi(t)<4\varphi(x)} \Omega_\tau(x-t) f(t)\,dt\Big|^2\,dx \leqslant c(1+|\tau|^n)^2 \int_{\|t\|\geqslant 1} |f|^2\,dt. \end{aligned}$$

Remark 1.6. We can generalize these results to the following class of kernels. Let $k(t; 0) \equiv 1$ and $k(t; \tau) \in L^\infty$ for all t and τ. Next suppose $\gamma_\tau(t)$

satisfies Lemma A, Lemma B and Lemma 1.2 in place of Ω_τ, where

$$(1.6)\qquad \begin{gathered} \gamma_\tau(t) = k(t;\tau)\,g_{a,b}(t), \qquad \gamma_{\tau,a}(t) = k(t;\tau)\,g_{a,1-a/2}(t),\\ g_{a,b}\in L^\infty, \qquad \int\limits_{\varphi(t)\leqslant u} |g_{a,b}(t)|\,dt \leqslant cu^{n(1-b)}. \end{gathered}$$

We have replaced $(1+|t|^n)^{-i\tau}$ by $k(t;\tau)$ and $K_{a,b}$ by $g_{a,b}$ in comparison with (1.0).

By putting logarithmic terms into (0.1) we would obtain examples for this general class.

§ 2. General principles. In this section, we are concerned with stating the interpolation principles, the general duality results with weights as well as the Bradley type estimates [1] and putting them into a form that will be easily applicable to our case.

Here, we let Tf stand for our linear operator. For $\lambda > 0$ and $u \geqslant 1$, we define $E_\lambda = \{x\colon |Tf(x)| > \lambda\}$ and $F_u = \{x\colon u/4 \leqslant \varphi(x) \leqslant 4u\}$. We also let dv, $d\mu$ denote nonnegative measures which are absolutely continuous except at the origin, and we write $v(a) = \int_{|x|\geqslant a} dv$ for $a>0$. We begin with the interpolation results.

Proposition 2.1. *Assume* $\lim_{a\to+\infty} v(a) = 0$ *and* $\lim_{a\to+0} v(a) = +\infty$. *If*

$$\text{(i)}\qquad |Tf(x)|\cdot|v(|x|)| \leqslant c\,\|f\|_{1,\mu} \quad \textit{for all } x\neq 0,$$

then

$$v(E_\lambda) \leqslant \frac{c}{\lambda}\|f\|_{1,\mu}.$$

Proof. We may assume that $\|f\|_{1,\mu}>0$. We get by (i)

$$(2.1)\qquad |v(|x|)| \leqslant \frac{c}{\lambda}\|f\|_{1,\mu} \quad \text{for } x\in E_\lambda,\ x\neq 0.$$

By continuity, there is an $a>0$ so that

$$(2.2)\qquad \Big\{x\neq 0\colon |v(|x|)| \leqslant \frac{c}{\lambda}\|f\|_{1,\mu}\Big\} = \{x\colon |x|\geqslant a\}.$$

By (2.1) we get

$$\int\limits_{E_\lambda} dv \leqslant \int\limits_{\{x\colon |v(|x|)|\leqslant (c/\lambda)\|f\|_{1,\mu}\}} dv$$

and so by (2.2) we get

$$\int\limits_{E_\lambda} dv \leqslant \int\limits_{|x|\geqslant a} dv = v(a).$$

But by (2.2), $|v(a)| \leqslant (c/\lambda)\|f\|_{1,\mu}$, and the proof is complete.

We also need the next result.

Proposition 2.2. *Let $h(t)\geqslant 0$ (measurable) be given and $u\geqslant 1$. Suppose Tf is an integral operator (depending on u) and $dv = v'(|x|)\,dx$ for $x\neq 0$. If*

(i) $v(|x|) \leqslant c_1 v(|t|)$ *and* $v'(|x|) \leqslant c_2 v'(|t|)$ *for* $|t|/2 \leqslant |x| \leqslant 2|t|$,

(ii) $v(|x|)\,|Tf(x)| \leqslant c \int\limits_{u\leqslant\varphi(t)\leqslant 2u} h(t)\,|K(x,t)|\,|f(t)|\,d\mu$, *and*

(iii) $v'(u)\,h(t) \int\limits_{F_u} |K(x,t)|\,dx\cdot\chi(u\leqslant\varphi(t)\leqslant 2u) \leqslant cv(u)$,

then

$$v(E_\lambda\cap F_u) \leqslant \frac{c}{\lambda}\int\limits_{u\leqslant\varphi(t)\leqslant 2u} |f|\,d\mu.$$

Proof. We get from (i) and (ii)

$$\begin{aligned} \int\limits_{E_\lambda\cap F_u} du &\leqslant \frac{c}{v(u)}\int\limits_{E_\lambda\cap F_u} dv\cdot\frac{1}{\lambda}\int\limits_{u\leqslant\varphi(t)\leqslant 2u} d\mu\, h(t)\,|K(x,t)|\,|f(t)|\\ &\leqslant \frac{c}{v(u)}\cdot\frac{1}{\lambda}\int\limits_{u\leqslant\varphi(t)\leqslant 2u} d\mu\, h(t)\,|f(t)| \int\limits_{E_\lambda\cap F_u} dx\, v'(|x|)\,|K(x,t)|\\ &\leqslant c\frac{v'(u)}{v(u)}\cdot\frac{1}{\lambda}\int\limits_{u\leqslant\varphi(t)\leqslant 2u} d\mu\, h(t)\,|f(t)|\int\limits_{F_u}|K(x,t)|\,dx, \end{aligned}$$

and now our result follows by (iii).

We now consider duality with weights in the p, q cases. The proof is standard and will be omitted. Generally let $1/s+1/s' = 1$, $1<s<\infty$.

Proposition 2.3. *Let Tf be a convolution operator and assume $w>0$ and $v>0$. If $\|Tf\|_{q,w} \leqslant c\|f\|_{p,v}$, then*

$$\|Tf\|_{p',v^{1-p'}} \leqslant c\|f\|_{q',w^{1-q'}}.$$

The Bradley type estimates [1] also apply in our case. In n dimensions, see Theorem on p. 260 of [4].

Proposition 2.4. *Let*

$$Af(t) = |t|^{-n}\int\limits_{|v|\leqslant|t|} |f|\,dv$$

and let $u(t)\geqslant 0$ be a radial function. Then

$$\text{(a)}\qquad \int\limits_{|t|\leqslant 1} (Af)^q|t|^{n(q-2)}u(t)\,dt \leqslant c\int\limits_{|t|\leqslant 1} |f|^q|t|^{n(q-2)}u(t)\,dt \Leftrightarrow$$

$$(2.3)\qquad \sup_{0<s\leqslant 1}\Big(\int\limits_{s\leqslant|t|\leqslant 1}\frac{u(t)}{|t|^{2n}}\,dt\Big)^{1/q}\Big(\int\limits_{|t|\leqslant s}\frac{|t|^{n(q'-2)}}{(u(t))^{q'-1}}\,dt\Big)^{1/q'} < \infty.$$

(b) $\int\limits_{|t|\geqslant 1}(Af)^q|t|^{n(q-a)}u(t)\,dt\leqslant c\int\limits_{|t|\geqslant 1}|f|^q|t|^{n(q-a)}u(t)\,dt \Leftrightarrow$

$$\sup_{s\geqslant 1}\Big(\int\limits_{|t|\geqslant s}\frac{u(t)}{|t|^{na}}\,dt\Big)^{1/q}\Big(\int\limits_{1\leqslant|t|\leqslant s}\frac{|t|^{n(q'-1)(a-1)}}{|t|^n\big(u(t)\big)^{q'-1}}\,dt\Big)^{1/q'}<\infty. \tag{2.4}$$

§ 3. **Weighted L^q estimates.** In this section, we shall prove the sufficiency in Corollary 0.2 for the operator $T_0 f$ defined there. We begin with

LEMMA 3.1. *Let $q\geqslant 2$, $a>1$, $b\leqslant 1-a/2$ and $n=1$ or 2. Let $w(t)\geqslant 0$. Then*

$$\int\limits_{\varphi(x)\geqslant 1}\Big|\int\limits_{\varphi(x)/4<\varphi(t)<4\varphi(x)}\Omega_\tau(x-t)f(t)\,dt\Big|^q dx \leqslant c(1+|\tau|^n)^q\int\limits_{\|t\|\geqslant 1}|f|^q|t|^{n(q-bq-a)}\,dt. \tag{A$_0$}$$

Furthermore, if w satisfies Definition 0.1(i), *then*

$$\int\limits_{\varphi(x)\geqslant 1}\Big|\int\limits_{\varphi(x)/4<\varphi(t)<4\varphi(x)}\Omega_\tau(x-t)f(t)\,dt\Big|^q w(x)\,dx \leqslant c(1+|\tau|^n)^q\int\limits_{1\leqslant\|t\|}|f|^q|t|^{n(q-bq-a)}w(t)\,dt. \tag{A$_{0,w}$}$$

Proof. Set $d\nu=|x|^{-2n}dx$ and $d\mu=|t|^{n(a-2)}dt$ and note that $\nu(|x|)=c|x|^{-n}$. Now define

$$Tf(x)=\chi\big(u\leqslant\varphi(x)\leqslant 2u\big)\cdot|x|^n\int\limits_{\varphi(x)/4<\varphi(t)<4\varphi(x)}\Omega_\tau(x-t)f(t)|t|^{nb}\,d\mu.$$

