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Local Banach algebras as Henselian rings

by

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Abstract. We use methods of analysis to show that local algebras which are homomorphic images of Banach algebras or which are projective or injective limits of Banach algebras are Henselian. Many of the standard examples from algebra are shown to be Henselian by these methods, and a number of further examples, not accessible to classical algebra, are given.

1. In the theory of local rings, there is a notion of a *Henselian ring* ([5, § 16], [11, § 30]). The condition is important because it gives a reducibility criterion for polynomials over the ring. A form of *Hensel's lemma*—for example, the form given in [13, VIII, Theorem 17]—is that each complete Noetherian local ring is Henselian. In this paper, it is shown that rings which are complete in another sense are also Henselian: each local Banach algebra is Henselian. More generally, we prove that a local algebra which is a homomorphic image of a Banach algebra, or which is a projective limit of Banach algebras, or which is an inductive limit of Banach algebras is Henselian.

These results are sufficient to cover the standard examples of Henselian rings, such as the algebras $C[[X]]$ and $C\langle\langle X\rangle\rangle$ of formal and of convergent power series in one indeterminate, and they cover a number of other examples, some of which do not seem to be easily accessible to classical algebraic methods.

This paper is written for analysts: we shall assume that the reader is familiar with commutative Banach algebra theory, but we shall give some algebraic details which the experienced reader of, say, Nagata's "Local Rings" would find to be elementary.

In § 2, we shall first give an algebraic condition equivalent to the fact that a local algebra is Henselian. Unfortunately, the proof of this equivalence involves some rather deep algebra. To avoid reliance on this, and to make this paper self-contained, it will be proved that a formally stronger condition implies that a local algebra is Henselian. This latter condition will be applied in § 3 to show that each local algebra which is the homomorphic image of a Banach algebra or which is a complete LMC algebra, or which is a pseudo-Banach algebra is Henselian. We shall conclude in § 4 with some examples and with some comparisons between our results and standard theorems.

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2. We begin with some basic definitions. It should be noted that there is considerable variation over the very definitions of "local ring", "Hensel's lemma" and "Henselian" in the standard algebra books, and so some care must be taken.

Since we are concerned with Banach algebras, our particular simplification will be to state the definitions for algebras over the complex field C, rather than for rings. Thus, throughout, the underlying field of each algebra is C.

Let $\mathfrak{A}$ be a commutative unital algebra. The identity of $\mathfrak{A}$ is denoted by $e_{\mathfrak{A}}$ or by e, the set of invertible elements of $\mathfrak{A}$ by $\operatorname{Inv}\mathfrak{A}$, the Jacobson radical by $\operatorname{rad}\mathfrak{A}$, and the set of characters on $\mathfrak{A}$ by $\Phi_{\mathfrak{A}}$.

2.1. Definition. An algebra $\mathfrak{A}$ is a *local algebra* if it is commutative and unital and has a unique maximal ideal.

Thus, following Bourbaki [3, II, 3.1] and Jacobson [8, Definition 3.1], but in distinction to Nagata [11, § 5] and Zariski and Samuel [13], we do not require that a local algebra be Noetherian. Further, a local algebra is not necessarily an integral domain.

The maximal ideal of a local algebra $\mathfrak{A}$ is denoted by $M_{\mathfrak{A}}$; we have $\operatorname{rad}\mathfrak{A} = M_{\mathfrak{A}}$ and $\operatorname{Inv}\mathfrak{A} = \mathfrak{A}\backslash M_{\mathfrak{A}}$. The quotient field $\mathfrak{A}/M_{\mathfrak{A}}$ is the *residue field* of $\mathfrak{A}$, and we write $\pi\colon \mathfrak{A}\to\mathfrak{A}/M_{\mathfrak{A}}$ for the quotient map.

The algebras of polynomials in one and n indeterminates with coefficients in a commutative unital algebra $\mathfrak{A}$ are denoted by $\mathfrak{A}[X]$ and $\mathfrak{A}[X_1, \ldots, X_n]$, respectively; we shall also write $\mathfrak{A}[X]$ for the latter algebra. Let $p\in\mathfrak{A}[X]$, say $p = a_0+a_1X+\ldots+a_nX^n$, where $a_0, \ldots, a_n\in\mathfrak{A}$ and $a_n\neq 0$. Then a_n is the *leading coefficient* of p, and the degree ∂p of p is n; p is a *monic polynomial* if a_n is the identity of $\mathfrak{A}$. Two polynomials p and q in $\mathfrak{A}[X]$ are *coprime* if there is no polynomial r in $\mathfrak{A}[X]$ with $\partial r\geqslant 1$ which divides both p and q.

Let $\mathfrak{A}$ be a local algebra with residue field k, and let $p = \sum_{j=0}^{n} a_jX^j\in\mathfrak{A}[X]$. Then

$$\pi(p) = \sum_{j=0}^{n}\pi(a_j)X^j\in k[X].$$

2.2. Definition. A local algebra $\mathfrak{A}$ is *Henselian* if, for each monic polynomial $p\in\mathfrak{A}[X]$ and each factorization $\pi(p) = fg$, where f and g are coprime monic polynomials in $k[X]$, there exist polynomials q and r in $\mathfrak{A}[X]$ such that $p = qr$ and $\pi(q) = f$ and $\pi(r) = g$.

This is equivalent to the definition given in [11, § 30], and is exactly the definition given in [3, III.4, Ex. 3]. Raynaud [12] defines a local algebra $\mathfrak{A}$ to be Henselian if each commutative algebra over $\mathfrak{A}$ which is a finitely generated $\mathfrak{A}$-module is a direct product of $\mathfrak{A}$-algebras which are local algebras; by [12, Proposition 5], this is equivalent to our definition.

2.3. Proposition. *Let $\mathfrak{A}$ be a local algebra. Then $\mathfrak{A}$ is Henselian if and only if each polynomial p in $\mathfrak{A}[X]$ of the form*

$$p = a_0+X+a_2X^2+\ldots+a_kX^k, \tag{1}$$

where $a_0\in\operatorname{rad}\mathfrak{A}$, has a root in $\operatorname{rad}\mathfrak{A}$.

Proof. Suppose that $\mathfrak{A}$ is Henselian, and that $p\in\mathfrak{A}[X]$ has the form specified in (1). Set

$$q(Y) = a_0^{k-1}a_k+a_0^{k-2}a_{k-1}Y+\ldots+a_0a_2Y^{k-2}+Y^{k-1}+Y^k.$$

Then q is a monic polynomial over $\mathfrak{A}$, and

$$(\pi q)(Y) = Y^{k-1}+Y^k$$

because $a_0\in\operatorname{rad}\mathfrak{A}$. We have $(\pi q)(Y) = Y^{k-1}(1+Y)$, and so, since $\mathfrak{A}$ is Henselian, $q = rs$, where r and s are monic polynomials over $\mathfrak{A}$, $(\pi r)(Y) = Y^{k-1}$ and $(\pi s)(Y) = 1+Y$, say $s(Y) = Y-b_0$, where $\pi(b_0) = -1$. Then $b_0\in\operatorname{Inv}\mathfrak{A}$. Set $a = a_0b_0^{-1}$. Since $a_0\in\operatorname{rad}\mathfrak{A}$, $a\in\operatorname{rad}\mathfrak{A}$, and we have

$$b_0^kp(a) = a_0(b_0^k+b_0^{k-1}+a_0a_2b_0^{k-2}+\ldots+a_0^{k-1}a_k) = a_0q(b_0) = 0.$$

Since b_0 is invertible, it follows that $p(a) = 0$, and so p has a root in $\operatorname{rad}\mathfrak{A}$.

For the converse, we apply Proposition 3, p. 76, of [12]. We have already noted that the definition of a Henselian algebra in [12] coincides with our definition. The cited proposition states that $\mathfrak{A}$ is Henselian if each monic polynomial p in $\mathfrak{A}[X]$ such that p has a simple root α in $k[X]$ has a root a in $\mathfrak{A}$ with $\pi(a) = \alpha$. Let p be such a polynomial, and let $q(X) = p(X+\alpha)$. Then X is a factor of $\pi(q)$ and X^2 is not a factor of $\pi(q)$, and so q has the form $q = a_0+(\beta+a_1)X+\ldots$, where $a_0, a_1\in\operatorname{rad}\mathfrak{A}$ and $\beta\neq 0$. Since $\beta+a_1\in\operatorname{Inv}\mathfrak{A}$, the hypothesis shows that q has a root, say b, in $\operatorname{rad}\mathfrak{A}$, and $a = b-\alpha$ is a root of p with $\pi(a) = \alpha$. ■