By Lemma 1.4(a) we get

$$\int|Tf(x)|^2\,d\nu=\int\limits_{u\leqslant\varphi(x)\leqslant 2u}\Big||x|^n\int\limits_{\varphi(x)/4<\varphi(t)<4\varphi(x)}\Omega_\tau(x-t)f(t)|t|^{nb}\,d\mu\Big|^2\frac{dx}{|x|^{2n}} \leqslant c(1+|\tau|^n)^2\int\limits_{u/4\leqslant\varphi(t)\leqslant 8u}|f|^2\,d\mu. \tag{3.1}$$

By Proposition 2.2 ($F_u=\{x\colon u<\varphi(x)<2u\}$)

$$\nu(E_\lambda\cap F_u)\leqslant\frac{c}{\lambda}\int\limits_{u\leqslant\varphi(t)\leqslant 2u}|f|\,d\mu. \tag{3.2}$$

Now using (3.1), (3.2) and interpolation we get (with f supported in $u\leqslant\varphi(t)\leqslant 2u$)

$$\|Tf\|_{p,\nu}\leqslant c(1+|\tau|^n)\|f\|_{p,\mu}\quad\text{for } 1<p\leqslant 2. \tag{3.3}$$

Unraveling (3.3) and using Proposition 2.3 gives $(1/p+1/q=1)$

$$\int\limits_{u\leqslant\varphi(x)\leqslant 2u}\Big|\int\limits_{\varphi(x)/4<\varphi(t)<4\varphi(x)}\Omega_\tau(x-t)f(t)\,dt\Big|^q|x|^{nbq+n(a-2)}\,dx \leqslant c(1+|\tau|^n)^q\int\limits_{u\leqslant\varphi(t)\leqslant 2u}|f|^q|t|^{n(q-2)}\,dt. \tag{3.4}$$

Now it follows from (3.4) that

$$\int\limits_{u\leqslant\varphi(x)\leqslant 2u}\Big|\int\limits_{\varphi(x)/4<\varphi(t)<4\varphi(x)}\Omega_\tau(x-t)f(t)\,dt\Big|^q dx \leqslant c(1+|\tau|^n)^q\int\limits_{u\leqslant\varphi(t)\leqslant 2u}|f|^q|t|^{n(q-bq-a)}\,dt. \tag{3.5}$$

Summing (3.5) we get (A$_0$).

If w satisfies Definition 0.1(i) then

$$\int\limits_{u\leqslant\varphi(x)\leqslant 2u}\Big|\int\limits_{\varphi(x)/4<\varphi(t)<4\varphi(x)}\Omega_\tau(x-t)f(t)\,dt\Big|^q w(x)\,dx \leqslant c(1+|\tau|^n)^q\int\limits_{u\leqslant\varphi(t)\leqslant 2u}|f|^q|t|^{n(q-bq-a)}w(t)\,dt. \tag{3.6}$$

Summing (3.6) gives (A$_{0,w}$).

Next we estimate the corresponding pieces.

LEMMA 3.2. *Let $q\geqslant 2$, $a>1$ and $n=1$ or 2. Then*

$$\int\limits_{\varphi(x)\geqslant 1}\Big|\int\limits_{\varphi(t)\leqslant\varphi(x)/4}\Omega_\tau(x-t)f(t)\,dt\Big|^q|x|^{n(bq+a-2)}\,dx \leqslant c(1+|\tau|^n)^q\int\limits_{\varphi(t)\geqslant 1}|f|^q|t|^{n(q-2)}\,dt, \tag{B$_0$}$$

$$\int\limits_{\varphi(x)\geqslant 1}\Big|\int\limits_{\varphi(t)\geqslant 4\varphi(x)}\Omega_\tau(x-t)f(t)\,dt\Big|^q dx\leqslant c(1+|\tau|^n)^q\int\limits_{\|t\|\geqslant 1}|f|^q|t|^{n(q-bq-a)}\,dt, \tag{C$_0$}$$

$$\int\limits_{|x|\geqslant 1}\Big|\int\limits_{|t|\leqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^q|x|^{n(bq+a-2)}\,dx \leqslant c(1+|\tau|^n)^q\int\limits_{|t|\leqslant 1}|f|^q|t|^{n(q-2)}\,dt, \tag{B$'_0$}$$

$$\int\limits_{|x|\leqslant 1}\Big|\int\limits_{|t|\geqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^q dx\leqslant c(1+|\tau|^n)^q\int\limits_{|t|\geqslant 1}|f|^q|t|^{n(q-bq-a)}\,dt. \tag{C$'_0$}$$

Proof. We begin by showing (C$_0$) and (C$'_0$). In the case of (C$_0$) we set

$$T_1 f(x)=|x|^{n(b+a-1)}\Big(\int\limits_{\varphi(t)\leqslant\varphi(x)/4}\Omega_\tau(x-t)f(t)\,dt\Big)\chi\big(\varphi(x)\geqslant 1\big),$$

while in the case of (C$'_0$) we take

$$T_2 f(x)=|x|^{n(b+a-1)}\Big(\int\limits_{|t|\leqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big)\chi(|x|\geqslant 1).$$

Set $d\nu=|x|^{-na}dx$; that implies $\nu(|x|)=c|x|^{n-na}$. Now using (1.1) for T_1 and (1.3) for T_2 we get

$$\|T_j f\|_{2,\nu}\leqslant c(1+|\tau|^n)\|f\|_2\quad\text{for } j=1,2,$$

and by Proposition 2.1

$$\sup_{\lambda>0}\lambda|E_\lambda|=\|T_j f\|^*_{1,\nu}\leqslant c\|f\|_1.$$

Putting these estimates together yields

$$\|T_j f\|_{p,v} \leqslant c(1+|\tau|^n)\|f\|_p \quad \text{for } 1<p\leqslant 2,\ j=1,2. \tag{3.7}$$

Now unraveling (3.7) and using Proposition 2.3 we get (C_0) in case $j=1$ and (C_0') in case $j=2$.

Next, to see $(\mathbf{B}_0)$ and $(\mathbf{B}_0')$, this time we define

$$T_1 f(x) = |x|^n \Big(\int_{\varphi(t)\geqslant 4\varphi(x)} \Omega_\tau(x-t) f(t)|t|^{n(a-2+b)}\,dt\Big)\chi(\varphi(x)\geqslant 1)$$

in the case of (B_0), and

$$T_2 f(x) = |x|^n \Big(\int_{|t|\geqslant 1} \Omega_\tau(x-t) f(t)|t|^{n(a-2+b)}\,dt\Big)\chi(|x|\leqslant 1)$$

in the case of (B_0'). Set $dv=|x|^{-2n}dx$ and $d\mu=|t|^{n(a-2)}dt$. Using (1.2) for T_1 and (1.4) for T_2 we get $\|T_j f\|_{2,v}\leqslant c(1+|\tau|^n)\|f\|_{2,\mu}$ for $j=1,2$, and by Proposition 2.1 it also follows that $\|T_j f\|_{1,v}^{*}\leqslant c\|f\|_{1,\mu}$ for $j=1,2$. This implies that

$$\|T_j f\|_{p,v} \leqslant c(1+|\tau|^n)\|f\|_{p,\mu} \quad \text{for } 1<p\leqslant 2,\ j=1,2. \tag{3.8}$$

Now unraveling (3.8) we get (B_0) and (B_0') by Proposition 2.3.

LEMMA 3.3. *Let $q\geqslant 2$, $a>1$ and $n=1$ or 2.*

(a) *If* (i) $w\geqslant 0$ *and is radial,* (ii) $w(t)\nearrow\nearrow$ *for* $\|t\|\geqslant 1$, *then*

$$(\mathrm{C}_{0,w})\qquad \int_{1\leqslant\varphi(x)} \Big|\int_{\varphi(t)\geqslant 4\varphi(x)} \Omega_\tau(x-t) f(t)\,dt\Big|^q w(x)\,dx \leqslant c(1+|\tau|^n)^q \int_{\|t\|\geqslant 1} |f|^q |t|^{n(q-bq-a)} w(t)\,dt.$$

(b) *If* (i) (a) *holds and* (ii) $w(t)/|t|^{n(bq+a-2)}\searrow\searrow$ *for* $\|t\|\geqslant 1$, *then*

$$(\mathrm{B}_{0,w}')\qquad \int_{1\leqslant\varphi(x)} \Big|\int_{1\leqslant\varphi(t)\leqslant\varphi(x)/4} \Omega_\tau(x-t) f(t)\,dt\Big|^q w(x)\,dx \leqslant c(1+|\tau|^n)^q \int_{\|t\|\geqslant 1} |f|^q |t|^{n(q-bq-a)} w(t)\,dt.$$