The non-trivial part of Proposition 2.3 can also be deduced from results in [11]. In [11, § 43], the Henselization $\mathfrak{A}^*$ of a local algebra $\mathfrak{A}$ is defined. By (43.3), $\mathfrak{A}^*$ is Henselian and $\mathfrak{A}^*$ is unramified over $\mathfrak{A}$, which implies that the natural homomorphism from $\mathfrak{A}$ into $\mathfrak{A}^*$ is injective; we regard $\mathfrak{A}$ as a subalgebra of $\mathfrak{A}^*$. Let $b\in\mathfrak{A}^*$. Then, by (43.9), there is an element a of the maximal ideal $M_{\mathfrak{A}^*}$ of $\mathfrak{A}^*$ and a monic polynomial

$$p = c_0+c_1X+\ldots+c_{k-1}X^{k-1}+X^k$$

in $\mathfrak{A}[X]$ such that $c_0\in M_{\mathfrak{A}}$, $c_1\in\operatorname{Inv}\mathfrak{A}$ and $p(a) = 0$, and such that the algebra $\mathfrak{A}'$ which is the localization of the algebra $\mathfrak{A}[a]$ at the prime ideal $M_{\mathfrak{A}}+a\mathfrak{A}[a]$ contains b. Now, by the hypothesis in 2.3, the polynomial p has a root c in rad $\mathfrak{A}$. But p has at most one root in $M_{\mathfrak{A}^*}$, and so $c = a$. Thus $a \in M_{\mathfrak{A}}$, $M_{\mathfrak{A}} + a\mathfrak{A}[a] = M_{\mathfrak{A}}$, and the localization $\mathfrak{A}'$ is just $\mathfrak{A}[a]$. This shows that $b \in \mathfrak{A}[a] \subset \mathfrak{A}$, and hence that $\mathfrak{A}^* = \mathfrak{A}$. Thus $\mathfrak{A}$ is Henselian.

I find it surprising that the condition of Proposition 2.3 does not seem to be explicitly stated in the standard algebra texts for an arbitrary local algebra; a self-contained, elementary proof would be of interest.

To establish then that an algebra is Henselian, it suffices to verify the condition of 2.3. However, we now wish to introduce a formally stronger condition, and to show that, to prove that an algebra is Henselian, it suffices to verify this stronger condition. In this way, we avoid reliance on the unproved results.

Let $\mathfrak{A}$ be a commutative unital algebra with identity e. The algebra of $n \times n$-matrices with coefficients in $\mathfrak{A}$ is $\mathfrak{M}_n(\mathfrak{A})$, and the identity matrix in $\mathfrak{M}_n(\mathfrak{A})$ is I_n. The determinant of an element A of $\mathfrak{M}_n(\mathfrak{A})$ is $\det A$: we have $A \in \operatorname{Inv} \mathfrak{M}_n(\mathfrak{A})$ if and only if $\det A \in \operatorname{Inv} \mathfrak{A}$.

Let $p_1, \ldots, p_n \in \mathfrak{A}[X_1, \ldots, X_n] = \mathfrak{A}[X]$, and set $\boldsymbol{p} = (p_1, \ldots, p_n)$; we regard $\boldsymbol{p}$ as a map from $\mathfrak{A}^{(n)}$ into $\mathfrak{A}^{(n)}$. The *Jacobian matrix* of $\boldsymbol{p}$ at the element $\boldsymbol{a}$ of $\mathfrak{A}^{(n)}$ is

$$\boldsymbol{p}'(\boldsymbol{a}) = \left(\frac{\partial p_i}{\partial X_j}(\boldsymbol{a}):\ i, j = 1, \ldots, n\right) \in \mathfrak{M}_n(\mathfrak{A}).$$

The map $\boldsymbol{p}$ is *non-singular* at $\boldsymbol{a}$ if $\boldsymbol{p}'(\boldsymbol{a}) \in \operatorname{Inv} \mathfrak{M}_n(\mathfrak{A})$.

2.4. Definition. Let $\mathfrak{A}$ be a commutative unital algebra, and let $n \in \mathbf{N}$. Then $\mathfrak{A}$ satisfies *condition* $\mathscr{P}_n$ if, for each $\boldsymbol{p} \in (\mathfrak{A}[X])^{(n)}$ which is non-singular at $\boldsymbol{0}$ and which is such that $\boldsymbol{p}(\boldsymbol{0}) \in (\operatorname{rad} \mathfrak{A})^{(n)}$, there exists $\boldsymbol{x} \in (\operatorname{rad} \mathfrak{A})^{(n)}$ with $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{0}$.

Clearly, when considering the existence or uniqueness of solutions of the equation $\boldsymbol{p}(X) = \boldsymbol{0}$, where $\boldsymbol{p}$ is as above, it is sufficient to suppose, further, that $\boldsymbol{p}'(\boldsymbol{0}) = I_n$.

Proposition 2.3 asserts that a local algebra is Henselian if and only if it satisfies $\mathscr{P}_1$.

2.5. Proposition. *Let $\boldsymbol{p} \in (\mathfrak{A}[X])^{(n)}$ be as specified in 2.4. Then there is at most one element $\boldsymbol{x} \in (\operatorname{rad} \mathfrak{A})^{(n)}$ with $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{0}$.*

Proof. We suppose that $\boldsymbol{p}'(\boldsymbol{0}) = I_n$. Set $\boldsymbol{p} = (p_1, \ldots, p_n)$, and set $p_j(X) = a_j + X_j + q_j(X)$, so that $a_j \in \operatorname{rad} \mathfrak{A}$ and each non-zero monomial in each q_j has degree at least two.

Suppose that $\boldsymbol{x}, \boldsymbol{y} \in (\operatorname{rad} \mathfrak{A})^{(n)}$ with $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{y}) = \boldsymbol{0}$, and set $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$. Then

$$(2) \qquad z_j + q_j(\boldsymbol{x}) - q_j(\boldsymbol{y}) = 0 \quad (j = 1, \ldots, n).$$

Now $q_j(\boldsymbol{x}) - q_j(\boldsymbol{y}) \in \sum_{i=1}^{n} z_i \mathfrak{A}$ $(j = 1, \ldots, n)$, and so equations (2) can be written in the form $T\boldsymbol{z} = \boldsymbol{0}$, where $T \in \mathfrak{M}_n(\mathfrak{A})$ and

$$\det T \in e + \operatorname{rad} \mathfrak{A} \subset \operatorname{Inv} \mathfrak{A}.$$

It follows that $\boldsymbol{z} = \boldsymbol{0}$, and hence that $\boldsymbol{y} = \boldsymbol{x}$. ■

The following result is the implication (d′)$\Rightarrow$(e) in [10, Chapter 1, Theorem 4.2]; we sketch the proof.

2.6. Proposition. *Let $\mathfrak{A}$ be a local algebra which satisfies $\mathscr{P}_n$ for each n. Then $\mathfrak{A}$ is Henselian.*

Proof. Let the residue field of $\mathfrak{A}$ be k. Let $p \in \mathfrak{A}[X]$ be a monic polynomial, and let $f, g \in k[X]$ be coprime, monic polynomials such that $\pi(p) = fg$, say

$$p(X) = (\alpha_1 + a_1) + \ldots + (\alpha_{m+n} + a_{m+n}) X^{m+n-1} + X^{m+n},$$

$$f(X) = \beta_1 + \ldots + \beta_m X^{m-1} + X^m,$$

$$g(X) = \gamma_1 + \ldots + \gamma_n X^{n-1} + X^n,$$

where $\alpha_1, \ldots, \alpha_{m+n}, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n \in k$ and $a_1, \ldots, a_{m+n} \in M_{\mathfrak{A}}$. We must find $b_1, \ldots, b_m, c_1, \ldots, c_n \in M_{\mathfrak{A}}$ such that $p = qr$, where

$$q(X) = (\beta_1 + b_1) + \ldots + (\beta_m + b_m) X^{m-1} + X^m,$$

$$r(X) = (\gamma_1 + c_1) + \ldots + (\gamma_n + c_n) X^{n-1} + X^n.$$

Thus we must find a solution in $M_{\mathfrak{A}}^{(m+n)}$ of the equations

$$p_i(X_1, \ldots, X_{m+n}) = 0 \quad (i = 1, \ldots, m+n)$$

given by equating the coefficients of $X^0, \ldots, X^{m+n-1}$ in the equation $p(X) = q(X) r(X)$.

Clearly, $p_i(\boldsymbol{0}) = a_i$ $(i = 1, \ldots, m+n)$, and an easy calculation shows that $\boldsymbol{p}'(\boldsymbol{0}) \in \mathfrak{M}_{m+n}(k)$ and that $\det \boldsymbol{p}'(\boldsymbol{0}) = R(f, g)$, the resultant of f and g. Since f and g are coprime, $R(f, g) \neq 0$, and so $\boldsymbol{p}$ is non-singular at $\boldsymbol{0}$. The existence of the required solution in $M_{\mathfrak{A}}^{(m+n)}$ follows because $\mathfrak{A}$ satisfies $\mathscr{P}_{m+n}$. ■

In fact, each Henselian algebra satisfies $\mathscr{P}_n$ for each n. This is proved in [10, Theorem 4.2], but the proof requires a certain fluency in étale cohomology.