Proof. We first show $(\mathrm{C}_{0,w})$. The constant case follows from (C_0), hence we can assume that $w(t)\to\infty$ as $|t|\to\infty$; thus let $1<\lambda_m\nearrow\infty$ so that $w(x)\sim 2^m$ (i.e. $c_1 2^m\leqslant w(x)\leqslant c_2 2^m$) in case $\lambda_m\leqslant\varphi(x)\leqslant\lambda_{m+1}$. By (C_0) we have

$$\sum_{m\geqslant 0}^{\infty} \int_{\lambda_m\leqslant\varphi(x)\leqslant\lambda_{m+1}} \Big|\int_{\varphi(t)\geqslant 4\varphi(x)} \Omega_\tau(x-t) f(t)\,dt\Big|^q w(x)\,dx \tag{3.9}$$
$$\leqslant c(1+|\tau|^n)^q \sum_{m=0}^{\infty} 2^m \int_{\varphi(t)\geqslant 4\lambda_m} |f|^q |t|^{n(q-bq-a)}\,dt$$
$$\leqslant c(1+|\tau|^n)^q \int_{\|t\|\geqslant 1} |f|^q |t|^{n(q-bq-a)} \sum_{\lambda_m\leqslant\varphi(t)} 2^m$$

where we are summing over all values of m so that $\lambda_m\leqslant\varphi(t)$. Suppose that $\lambda_{m'}\leqslant\varphi(t)<\lambda_{m'+1}$. Since w satisfies (a)(ii) we get

$$\sum_{\lambda_m\leqslant\varphi(t)} 2^m = \sum_{m=0}^{m'} 2^m \leqslant c2^{m'} \leqslant cw(t),$$

and $(\mathrm{C}_{0,w})$ follows by (3.9).

Now to see $(\mathrm{B}_{0,w}')$, we begin with

$$\mathrm{I} = \sum_{m=0}^{\infty} \int_{\lambda_m\leqslant\varphi(x)\leqslant\lambda_{m+1}} \Big|\int_{1\leqslant\varphi(t)\leqslant\varphi(x)/4} \Omega_\tau(x-t) f(t)\,dt\Big|^q |x|^{n(bq+a-2)} w_1(x)\,dx$$

where $w_1(x)=w(x)/|x|^{n(bq+a-2)}$. Since $w_1(x)\searrow\searrow$ we similarly obtain for $1<\lambda_m\nearrow\infty$ that $w_1(x)\sim 2^{-m}$ if $\lambda_m\leqslant\varphi(x)\leqslant\lambda_{m+1}$. Now we get

$$\mathrm{I}\leqslant c\sum_{m=0}^{\infty} 2^{-m} \int_{\lambda_m\leqslant\varphi(x)\leqslant\lambda_{m+1}} \Big|\int_{1\leqslant\varphi(t)\leqslant\varphi(x)/4} \Omega_\tau(x-t) f(t)\,dt\Big|^q |x|^{n(bq+a-2)}\,dx$$
$$\leqslant c(1+|\tau|^n)^q \sum_{m=0}^{\infty} 2^{-m} \int_{1\leqslant\varphi(t)\leqslant\lambda_{m+1}/4} |f|^q |t|^{n(q-2)}\,dt$$

where the last inequality follows from (B_0). Hence

$$\mathrm{I}\leqslant c(1+|\tau|^n)^q \int_{\|t\|\geqslant 1} |f|^q |t|^{n(q-2)} \sum_{\varphi(t)\leqslant\lambda_{m+1}} 2^{-m},$$

where we are summing over all values of m so that $\varphi(t)\leqslant\lambda_{m+1}$. Now let m' be such that $\lambda_{m'}<\varphi(t)\leqslant\lambda_{m'+1}$. Then

$$\mathrm{I}\leqslant c(1+|\tau|^n)^q \int_{\|t\|\geqslant 1} |f|^q |t|^{n(q-2)} \sum_{m=m'}^{\infty} 2^{-m};$$

but $2^{-m'}\leqslant cw_1(t)$, and so we have completed the estimate for $(\mathrm{B}_{0,w}')$.

The next two lemmas are primarily concerned with where $\varphi(x)\leqslant 1$ (the outer limits on the integral) or $\varphi(t)\leqslant 1$ (the inner limits). We begin with

LEMMA 3.4. *Let $q\geqslant 2$, $a>1$ and assume $n=1$ or 2. If the conditions* (i) $w(t)/|t|^{n(bq+a-2)}\searrow\searrow$, (ii) $0\leqslant w(t)=w(|t|)$ *hold for* $|t|\geqslant 1$, *then*

$$\int_{1\leqslant|x|} \Big|\int_{|t|\leqslant 1} \Omega_\tau(x-t) f(t)\,dt\Big|^q w(x)\,dx \leqslant c(1+|\tau|^n)^q \int_{|t|\leqslant 1} (|f|+Af)^q |t|^{n(q-2)} |t|^{-n(bq+a-2)/(1-a)} w(|t|^{1/(1-a)})\,dt.$$

Proof. First set $w(x)=|x|^{n(bq+a-2)}w_1(x)$; then $w_1(x)\searrow\searrow$ for $|x|\geqslant 1$. That means that for $1\leqslant\lambda_m\nearrow\infty$ we have $w_1(x)\sim 2^{-m}$ for $\lambda_m\leqslant|x|\leqslant\lambda_{m+1}$, the case $w_1(x) \sim 1$ being covered by (B_0'). Hence

$$\sum_{m=0}^{\infty} \int_{\lambda_m \leqslant |x| \leqslant \lambda_{m+1}} \Big| \int_{|t| \leqslant 1} \Omega_\tau(x-t) f(t)\,dt \Big|^q |x|^{n(bq+a-2)} w_1(x)\,dx$$

$$\leqslant c \sum_{m=0}^{\infty} 2^{-m} \int_{\lambda_m \leqslant |x| \leqslant \lambda_{m+1}} \Big| \int_{|t| \leqslant \lambda_{m+1}^{1-a}} \Omega_\tau(x-t) f(t)\,dt \Big|^q |x|^{n(bq+a-2)}\,dx$$

$$+ c \sum_{m=0}^{\infty} 2^{-m} \int_{\lambda_m \leqslant |x| \leqslant \lambda_{m+1}} \Big| \int_{\lambda_{m+1}^{1-a} \leqslant |t| \leqslant 1} \Omega_\tau(x-t) \times f(t)\,dt \Big|^q |x|^{n(bq+a-2)}\,dx$$

$$= \mathrm{I} + \mathrm{II}.$$

We have

$$\mathrm{I} \leqslant c \sum_{m=0}^{\infty} 2^{-m} \int_{\lambda_m \leqslant |x| \leqslant \lambda_{m+1}} \Big(\int_{|t| \leqslant \lambda_{m+1}^{1-a}} |f|\,|x|^{-nb}\,dt \Big)^q |x|^{n(bq+a-2)}\,dx$$

and so

$$\mathrm{I} \leqslant \sum_{m=0}^{\infty} 2^{-m} \int_{\lambda_m \leqslant |x| \leqslant \lambda_{m+1}} \big(Af(|x|^{1-a})\big)^q |x|^{n(a-2)} |x|^{(1-a)nq}\,dx.$$

Since $w_1(x) \sim 2^{-m}$ for $\lambda_m \leqslant |x| \leqslant \lambda_{m+1}$ we get

$$\mathrm{I} \leqslant c \sum_{m=0}^{\infty} \int_{\lambda_m \leqslant |x| \leqslant \lambda_{m+1}} \big(Af(|x|^{1-a})\big)^q |x|^{n(a-2)} |x|^{(1-a)nq} w_1(x)\,dx.$$

Now using polar coordinates with $s = r^{1-a}$ gives

$$\int_{\lambda_m}^{\lambda_{m+1}} \big(Af(r^{1-a})\big)^q r^{n(a-2)} r^{(1-a)nq} r^{n-1} w_1(r)\,dr$$

$$= c \int_{\lambda_{m+1}^{1-a} \leqslant |t| \leqslant \lambda_m^{1-a}} \big(Af(t)\big)^q |t|^{n(q-2)} w_1\big(|t|^{1/(1-a)}\big)\,dt.$$

Putting all this together we get

$$\mathrm{I} \leqslant c \int_{|t| \leqslant 1} (Af)^q |t|^{n(q-2)} w_1\big(|t|^{1/(1-a)}\big)\,dt. \tag{3.10}$$

We now estimate the term II. We get by (B_0') with f supported in $\lambda_{m+1} \leqslant |t| \leqslant 1$

$$\int_{|x| \geqslant 1} \Big| \int_{\lambda_{m+1}^{1-a} \leqslant |t| \leqslant 1} \Omega_\tau(x-t) f(t)\,dt \Big|^q |x|^{n(bq+a-2)}\,dx \tag{3.11}$$

$$\leqslant c(1+|\tau|^n)^q \int_{\lambda_{m+1}^{1-a} \leqslant |t| \leqslant 1} |f|^q |t|^{n(q-2)}\,dt.$$

From (3.11) it follows that

$$\mathrm{II} \leqslant c(1+|\tau|^n)^q \int_{|t| \leqslant 1} |f|^q |t|^{n(q-2)} \sum_{\lambda_{m+1}^{1-a} \leqslant |t|} 2^{-m}.$$

But

$$\sum_{\lambda_{m+1} \geqslant |t|^{1/(1-a)}} 2^{-m} \leqslant c \sum_{m=m'}^{\infty} 2^{-m} \leqslant c 2^{-m'} \leqslant c w_1\big(|t|^{1/(1-a)}\big),$$

and so

$$\mathrm{II} \leqslant c(1+|\tau|^n)^q \int_{|t| \leqslant 1} |f|^q |t|^{n(q-2)} w_1\big(|t|^{1/(1-a)}\big)\,dt. \tag{3.12}$$

Now by (3.10) and (3.12) we get our result.