3. We now prove that various local algebras related to Banach algebras satisfy $\mathscr{P}_n$ for each n, and hence that they are Henselian. That local Banach algebras themselves satisfy condition $\mathscr{P}_1$ is exactly Lemma 3.2.8 of [7], and a similar argument shows that they satisfy $\mathscr{P}_n$. We give a different proof which leads more easily to the generalizations that we need to consider.

Let $\mathfrak{A}$ be a commutative Banach algebra. The spectral radius of an element $a \in \mathfrak{A}$ is denoted by $\nu(a)$, so that

$$\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}, \qquad \operatorname{rad}\mathfrak{A} = \{a \in A\colon \nu(a) = 0\} = \bigcap \{\ker\varphi\colon \varphi \in \Phi_{\mathfrak{A}}\}.$$

3.1. Lemma. *Let $\mathfrak{A}$ be a commutative Banach algebra with identity e, let $a_1, \ldots, a_k \in \mathfrak{A}$, and take $\varepsilon > 0$. Then there is an algebra norm $\|\cdot\|$ on $\mathfrak{A}$ which is equivalent to the given norm and such that $\|e\| = 1$ and $\|a_j\| \leqslant \nu(a_j) + \varepsilon$ $(j = 1, \ldots, k)$.*

Proof. Set $b_j = a_j/(\nu(a_j)+\varepsilon)$ $(j = 1, \ldots, k)$. Then $\nu(b_j) < 1$, and so $\{\|b_j^n\|\colon n \in \mathbf{Z}^+\}$ is a bounded set. Thus the semigroup

$$S = \{b_1^{n_1} \ldots b_k^{n_k}\colon n_1, \ldots, n_k \in \mathbf{Z}^+\}$$

is bounded in $\mathfrak{A}$. By [2, 5.1], there is an algebra norm $\|\cdot\|$ on $\mathfrak{A}$, equivalent to the given norm, such that $\|e\| = 1$ and $\|x\| \leqslant 1$ $(x \in S)$. This norm has the required properties. ∎

Let E be a Banach space. Then $E^{(n)}$ is a Banach space with respect to the norm

$$\|\boldsymbol{x}\|_\infty = \max\{\|x_j\|\colon \boldsymbol{x} = (x_1, \ldots, x_n) \in E^{(n)}\}.$$

Let $\mathfrak{A}$ be a commutative Banach algebra, let $p_1, \ldots, p_n \in \mathfrak{A}[X]$, and let $\boldsymbol{p} = (p_1, \ldots, p_n)$. We associate a constant $C(\boldsymbol{p})$ with $\boldsymbol{p}$ as follows. Let S be the set of non-zero elements of $\mathfrak{A}$ which are the coefficients of at least one monomial of degree at least two in at least one of the p_j's, let N be the total number of non-zero coefficients in all of the p_j's, and let M be the maximum degree of any of the monomials appearing in any of the p_j's; we set

$$C(\boldsymbol{p}) = n^2 + nNM\max\{\|a\|+1\colon a \in S\}.$$

(Take $C(\boldsymbol{p}) = n^2$ if $S = \emptyset$.)

Except for the estimate on the size of ε, the following lemma is a consequence of the local inversion theorem for Banach spaces ([4], 10.2.5]).

3.2. Lemma. *Let $\mathfrak{A}$ be a commutative unital Banach algebra, let $q_1, \ldots, q_n \in \mathfrak{A}[X]$ be polynomials in which each non-zero monomial has degree at least two, and let $\boldsymbol{q} = (q_1, \ldots, q_n)$. Take $\varepsilon > 0$ with $4\varepsilon C(\boldsymbol{q}) < 1$. Then, for each $\boldsymbol{a} \in \mathfrak{A}^{(n)}$ with $\|\boldsymbol{a}\|_\infty < \varepsilon$, there exists $\boldsymbol{x} \in \mathfrak{A}^{(n)}$ with*

$$(3) \qquad a_j + x_j + q_j(\boldsymbol{x}) = 0 \qquad (j = 1, \ldots, n),$$

and $\boldsymbol{x} = T\boldsymbol{a}$ for some $T \in \operatorname{Inv}\mathfrak{M}_n(\mathfrak{A})$.

Proof. Let $C = C(\boldsymbol{q})$. Take $\boldsymbol{a} \in \mathfrak{A}^{(n)}$ with $\|\boldsymbol{a}\|_\infty < \varepsilon$, and set

$$B = \{\boldsymbol{x} \in \mathfrak{A}^{(n)}\colon \|\boldsymbol{x} - \boldsymbol{a}\|_\infty \leqslant \varepsilon\}.$$

Then B is a closed subset of $\mathfrak{A}^{(n)}$, and so B is a complete metric space for the metric $d\colon (\boldsymbol{x}, \boldsymbol{y}) \mapsto \|\boldsymbol{x} - \boldsymbol{y}\|_\infty$.

For $j = 1, \ldots, n$ and $\boldsymbol{x} \in B$, set

$$f_j(\boldsymbol{x}) = -a_j - q_j(\boldsymbol{x}), \qquad F(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})),$$

so that $F\colon B \to \mathfrak{A}^{(n)}$ is a map. For $\boldsymbol{x} \in B$,

$$(4) \qquad \|x_j\| \leqslant \|a_j\| + \varepsilon < 2\varepsilon < 1$$

and

$$\|f_j(\boldsymbol{x}) + a_j\| = \|q_j(\boldsymbol{x})\| \leqslant 4C\varepsilon^2 < \varepsilon,$$

and so $F(B) \subset B$. Take $\boldsymbol{x}, \boldsymbol{y} \in B$ with $d(\boldsymbol{x}, \boldsymbol{y}) = \delta$, say. Then $f_j(\boldsymbol{x}) - f_j(\boldsymbol{y}) = q_j(\boldsymbol{y}) - q_j(\boldsymbol{x})$. A typical term in $q_j(\boldsymbol{y}) - q_j(\boldsymbol{x})$ is

$$a(y_1^{k_1} \ldots y_n^{k_n} - x_1^{k_1} \ldots x_n^{k_n}),$$

where $k_1 + \ldots + k_n \geqslant 2$ and $a \in S$. Now

$$y_1^{k_1} \ldots y_n^{k_n} - x_1^{k_1} \ldots x_n^{k_n} = \sum_{r=1}^{n} x_1^{k_1} \ldots x_{r-1}^{k_{r-1}} (y_r^{k_r} - x_r^{k_r}) y_{r+1}^{k_{r+1}} \ldots y_n^{k_n},$$

and, using (4), $\|y_r^{k_r} - x_r^{k_r}\| \leqslant M\delta$ whenever $k_r \geqslant 1$. Thus

$$\|f_j(\boldsymbol{x}) - f_j(\boldsymbol{y})\| \leqslant 2\varepsilon C\delta \leqslant \tfrac{1}{2}\delta,$$

and so $d(F(\boldsymbol{x}), F(\boldsymbol{y})) \leqslant \frac{1}{2} d(\boldsymbol{x}, \boldsymbol{y})$. Hence $F\colon B \to B$ is a contraction mapping.

By the contraction mapping principle, F has a fixed point, say $\boldsymbol{x}$, in B. Clearly $\boldsymbol{x}$ satisfies equations (3).

Since each non-zero term in each q_j has degree at least two, we can write $\boldsymbol{q}(\boldsymbol{x}) = U\boldsymbol{x}$, where $U = (u_{ij}) \in \mathfrak{M}_n(\mathfrak{A})$ and $\|u_{ij}\| < 2\varepsilon$ $(i, j = 1, \ldots, n)$. We have $\boldsymbol{a} = (I_n + U)\boldsymbol{x}$, and so

$$\|\det(I_n + U) - e_{\mathfrak{A}}\| < 2\varepsilon n^2 < 1.$$

Thus $\det(I_n + U) \in \operatorname{Inv}\mathfrak{A}$, and so $I_n + U \in \operatorname{Inv}\mathfrak{M}_n(\mathfrak{A})$. We have $\boldsymbol{x} = (I_n + U)^{-1}\boldsymbol{a}$. ∎

3.3. Lemma. *Let I be a proper ideal in a commutative Banach algebra $\mathfrak{A}$, let J be the closure of I, and let $\pi\colon \mathfrak{A}/I \to \mathfrak{A}/J$ be the quotient map. Then $\pi(\operatorname{rad}(\mathfrak{A}/I)) = \operatorname{rad}(\mathfrak{A}/J)$.*

Proof. Since maximal modular ideals in $\mathfrak{A}$ are closed, each maximal modular ideal which contains I also contains J. ∎

We now prove that various algebras satisfy condition $\mathscr{P}_n$.