Lemma 3.5. *Let $q \geqslant 2$, $a > 1$, and assume $n = 1$ or 2. If the conditions (i) $w(t) \searrow\searrow$, (ii) $0 \leqslant w(t) = w(|t|)$ hold for $|t| \leqslant 1$, then*

$$\int_{|x| \leqslant 1} \Big| \int_{|t| \geqslant 1} \Omega_\tau(x-t) f(t)\,dt \Big|^q w(x)\,dx$$

$$\leqslant c(1+|\tau|^n)^q \int_{|t| \geqslant 1} \big(|f| + (Af)_b\big)^q |t|^{n(q-bq-a)} w\big(|t|^{1-a}\big)\,dt,$$

where $(Af)_b(t) = |t|^{n(b-1)} \int_{1 \leqslant |s| \leqslant |t|} |f|\,|s|^{-bn}\,ds.$

Proof. We suppose that $\lambda_{m+1} \leqslant \lambda_m \leqslant 1$ and $\lambda_m \searrow 0$, where $w(t) \sim 2^m$ if $\lambda_{m+1} \leqslant |t| \leqslant \lambda_m$; this last fact is consistent with supposing that $w(t) \searrow\searrow$, the case $w \sim 1$ being covered by (C_0'). We now get

$$\sum_{m=0}^{\infty} \int_{\lambda_{m+1} \leqslant |x| \leqslant \lambda_m} \Big| \int_{|t| \geqslant 1} \Omega_\tau(x-t) f(t)\,dt \Big|^q w(x)\,dx$$

$$\leqslant c \sum_{m=0}^{\infty} w(\lambda_m) \int_{\lambda_{m+1} \leqslant |x| \leqslant \lambda_m} \Big| \int_{|t| \geqslant 1} \Omega_\tau(x-t) f(t)\,dt \Big|^q\,dx$$

$$\leqslant c(1+|\tau|^n)^q \Big\{ \sum_{m=0}^{\infty} w(\lambda_m) \int_{\lambda_{m+1} \leqslant |x| \leqslant \lambda_m} \Big(\int_{1 \leqslant |t| \leqslant \lambda_m^{1/(1-a)}} |f|\,|t|^{-bn}\,dt \Big)^q$$

$$+ \sum_{m=0}^{\infty} w(\lambda_m) \int_{|t| \geqslant \lambda_m^{1/(1-a)}} |f|^q |t|^{n(q-bq-a)}\,dt \Big\} = \mathrm{I} + \mathrm{II}.$$

The estimate for obtaining I is straightforward, while the estimate for II follows from (C_0'). Using polar coordinates we get

$$\mathrm{I} \leqslant c(1+|\tau|^n)^q \int_{|t| \geqslant 1} \big((Af)_b(t)\big)^q |t|^{n(q-bq-a)} w\big(|t|^{1-a}\big)\,dt. \tag{3.13}$$

For II, we obtain

$$\mathrm{II} \leqslant c(1+|\tau|^n)^q \int_{|t|\geqslant 1} |f|^q |t|^{n(q-bq-a)} \sum_{|t|\geqslant \lambda_m^{1/(1-a)}} w(\lambda_m),$$

where we are summing over all those m values for which $\lambda_m^{1/(1-a)} \leqslant |t|$. There is an m' so that $\lambda_{m'}^{1/(1-a)} \leqslant |t| \leqslant \lambda_{m'+1}^{1/(1-a)}$. But

$$\sum_{m=0}^{m'} 2^m \leqslant c2^{m'} \leqslant cw(\lambda_{m'})$$

and hence

$$\mathrm{II} \leqslant c(1+|\tau|^n)^q \int_{|t|\geqslant 1} |f|^q |t|^{n(q-bq-a)} w(|t|^{1-a})\,dt. \tag{3.14}$$

Putting (3.13) and (3.14) together we get our result.

We immediately see that the following holds:

Proposition 3.6. *Let $w(t)\geqslant 0$. If $\int_{|t|\leqslant 1} w(t)\,dt < \infty$, then*

$$\int_{|x|\leqslant 1} \Big| \int_{|t|\leqslant 1} \Omega_\tau(x-t) f(t)\,dt\Big|^q w(x)\,dx \leqslant c\|f\|_1^q,$$

where f is supported in $|t|\leqslant 1$.

Now we are in a position to show our first major result.

Theorem 3.7. *Let $b\leqslant 1-a/2$, $a>1$, $q\geqslant 2$ and $n=1$ or 2. Suppose that*

(i) *$w\geqslant 0$ is radial,*

(ii) *w satisfies Definition 0.1(i),*

(iii) *$w(t)/|t|^{n(bq+a-2)} \searrow\searrow$ for $|t|\geqslant 1$,*

(iv) *$w(t)\nearrow\nearrow$ for $\|t\|\geqslant 1$,*

(v) *$w(t)\searrow\searrow$ for $|t|\leqslant 1$, and*

(vi) *$\int_{|t|\leqslant 1} w(t)\,dt<\infty$.*

Then

$$\|\Omega_\tau * f\|_{q,w}^q \leqslant c(1+|\tau|^n)^q \Big\{\|f\|_{q,v}^q + \int_{|t|\leqslant 2} (Af)^q v(t)\,dt + \int_{|t|\geqslant 2} (Af)_0^q v(t)\,dt\Big\},$$

where

$$v(t)\geqslant c\,\{|t|^{n(q-bq-a)} \max\big(w(t), w(|t|^{1-a})\big)\chi(\|t\|\geqslant 1)$$
$$+|t|^{n(q-2)}|t|^{-n(bq+a-2)/(1-a)} w(|t|^{1/(1-a)})\chi(\|t\|\leqslant 1)\}.$$

Proof. By Lemmas 3.1–3.5 and Proposition 3.6 we get

$$\|\Omega_\tau * f\|_{q,w}^q \leqslant c(1+|\tau|^n)^q \Big\{\|f\|_{q,v}^q + \int_{\|t\|\leqslant 2} (Af)^q v(t)\,dt + \int_{\|t\|\geqslant 2} (Af)_0^q v(t)\,dt$$
$$+\Big(\int_{\|t\|\leqslant 1} |f|\,dt\Big)^q\Big\};$$

but notice that

$$\Big(\int_{|t|\leqslant 1} |f|\,dt\Big)^q \leqslant c \int_{1\leqslant |t|\leqslant 2} (Af)^q v(t)\,dt$$

since here $v(t)\chi(2\geqslant |t|\geqslant 1)$ is bounded away from zero, since we may assume $w>0$.

Because of (iii) and (iv), Theorem 3.7 works only when $bq+a-2\geqslant 0$. We shall now do the contrary case, when $bq+a-2\leqslant 0$.

Theorem 3.8. *Let $b\leqslant 1-a/2$, $a>1$, $q\geqslant 2$ and $n=1$ or 2. Suppose that*

(i) *$w\geqslant 0$ is radial,*

(ii) *$w(t)/|t|^{n(bq+a-2)}\searrow\searrow$ for $\|t\|\geqslant 1$,*

(iii) *$w(t)\searrow\searrow$ for $|t|\leqslant 1$,*

(iv) *$w(t)\chi(\|t\|\geqslant 1)\in L^\infty$, and*

(v) *$\int_{|t|\leqslant 2} w(t)\,dt<\infty$.*

Then

$$\|\Omega_\tau * f\|_{q,w}^q \leqslant c(1+|\tau|^n)^q \Big\{\|f\|_{q,v}^q + \int_{|t|\leqslant 2} (Af)^q v(t)\,dt + \int_{|t|\geqslant 2} (Af)_0^q v(t)\,dt\Big\},$$

where

$$v(t)\geqslant c\,\{|t|^{n(q-bq-a)}\max\big(w(t), w(\|t\|^{1-a}), 1\big)\chi(\|t\|\geqslant 1)$$
$$+|t|^{n(q-2)}|t|^{-n(bq+a-2)/(1-a)} w(|t|^{1-a})\chi(|t|\leqslant 1)\}.$$

Proof. Here we get our result by (A_0) of Lemma 3.1, Lemmas 3.2–3.5 and Proposition 3.6.

§ 4. (L_w^q, L_v^p) estimates.

For the most part in this section

$$K(t) = \frac{e^{i|t|^a}}{(1+|t|^n)^{b(1-\lambda)}},\qquad \frac{1}{q}=\frac{1}{p}-\lambda,\qquad 0\leqslant\lambda\leqslant 1.$$

We set

$$Tf = K*f \tag{4.0}$$

and we wish to determine those weights v, w for which

$$\|Tf\|_{q,w}\leqslant c\|f\|_{p,v},\qquad 1<p\leqslant q<\infty. \tag{4.1}$$

By Proposition 2.4(a) and Lemma 3.4 we get

Proposition 4.1. *Let $q\geqslant 2$, $a>1$, $n=1$ or 2, and $1/q+1/q'=1$. Suppose that*

(i) *$w(t)/|t|^{n(bq+a-2)}\searrow\searrow$ for $|t|\geqslant 1$,*

(ii) *$0\leqslant w(t)$ for $|t|\geqslant 1$ and $w(t)$ is radial, and*

(iii) $$\sup_{0<s\leqslant 1}\Big(\int_{1\leqslant |t|\leqslant s^{1/(1-a)}} \frac{w(t)}{|t|^{nbq}}\,dt\Big)^{1/q}\Big(\int_{s^{1/(1-a)}\leqslant |t|} \frac{|t|^{(q'-1)n(bq-1)}}{(w(t))^{q'-1}}\frac{dt}{|t|^n}\Big)^{1/q'}<\infty.$$

Then

$$(4.2)\quad \int_{1\leqslant|x|}\Big|\int_{|t|\leqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^q w(x)\,dx \leqslant c(1+|\tau|^n)^q\int_{|t|\leqslant 1}|f|^q|t|^{n(q-2)}|t|^{n(bq+a-2)/(a-1)}w(|t|^{1/(1-a)})\,dt.$$

By Proposition 2.4(b) and Lemma 3.5 we get

PROPOSITION 4.2. *Let $q\geqslant 2$, $a>1$, $n=1$ or 2, and $1/q+1/q'=1$. Suppose that*

(i) $w(t)\searrow\searrow$ *for* $|t|\leqslant 1$,
(ii) $0\leqslant w(t)$ *and* $w(t)$ *is radial, and*
(iii) $\sup_{s\geqslant 1}\Big(\int_{|t|\leqslant s^{1-a}}w(t)\,dt\Big)^{1/q}\Big(\int_{s^{1-a}\leqslant|t|\leqslant 1}\frac{dt}{|t|^{nq'}(w(t))^{q'-1}}\Big)^{1/q'}<\infty.$

Then

$$(4.3)\quad \int_{|x|\leqslant 1}\Big|\int_{|t|\geqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^q w(x)\,dx \leqslant c(1+|\tau|^n)^q\int_{|t|\geqslant 1}|f|^q|t|^{n(q-bq-a)}w(|t|^{1-a})\,dt.$$

We now define

$$(4.4)\quad \begin{aligned} T_z f(x) &= (w(x))^{z/q^*}\int h_z(x,t)f(t)\,dt,\\ h_z(x,t) &= \frac{g(x,t)}{(1+|x-t|^n)^{bz}},\qquad z=\sigma+i\tau,\ 0\leqslant\sigma\leqslant 1,\ -\infty<\tau<\infty.\end{aligned}$$

PROPOSITION 4.3. *Let $\alpha(1-q^*)=\beta$, $d\mu=|t|^\alpha dt$ and let T_z be defined by (4.4). If $g\in L^\infty$ and*

$$\|T_{1+i\tau}f\|_{q^*}\leqslant c(1+|\tau|^n)\Big(\int|f|^{q^*}|t|^\beta v(t)\,dt\Big)^{1/q^*},$$

then for $w>0$, $v>0$

$$\|T_{1-\lambda}f\|_q\leqslant c\|f\|_{p,|t|^{\alpha(1-p)}(v(t))^{p/q}}$$

where $1/p=1/q+\lambda$ and $1/q=(1-\lambda)/q^*$.

Proof. Consider the operator

$$\mathcal{I}_z f = T_z\big(f|t|^\alpha(v(t))^{-z/q^*}\big).$$

Note that since $g\in L^\infty$,

$$\text{(i)}\qquad \|\mathcal{I}_{i\tau}f\|_\infty\leqslant c\|f\|_{1,\mu}$$

and by hypothesis we get

$$\text{(ii)}\qquad \|\mathcal{I}_{1+i\tau}f\|_{q^*}\leqslant c(1+|\tau|^n)\Big(\int|f|^{q^*}|t|^{\alpha q^*}(v(t))^{-1}|t|^\beta v(t)\,dt\Big)^{1/q^*} = c(1+|\tau|^n)\|f\|_{q^*,\mu}.$$

(i) and (ii) imply that

$$\text{(iii)}\qquad \|\mathcal{I}_{1-\lambda}f\|_q\leqslant c\|f\|_{p,\mu},\quad\text{where } \frac{1}{p}=\frac{1}{q}+\lambda,\ \frac{1}{q}=\frac{1-\lambda}{q^*},$$

and by unraveling (iii) we get our result.

We have ($1<p,\ q<\infty$ is assumed throughout)

LEMMA 4.4. *Let $1/p+1/q\leqslant 1$, $a>1$, $b\leqslant 1-a/2$ and $n=1$ or 2. Suppose that*

(i) $w\geqslant 0$ *is radial,*
(ii) w *satisfies Definition* 0.1(i),
(iii) $w(t)\nearrow\nearrow$ *for* $\|t\|\geqslant 1$, *and*
(iv) $w(t)/|t|^{n(b(q-q/p+1)+a-2)}\searrow\searrow$ *for* $\|t\|\geqslant 1$.

Then for $1/q=1/p-\lambda$, $0\leqslant\lambda\leqslant 1$,

$$(4.5)\quad \Big(\int_{1\leqslant\varphi(x)}\Big|\int_{1\leqslant\varphi(t)}K(x-t)f(t)\,dt\Big|^q w(x)\,dx\Big)^{1/q} \leqslant c\Big(\int_{1\leqslant\|t\|}|f|^p|t|^{\alpha(1-p)}(w(t))^{p/q}\,dt\Big)^{1/p},$$

where $\alpha=\dfrac{n[(1-b)(pq-q+p)-ap]}{q(1-p)}$.

Proof. We get by $(A_{0,w})$, $(B'_{0,w})$ and $(C_{0,w})$

$$\int_{\varphi(x)\geqslant 1}\Big|\int_{\varphi(t)\geqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^{q^*}w(x)\,dx \leqslant c(1+|\tau|^n)^{q^*}\int_{\|t\|\geqslant 1}|f|^{q^*}|t|^{n(q^*-bq^*-a)}w(t)\,dt,$$

where $1/q=(1-\lambda)/q^*$ and $1/p=1/q+\lambda$. The hypotheses of Proposition 4.3 are satisfied with $g(x,t)=e^{i|x-t|^a}\chi(\varphi(x)\geqslant 1$ and $\varphi(t)\geqslant 1)$, $\beta=n(q^*-bq^*-a)$ with $q^*\geqslant 2$ and $v(t)=w(t)$, and our result follows from that proposition.

LEMMA 4.5. *Let $1/p+1/q\leqslant 1$, $a>1$, $b\leqslant 1-a/2$, and $n=1$ or 2. Suppose that*

(i) $w\geqslant 0$ *is radial,*
(ii) $w(t)\chi(\|t\|\geqslant 1)\in L^\infty$, *and*
(iii) $w(t)/|t|^{n(b(q-q/p+1)+a-2)}\searrow\searrow$ *for* $\|t\|\geqslant 1$.

Then for $1/q=1/p-\lambda$, $0\leqslant\lambda\leqslant 1$,

$$(4.6)\quad \Big(\int_{1\leqslant\varphi(x)}\Big|\int_{1\leqslant\varphi(t)}K(x-t)f(t)\,dt\Big|^q w(x)\,dx\Big)^{1/q} \leqslant c\Big(\int_{\|t\|\geqslant 1}|f|^p|t|^{\alpha(1-p)}\max\big(1,(w(t))^{p/q}\big)\,dt\Big)^{1/p},$$

where $\alpha=\dfrac{n[(1-b)(pq-q+p)-ap]}{q(1-p)}$.

Proof. We may assume $w>0$ by slightly increasing w. We get by (A_0), $(B'_{0,w})$ and (C_0)

$$\int_{\varphi(x)\geqslant 1}\Big|\int_{\varphi(t)\geqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^{q^*}w(x)\,dx$$
$$\leqslant c(1+|\tau|^n)^{q^*}\int_{\|t\|\geqslant 1}|f|^{q^*}|t|^{n(q^*-bq^*-a)}\max(1,\,w(t))\,dt$$

where $1/q=(1-\lambda)/q^*$ and $1/p=1/q+\lambda$. If we now take $g(x,t)=e^{i|x-t|^a}\chi(\varphi(x)\geqslant 1$ and $\varphi(t)\geqslant 1)$, $\beta=n(q^*-bq^*-a)$ with $q^*\geqslant 2$ and $v=\max(1,w)$ we get our result from Proposition 4.3.