3.4. Theorem. *Let $\mathfrak{A}$ be a commutative unital Banach algebra, and let I be a proper ideal in $\mathfrak{A}$. Then $\mathfrak{A}/I$ satisfies condition $\mathscr{P}_n$ for each $n \in \mathbf{N}$.*

Proof. Let $\boldsymbol{p} \in ((\mathfrak{A}/I)[X])^{(n)}$ be such that $\boldsymbol{p}'(0)$ is the identity in $\mathfrak{M}_n(\mathfrak{A}/I)$ and $\boldsymbol{p}(\boldsymbol{0}) \in (\operatorname{rad}(\mathfrak{A}/I))^{(n)}$, say $\boldsymbol{p} = (p_1, \ldots, p_n)$. Then

$$p_j(\boldsymbol{X}) = a_j + X_j + q_j(\boldsymbol{X}) \qquad (j = 1, \ldots, n),$$

where $a_1, \ldots, a_n \in \operatorname{rad}(\mathfrak{A}/I)$ and $q_1, \ldots, q_n \in (\mathfrak{A}/I)[X]$ are polynomials in which each non-zero monomial has degree at least two. Set $\boldsymbol{a} = (a_1, \ldots, a_n)$, and let M, N, and S be associated with $\boldsymbol{p}$ as above.

Let J be the closure of I in $\mathfrak{A}$. Then $\mathfrak{A}/J$ is a commutative Banach algebra with identity $e_{\mathfrak{A}/J}$. Let $\pi_1\colon \mathfrak{A} \to \mathfrak{A}/I$, $\pi_2\colon \mathfrak{A} \to \mathfrak{A}/J$, and $\pi\colon \mathfrak{A}/I \to \mathfrak{A}/J$ be the canonical maps, so that $\pi_2 = \pi \circ \pi_1$. We can also regard these three maps as acting on $\mathfrak{A}[X]$, $\mathfrak{A}^{(n)}$, $\mathfrak{M}_n(\mathfrak{A})$, etc., in the obvious way.

Consider the element $\pi(\boldsymbol{p})$ of $((\mathfrak{A}/J)[X])^{(n)}$. By Lemma 3.3, $(\pi\boldsymbol{p})(0) \in (\operatorname{rad}(\mathfrak{A}/J))^{(n)}$. Let

$$C = n^2 + nNM \max\{\nu(\pi(a)) + 2\colon a \in S\},$$

and take $\varepsilon > 0$ with $4\varepsilon C < 1$. By Lemma 3.1, there is an algebra norm $\|\cdot\|$ on $\mathfrak{A}/J$, equivalent to the given norm, with $\|e_{\mathfrak{A}/J}\| = 1$, with $\|\pi(a)\|_\infty < \varepsilon$, and with $\|\pi(a)\| < \nu(\pi(a)) + 1$ $(a \in S)$. We have $C(\boldsymbol{p}) < C$, and so it follows from Lemma 3.2 that there exists $z \in (\mathfrak{A}/J)^{(n)}$ with $(\pi\boldsymbol{p})(z) = \boldsymbol{0}$ and with $z \in \pi(\boldsymbol{a}) \operatorname{Inv} \mathfrak{M}_n(\mathfrak{A}/J) \subset (\operatorname{rad}(\mathfrak{A}/J))^{(n)}$.

Choose $y \in \mathfrak{A}^{(n)}$ with $\pi_2(y) = z$, and take $\boldsymbol{P} \in (\mathfrak{A}[X])^{(n)}$ with $\pi_1(\boldsymbol{P}) = \boldsymbol{p}$ and with $\boldsymbol{P}'(0)$ equal to the identity of $\mathfrak{M}_n(\mathfrak{A})$. Then $\boldsymbol{P}(y) \in J^{(n)}$. Also,

$$\det \boldsymbol{P}'(y) \in e_{\mathfrak{A}} + \sum_{j=1}^{n} y_j \mathfrak{A},$$

and so

$$\pi_2(\det \boldsymbol{P}'(y)) \in e_{\mathfrak{A}/J} + \sum_{j=1}^{n} z_j(\mathfrak{A}/J) \subset e_{\mathfrak{A}/J} + \operatorname{rad}(\mathfrak{A}/J) \subset \operatorname{Inv}(\mathfrak{A}/J).$$

Since $\mathfrak{M}_n(\mathfrak{A}/J) \cong \mathfrak{M}_n(\mathfrak{A})/\mathfrak{M}_n(J)$, it follows that there exist $T \in \mathfrak{M}_n(\mathfrak{A})$ and $U = (u_{ij}) \in \mathfrak{M}_n(J)$ such that $\boldsymbol{P}'(y)\, T = I_n + U$.