Lemma 4.6. *Let* $1/p+1/q\leqslant 1$, $a>1$, $n=1$ *or* 2 *and* $1/p+1/p'=1$. *Suppose that*

(i) $w(t)/|t|^{n(b(qp-q+p)+(a-2)p)/p}\searrow\searrow$ *for* $|t|\geqslant 1$,

(ii) $0\leqslant w(t)\chi(|t|\geqslant 1)$ *and* $w(t)$ *is radial, and*

(iii) $$\sup_{0<s\leqslant 1}\Big(\int_{1\leqslant|t|\leqslant s^{1/(1-a)}}\frac{w(t)\,dt}{|t|^{nb(pq-q+p)/p}}\Big)^{1/q}\times\Big(\int_{s^{1/(1-a)}\leqslant|t|}\frac{|t|^{n(b(pq-q+p)-p)/[q(p-1)]}}{|t|^n(w(t))^{p/[q(p-1)]}}\Big)^{1/p'}<\infty.$$

Then for $1/q=1/p-\lambda$, $0\leqslant\lambda\leqslant 1$,

$$(4.7)\quad \Big(\int_{|x|\geqslant 1}\Big|\int_{|t|\leqslant 1}K(x-t)f(t)\,dt\Big|^q w(x)\,dx\Big)^{1/q}\leqslant c(1+|\tau|^n)\Big(\int_{|t|\leqslant 1}|f|^p|t|^\gamma\big(w(|t|^{1/(1-a)})\big)^{p/q}dt\Big)^{1/p},$$

where $\gamma=\dfrac{n}{q}\Big[(pq-q-p)+\dfrac{1}{a-1}\big(b(pq-q+p)+(a-2)p\big)\Big]$.

Proof. For $q^*\geqslant 2$, $1/q=(1-\lambda)/q^*$ and $1/p=1/q+\lambda$ we see that (4.2) is satisfied for q^*, i.e.

$$\int_{1\leqslant|x|}\Big|\int_{|t|\leqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^{q^*}w(x)\,dx$$
$$\leqslant c(1+|\tau|^n)^{q^*}\int_{|t|\leqslant 1}|f|^{q^*}|t|^{n(q^*-2)}|t|^{n(bq^*+a-2)/(a-1)}w(|t|^{1/(1-a)})\,dt.$$

Now if we take $g(x,t)=e^{i|x-t|^a}\chi(|x|\geqslant 1$ and $|t|\leqslant 1)$, $\beta=n[(q^*-2)+(a-1)^{-1}(bq^*+a-2)]$, $v(t)=w(|t|^{1/(1-a)})$ and define $\alpha(1-p)=\gamma$, then our result follows from Proposition 4.3.

Lemma 4.7. *Let* $1/p+1/q\leqslant 1$, $a>1$, $n=1$ *or* 2, *and* $1/p+1/p'=1$.

Suppose that

(i) $w(t)\searrow\searrow$ *for* $|t|\leqslant 1$,

(ii) $0\leqslant w(t)$ *and* $w(t)$ *is radial, and*

(iii) $$\sup_{s\geqslant 1}\Big(\int_{|t|\leqslant s^{1-a}}w(t)\,dt\Big)^{1/q}\times\Big(\int_{s^{1-a}\leqslant|t|\leqslant 1}\frac{dt}{|t|^{n(qp-q+p)/[q(p-1)]}(w(t))^{p/[q(p-1)]}}\Big)^{1/p'}<\infty.$$

Then for $1/q=1/p-\lambda$, $0\leqslant\lambda\leqslant 1$,

$$\Big(\int_{|x|\leqslant 1}\Big|\int_{|t|\geqslant 1}K(x-t)f(t)\,dt\Big|^q w(x)\,dx\Big)^{1/q}\leqslant c\Big(\int_{|t|\geqslant 1}|f|^p|t|^{\alpha(1-p)}\big(w(|t|^{1-a})\big)^{p/q}dt\Big)^{1/p},$$

where $\alpha=n\dfrac{(1-b)(pq-q+p)-ap}{q(1-p)}$.

Proof. By Proposition 4.2 we get for $q^*\geqslant 2$

$$\int_{|x|\leqslant 1}\Big|\int_{|t|\geqslant 1}\Omega_\tau(x-t)f(t)\,dt\Big|^{q^*}w(x)\,dx$$
$$\leqslant c(1+|\tau|^n)^{q^*}\int_{|t|\geqslant 1}|f|^{q^*}|t|^{n(q^*-bq^*-a)}w(|t|^{1-a})\,dt$$

where $1/q=(1-\lambda)/q^*$ and $1/p=1/q+\lambda$. Now with $g(x,t)=e^{i|x-t|^a}\chi(|x|\leqslant 1$ and $|t|\geqslant 1)$, $\beta=n(q^*-bq^*-a)$ and $v(t)\geqslant w(|t|^{1-a})$ we get our result.

Now we are in a position to prove the sufficiency in Corollary 0.2 in case $\beta>0$.

Theorem 4.8. *Assume the hypothesis of Corollary* 0.2 *with* $\beta>0$, *and let* $1/q=1/p-\lambda$ *for* $0\leqslant\lambda\leqslant 1$. *If* $v(t)\geqslant c_2v_0(t)$, *then* $\|T_\lambda f\|_{q,w}\leqslant c_1\|f\|_{p,v}$, *where* $T_\lambda f$ *and* v_0 *are defined in* (0.3).

Proof. We can suppose without loss of generality that w is not identical to zero. Next set (with $K(t)=e^{i|t|^a}/(1+|t|^n)^{b(1-\lambda)}$)

$$T_1f(x)=\Big(\int_{\varphi(t)\geqslant 1}K(x-t)f(t)\,dt\Big)\chi(\varphi(x)\geqslant 1),$$
$$T_2f(x)=\Big(\int_{|t|\leqslant 1}K(x-t)f(t)\,dt\Big)\chi(|x|\geqslant 1)$$
$$T_3f(x)=\Big(\int_{|t|\geqslant 1}K(x-t)f(t)\,dt\Big)\chi(|x|\leqslant 1),$$
$$T_4f(x)=\Big(\int_{|t|\leqslant 1}K(x-t)f(t)\,dt\Big)\chi(|x|\leqslant 1).$$

Notice that by (0.2), $\int_{|t|\leqslant 1}w(t)\,dt<\infty$, and so by Proposition 3.6

$$\|T_4f\|_{q,w}^q\leqslant c\int_{1\leqslant|t|\leqslant 2}(Af)^q\,dt\leqslant c\Big(\int_{|t|\leqslant 2}|f|^p\,dt\Big)^{q/p}.$$

Since $v(t)$ is bounded away from zero we get

$$\|T_4 f\|_{q,w} \leqslant c\Big(\int_{|t|\leqslant 2} |f|^p v(t)\,dt\Big)^{1/p},$$

and since

$$\|Tf\|_{q,w} \leqslant \sum_{j=1}^{4} \|T_j f\|_{q,w},$$

it follows from Lemmas 4.4, 4.6 and 4.7 that

$$\|Tf\|_{q,w} \leqslant c\,\|f\|_{p,v}.$$

Now we shall state and prove the sufficiency in Corollary 0.2 in case $\beta \leqslant 0$.

THEOREM 4.9. *Assume the hypothesis of Corollary* 0.2, $\beta \leqslant 0$, *and let* $1/q = 1/p - \lambda$ *for* $0 \leqslant \lambda \leqslant 1$. *If* $v(t) \geqslant c_2 v_0(t)$, *then* $\|T_\lambda f\|_{q,w} \leqslant c_1 \|f\|_{p,v}$, *where* $T_\lambda f$ *and* v_0 *are defined in* (0.3).

Proof. We argue as we did in Theorem 4.8, but this time we use Lemmas 4.5, 4.6 and 4.7 to get our result.

In the Fourier transform case (i.e. $a = 2$, $b = 0$) the hypothesis on the weight w reduces to $w \searrow\searrow$ since in this case $\beta = 0$ and $\delta = 0$, while the conditions (0.2) reduce to

$$(4.8)\quad \sup_{s>0}\Big(\int_{|t|\leqslant 1/s} w(t)\,dt\Big)^{1/q}\left(\int_{|t|\geqslant 1/s} \frac{dt}{|t|^{n(pq-q+p)/[q(p-1)]}\,(w(t))^{p/[q(p-1)]}}\right)^{1/p'} < \infty,$$

since $\varepsilon = n(pq-q+p)/[q(1-p)]$.

Now we obtain the dual results to Theorems 4.8 and 4.9. We employ Proposition 2.3 which says that if $\|Tf\|_{q,w} \leqslant c\|f\|_{p,v}$ then

$$\|Tf\|_{p',v^{1-p'}} \leqslant c\,\|f\|_{q',w^{1-q'}}.$$

Then we get

THEOREM 4.10. *Let* $b \leqslant 1-a/2$, $a > 1$, $1/p'+1/q' \geqslant 1$, $1 < q' \leqslant p' \leqslant 2$, $w_1 = v^{1-p'}$, $v_1 = w^{1-q'}$, $\beta > 0$, $n = 1$ *or* 2 *and* $v > 0$, $w > 0$. *Suppose that*

(i) v_1 *satisfies Definition* 0.1(i),

(ii) $v_1(t)/|t|^{n(b(-q'+q'/p-q'/q)+(a-2)(1-q'))} \nearrow\nearrow$ *for* $|t| \geqslant 1$,

(iii) $v_1(t) \searrow\searrow$ *for* $|t| \geqslant 1$,

(iv) $v_1(t) \nearrow\nearrow$ *for* $|t| \leqslant 1$,

(v)
$$\sup_{0<s\leqslant 1}\left(\int_{1<|t|\leqslant s^{1/(1-a)}} \frac{(v_1(t))^{1/(1-q')}\,dt}{|t|^{nb(q-q/p+1)}}\right)^{1/q} \times\left(\int_{s^{1/(1-a)}\leqslant|t|} \frac{(v_1(t))^{p/[q'(p-1)]}}{|t|^{n(pq-q+p)(1-b)/[q(p-1)]}}\,dt\right)^{1/p'} < \infty,$$

(vi)
$$\sup_{s\geqslant 1}\Big(\int_{|t|\leqslant s^{1-a}} (v_1(t))^{1/(1-q')}\,dt\Big)^{1/q}\left(\int_{s^{1-a}\leqslant|t|\leqslant 1} \frac{(v_1(t))^{p/[q'(p-1)]}\,dt}{|t|^{n(qp-q+p)/[q(p-1)]}}\right)^{1/p'} < \infty.$$

Then for $1/p' = 1/q' - \lambda$

$$\|T_\lambda f\|_{p',w_1} \leqslant c\,\|f\|_{q',v_1}$$

where

$$w_1(t) \leqslant c\,|t|^{\alpha}\chi(|t|\geqslant 1)\min\big((v_1(t))^{p'/q'},\,(v_1(|t|^{1-a}))^{p'/q'},\,1\big) + c\,|t|^{-\gamma(p'-1)}\big(v_1(|t|^{1/(1-a)})\big)^{p'/q'}\chi(|t|\leqslant 1),$$

and α, γ *are defined in Lemmas* 4.5 *and* 4.6 *respectively.*

The next result is dual to Theorem 4.9.

THEOREM 4.11. *Let* $b \leqslant 1-a/2$, $a > 1$, $1/p'+1/q' \geqslant 1$, $1 < q' \leqslant p' \leqslant 2$, $n = 1$ *or* 2, $w_1 = v^{1-p'}$, $v_1 = w^{1-q'}$, $\beta \leqslant 0$ *and* $v > 0$, $w > 0$. *Suppose*

(i) $v_1(t) \nearrow\nearrow$ *for* $|t| \leqslant 1$,

(ii) $v_1(t)/|t|^{n(b(-q'+q'/p-q'/q)+(a-2)(1-q'))} \nearrow\nearrow$ *for* $|t| \geqslant 1$,

and (v) *and* (vi) *of Theorem* 4.10 *are satisfied. Then for* $1/p' = 1/q' - \lambda$

$$\|T_\lambda f\|_{p',w_1} \leqslant c\,\|f\|_{q',v_1}$$

where

$$w_1(t) \leqslant c\,|t|^{\alpha}\chi(|t|\geqslant 1)\min\big((v_1(t))^{p'/q'},\,(v_1(|t|^{1-a}))^{p'/q'},\,1\big) + c\,|t|^{\gamma(1-p')}\big(v_1(|t|^{1/(1-a)})\big)^{p'/q'}\chi(|t|\leqslant 1),$$

and α, γ *appear in Lemmas* 4.5, 4.6 *respectively.*

COROLLARY 4.12. *Let* $0 \leqslant w$ *be radial,* $n = 1$ *or* 2 *and* $1 < p \leqslant q < \infty$. *Suppose that*

(i) $w(t) \searrow\searrow$,

(ii) w *satisfies* (4.8), *and*

(iii) $v(t) \geqslant c\,|t|^{n(pq-q-p)/q}(w(1/|t|))^{p/q}$.

Then

$$\|\hat{f}\|_{q,w} \leqslant c\,\|f\|_{p,v}.$$

Proof. In case $1/p+1/q \leqslant 1$, this follows from Theorem 4.9, and in case $1/p+1/q \geqslant 1$ from Theorem 4.11.

Notice this coincides with the sufficiency of Theorem 2 in [4] (here we use w, v and there we used w^q, v^p respectively). Also notice that for (5.1) in [4] with $(\operatorname{cap} W)\searrow$ and $(\operatorname{cap} V)\nearrow$ implies that

$$(V(r))^{1/p} \geqslant B r^{1-1/p-1/q}(W(1/r))^{1/q}.$$

§ 5. **Necessity results.** In Theorems 4.8, 4.9 we gave sufficient conditions in order that our operators map, i.e. lower bounds for the weight $v(t)$ in terms of $w(t)$. In this section, we show that these bounds are necessary. This then completes the proof of Corollary 0.2.

Furthermore, we shall analyze which conditions are necessary. For example, we shall show that Lemma 4.4(iii) is both necessary and sufficient for (4.5) to hold.

The case $a = 2$ and $b = 0$ is among the cases we analyze here. In Theorem 2 of [4], we showed that $Af(t)$ is needed in our estimates. That first appears here in Lemmas 3.4 and 3.5, i.e. when we are working around the origin in one of the variables. Thus conditions like (0.2) are needed in our results and that follows by [1].

Throughout this section, we shall use $T_\lambda f$, v_0, α and γ as defined earlier. In fact, $T_\lambda f$ and v_0 are defined in (0.3) and α, γ are defined in Lemmas 4.5, 4.6 respectively. Similarly, ε and δ will be as before, they appear in Corollary 0.2.

LEMMA 5.1. *Let $n \geqslant 1$, $a > 1$, $0 \leqslant \lambda \leqslant 1$, $1/q = 1/p - \lambda$ and $1 < p \leqslant q < \infty$. Assume that v, w are radial, positive and that they satisfy Definition 0.1(i). If*

$$\|T_\lambda f\|_{q,w} \leqslant c\|f\|_{p,v}, \tag{5.1}$$

then

$$\chi(|t| \leqslant 1)\, v(t) \geqslant c|t|^{\gamma}\big(w(|t|^{1/(1-a)})\big)^{p/q}\chi(|t| \leqslant 1), \tag{5.2}$$

$$\chi(|t| \geqslant 1)\, v(t) \geqslant c|t|^{\alpha(1-p)}\chi(|t| \geqslant 1)\max\big((w(|t|))^{p/q}, (w(|t|^{1-a}))^{p/q}, 1\big). \tag{5.3}$$

Proof. Throughout this argument we let $u, x, t, R, A \in \boldsymbol{R}^n$.

We begin by defining the following functions:

$$\begin{aligned} &(a) \quad f_1(t) = \chi(|R|^{1-a}/2 \leqslant |t| \leqslant |R|^{1-a}),\\ &(b) \quad f_2(t) = e^{-i|t-R|^a}\chi(|R|/4 \leqslant |t| \leqslant |R|/2),\\ &(c) \quad f_3(t) = e^{-i|t-A|^a}\chi(|R|^{1/(1-a)}/2 \leqslant |t| \leqslant |R|^{1/(1-a)}). \end{aligned} \tag{5.4}$$

Now by (5.1) it follows that

$$\Big(\int_{|x-R| \leqslant |R|/c} |T_\lambda f_1(x)\, e^{-i|x|^a}|^q\, w(x)\, dx\Big)^{1/q} \leqslant c\|f_1\|_{p,v}. \tag{5.5}$$

Next notice that for $|R| \geqslant 1$ with x as in (5.5) and t as in (5.4)(a)

$$\big||x-t|^a - |x|^a\big| = a\Big|\int_{|x-t|}^{|x|} v^{a-1}\, dv\Big| \leqslant c|R|^{a-1}|t| \leqslant 1. \tag{5.6}$$

Hence it follows from (5.5) and (5.6) that

$$w^{1/q}(R)|R|^{n/q}\frac{|R|^{(1-a)n}}{|R|^{nb(1-\lambda)}} \leqslant c\big(v(|R|^{1-a})\big)^{1/p}|R|^{(1-a)n/p}.$$

Take $|t| = |R|^{1-a}$; then it follows that

$$\chi(|t| \leqslant 1)\, v(t) \geqslant c|t|^{\gamma}\big(w(|t|^{1/(1-a)})\big)^{p/q}\chi(|t| \leqslant 1)$$

which is (5.2).

Again by (5.1) it follows that (with $|R| \geqslant 1$)

$$\Big(\int_{|x-R| \leqslant |R|^{1-a}/c} |Tf_2|^q\, w(x)\, dx\Big)^{1/q} \leqslant c\|f_2\|_{p,v}. \tag{5.7}$$

Just as in (5.6) we notice that

$$\big||x-t|^a - |t-R|^a\big| = a\Big|\int_{|t-R|}^{|x-t|} v^{a-1}\, dv\Big| \leqslant c|R|^{a-1}|x-R| \leqslant 1. \tag{5.8}$$

Hence by (5.7) and (5.8) we get

$$\frac{w^{1/q}(R)}{|R|^{nb(1-\lambda)}}|R|^n|R|^{(1-a)n/q} \leqslant v^{1/p}(R)|R|^{n/p},$$

and so if we take $|t| = |R|$ it follows that

$$\chi(|t| \geqslant 1)\, v(t) \geqslant c\big(w(|t|)\big)^{p/q}|t|^{\alpha(1-p)}\chi(|t| \geqslant 1). \tag{5.9}$$

Again by (5.1) it follows that

$$\Big(\int_{|x-R| \leqslant |R|/c} |Tf_3|^q\, w(x)\, dx\Big)^{1/q} \leqslant c\|f_3\|_{p,v}, \tag{5.10}$$

where here for f_3 we set $A = R$.

This time with $|R| \leqslant 1$ we get as before

$$\big||x-R|^a - |R-t|^a\big| = a\Big|\int_{|t-R|}^{|x-t|} v^{a-1}\, dv\Big| \leqslant c|R|^{-1}|x-R| \leqslant 1. \tag{5.11}$$

Hence it follows by (5.10) and (5.11) that

$$\frac{w^{1/q}(R)}{|R|^{nb(1-\lambda)/(1-a)}}|R|^{n/(1-a)}|R|^{n/q} \leqslant c\big(v(|R|^{1/(1-a)})\big)^{1/p}|R|^{n/[(1-a)p]},$$

and if we take $|t| = |R|^{1/(1-a)}$ then it follows that

$$\chi(|t| \geqslant 1)\, v(t) \geqslant \big(w(|t|^{1-a})\big)^{p/q}|t|^{\alpha(1-p)}\chi(|t| \geqslant 1). \tag{5.12}$$

Now as in (5.10) and (5.11), if we put A in place of R for f_3 and note by (5.1) that

$$\Big(\int_{|x-A| \leqslant |R|/c} |Tf_3|^q\, w(x)\, dx\Big)^{1/q} \leqslant c\|f_3\|_{p,v}, \tag{5.10'}$$

then if we assume that $|R| \leqslant |A| \leqslant |R|^{1/(1-a)}/4$ with $|R| \leqslant 1$, then

$$\frac{w^{1/q}(A)}{|R|^{nb(1-\lambda)/(1-a)}}|R|^{n/(1-a)}|R|^{n/q} \leqslant c|R|^{np/(1-a)}v^{1/p}(|R|^{1/(1-a)}).$$

Now take $|x| = |R|^{1/(1-a)}$; then it follows that

$$v(x) \geqslant c|x|^{\alpha(1-p)} w^{p/q}(A) \quad \text{as long as } |x|^{1-a} \leqslant |A| \leqslant |x|/4,$$

where $|x| \geqslant 1$, and since w satisfies Definition 0.1(i) it follows that

$$(5.13) \qquad v(x) \geqslant c|x|^{\alpha(1-p)} \sup (w(A))^{p/q}, \qquad |x|^{1-a} \leqslant |A| \leqslant |x|,\ |x| \geqslant 1.$$

But since $\|w\|_\infty \neq 0$ and w satisfies Definition 0.1(i) it follows from (5.13) that

$$(5.14) \qquad v(x) \geqslant c|x|^{\alpha(1-p)} \quad \text{for } |x| \geqslant 1,$$

and so our lower estimate (5.3) follows from (5.9), (5.12) and (5.14).

Proof of Corollary 0.2. The sufficiency follows from Theorems 4.8, 4.9. The necessity follows by Lemma 5.1.

Now we shall analyze whether the conditions on w are necessary.

PROPOSITION 5.2. *Let* $1 < p \leqslant q < \infty$, $1/p + 1/q \leqslant 1$, $a > 1$, $b \leqslant 1 - a/2$, *and* $n = 1$ *or* 2. *Assume* (i), (ii), *and* (iv) *of Lemma* 4.4. *Then for* $1/q = 1/p - \lambda$, $0 \leqslant \lambda \leqslant 1$,

(4.5) *holds* $\Leftrightarrow$ *condition* (iii) *of Lemma* 4.4 *holds*.

Proof. Because of Lemma 4.4, it suffices to do the necessity $\Rightarrow$.
If (4.5) holds then we get from (5.13)

$$v(x) \geqslant c|x|^{\alpha(1-p)} \sup_{1 \leqslant |A| \leqslant |x|} (w(A))^{p/q}$$

as long as $0 \leqslant v$ satisfies Definition 0.1(i) and is radial; but here $v(x) = |x|^{\alpha(1-p)}(w(x))^{p/q}$ and so it follows that w satisfies Lemma 4.4(iii). Note condition (iv) was not used in proving the necessity of (iii).

We notice that from Lemma 4.4(iii) and (iv) it follows that

$$(5.15) \qquad w(t) \leqslant c|t|^{n(b(q-q/p+1)+a-2)} \quad \text{for } |t| \geqslant 1.$$

We can ask whether (5.15) is also a necessary condition. To this end we show

PROPOSITION 5.3. *Let* $1/p = 1/q + \lambda$, $1/p + 1/q \leqslant 1$, $1 < p \leqslant q < \infty$, $a > 1$, $b \leqslant 1 - a/2$ *and* $n \geqslant 1$. *Suppose that* $0 \leqslant w$ *is radial and satisfies Definition* 0.1(i). *If* $\|T_\lambda f\|_{q,w} \leqslant c\|f\|_{p,v}$, *then* (5.15) *holds*.

Proof. Define $f_0(t) = e^{-i|R-t|^a} \chi(\|t-u\| \leqslant 1)$ where $2 \leqslant \|u\| \leqslant 3$ and $|R| \geqslant 1$ with $u, R \in R^n$. Now it follows that

$$(5.16) \qquad \Big(\int_{|x-R| \leqslant |R|^{2-a}/c} |T_\lambda f_0|^q w(x)\,dx\Big)^{1/q} \leqslant c\Big(\int_{\|t-u\| \leqslant 1} |f_0|^p v(t)\,dt\Big)^{1/p};$$

and so it follows from (5.16), once we show

$$(5.17) \qquad \big|(|x-t|^a - |R-t|^a) - (|x|^a - |R|^a)\big| \leqslant 1,$$

that

$$w^{1/q}(R)\,|R|^{n(2-a)/q} \frac{1}{|R|^{nb(1-\lambda)}} \leqslant c\Big(\int_{\|t-u\| \leqslant 1} v(t)\,dt\Big)^{1/p},$$

and so

$$w(R) \leqslant c_2 |R|^{n(b(q-q/p+1)+a-2)}.$$

Our proof is complete once we show (5.17) for the variables in (5.16).
We begin by setting $H(t) = |x-t|^a - |R-t|^a$ with $t \in R^n$. Then

$$H(t) = H(0) + \sum_{j=1}^{n} H_j(\xi t)\,t_j, \qquad 0 < \xi < 1,\ H_j = \frac{\partial H}{\partial t_j}.$$

Next notice that

$$H_j(\xi t) = a(x_j - \xi t_j)|x - \xi t|^{a-2} - a(R_j - \xi t_j)|R - \xi t|^{a-2} \quad \text{or}$$

$$H_j(\xi t) = a(x_j - R_j)|x - \xi t|^{a-2} + a(R_j - \xi t_j)(|x - \xi t|^{a-2} - |R - \xi t|^{a-2}),$$

and so

$$(H_j(\xi t))^2 \leqslant c\,[(x_j - R_j)^2 |R|^{2(a-2)} + (R_j^2 + t_j^2)|x - R|^2 |R|^{2(a-3)}],$$

and then

$$\sum_{j=1}^{n} (H_j(\xi t))^2 \leqslant c\Big(|R|^{2(a-2)} \sum_{j=1}^{n} (x_j - R_j)^2 + |R|^{2(2-a)} |R|^{2(a-3)} \sum_{j=1}^{n} R_j^2\Big) \leqslant c.$$

Hence

$$\Big|\sum_{j=1}^{n} H_j(\xi t)\,t_j\Big| \leqslant \Big(\sum_{j=1}^{n} (H_j(\xi t))^2\Big)^{1/2} \Big(\sum_{j=1}^{n} t_j^2\Big)^{1/2} \leqslant c.$$

This proves (5.17).

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Received March 20, 1986 (2156)

STUDIA MATHEMATICA, T. LXXXVII. (1987)

Local Banach algebras as Henselian rings

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Abstract. We use methods of analysis to show that local algebras which are homomorphic images of Banach algebras or which are projective or injective limits of Banach algebras are Henselian. Many of the standard examples from algebra are shown to be Henselian by these methods, and a number of further examples, not accessible to classical algebra, are given.

1. In the theory of local rings, there is a notion of a *Henselian ring* ([5, § 16], [11, § 30]). The condition is important because it gives a reducibility criterion for polynomials over the ring. A form of *Hensel's lemma*—for example, the form given in [13, VIII, Theorem 17]—is that each complete Noetherian local ring is Henselian. In this paper, it is shown that rings which are complete in another sense are also Henselian: each local Banach algebra is Henselian. More generally, we prove that a local algebra which is a homomorphic image of a Banach algebra, or which is a projective limit of Banach algebras, or which is an inductive limit of Banach algebras is Henselian.

These results are sufficient to cover the standard examples of Henselian rings, such as the algebras $C[[X]]$ and $C\langle\langle X\rangle\rangle$ of formal and of convergent power series in one indeterminate, and they cover a number of other examples, some of which do not seem to be easily accessible to classical algebraic methods.

This paper is written for analysts: we shall assume that the reader is familiar with commutative Banach algebra theory, but we shall give some algebraic details which the experienced reader of, say, Nagata's "Local Rings" would find to be elementary.

In § 2, we shall first give an algebraic condition equivalent to the fact that a local algebra is Henselian. Unfortunately, the proof of this equivalence involves some rather deep algebra. To avoid reliance on this, and to make this paper self-contained, it will be proved that a formally stronger condition implies that a local algebra is Henselian. This latter condition will be applied in § 3 to show that each local algebra which is the homomorphic image of a Banach algebra or which is a complete LMC algebra, or which is a pseudo-Banach algebra is Henselian. We shall conclude in § 4 with some examples and with some comparisons between our results and standard theorems.