Let $\boldsymbol{R}(X) = \boldsymbol{P}(T(X) + y)$, so that $\boldsymbol{R} \in (\mathfrak{A}[X])^{(n)}$. Then $\boldsymbol{R}(0) = \boldsymbol{P}(y) \in J^{(n)}$ and $\boldsymbol{R}'(0) = \boldsymbol{P}'(y)\, T = I_n + U \in \mathfrak{M}_n(\mathfrak{A})$.

Since I is dense in J, there exists $V = (v_{ij}) \in \mathfrak{M}_n(I)$ with

$$n^2 \|u_{ij} + v_{ij}\| < 1 \qquad (i, j = 1, \ldots, n).$$

Then $\|\det(I_n + U + V) - e_{\mathfrak{A}}\| < 1$, and so $I_n + U + V \in \operatorname{Inv} \mathfrak{M}_n(\mathfrak{A})$, say $W = (I_n + U + V)^{-1}$.

Now let

$$\boldsymbol{S}(X) = \boldsymbol{R}(W(X)) + VW(X),$$

so that $\boldsymbol{S} \in (\mathfrak{A}[X])^{(n)}$, $\boldsymbol{S}(0) = \boldsymbol{R}(0) \in J^{(n)}$, and $\boldsymbol{S}'(0) = (\boldsymbol{R}'(0) + V)\, W = I_n$. Let D be the constant associated with $\boldsymbol{S}$, and take $\eta > 0$ with $4\eta D < 1$. Then there exists $\boldsymbol{b} \in I^{(n)}$ with $\|\boldsymbol{S}(0) - \boldsymbol{b}\|_\infty < \eta$. By Lemma 3.2 again, there exists $c \in \mathfrak{A}^{(n)}$ with $\boldsymbol{S}(c) = \boldsymbol{b}$ and $c \in \boldsymbol{S}(0) \operatorname{Inv} \mathfrak{M}_n(\mathfrak{A}) \subset J^{(n)}$.

We have

$$\boldsymbol{P}(TW(c) + y) = \boldsymbol{R}(W(c)) = \boldsymbol{S}(c) - VW(c) = \boldsymbol{b} - VW(c).$$

Since $\boldsymbol{b} \in I^{(n)}$ and $V \in \mathfrak{M}_n(I)$, $\boldsymbol{P}(TW(c) + y) \in I^{(n)}$.

Finally, set $x = \pi_1(y + TW(c))$. Then $\boldsymbol{p}(x) = \boldsymbol{0}$ in $\mathfrak{A}/I$. Since $c \in J^{(n)}$, $\pi(x) = \pi_2(y) = z \in (\operatorname{rad}(\mathfrak{A}/J))^{(n)}$, and so, by 3.3, $x \in (\operatorname{rad}(\mathfrak{A}/I))^{(n)}$.

Thus $\mathfrak{A}/I$ satisfies condition $\mathscr{P}_n$. ∎

The natural proof that $\mathfrak{A}/I$ satisfies condition $\mathscr{P}_1$, the condition that arises in Proposition 2.3, is essentially the same as the above, but of course it is notationally a little cleaner.

A more general class of algebras than that of Banach algebras is the class of LMC algebras. The theory of these algebras is given in [9] and [14]. Briefly, it is as follows. Let $\mathfrak{A}$ be an algebra which is also a topological linear space. Then $\mathfrak{A}$ is an *LMC algebra* if the topology is given by a family of algebra seminorms, say $\{\|\cdot\|_\nu\}$. Let $\mathfrak{A}_\nu$ be the Banach algebra which is the completion of the normed algebra $\mathfrak{A}/(\ker\|\cdot\|_\nu)$. The family $\{\|\cdot\|_\nu\}$ can be taken to be a directed set, and there is a family $\{\pi_{\mu\nu}\colon \mu \leqslant \nu\}$ of norm-decreasing homomorphisms, $\pi_{\mu\nu}\colon \mathfrak{A}_\nu \to \mathfrak{A}_\mu$, such that $\{\mathfrak{A}_\nu;\ \pi_{\mu\nu}\}$ is a projective system. It is standard that each complete LMC algebra $\mathfrak{A}$ is algebraically and topologically isomorphic to the projective limit of the system $\{\mathfrak{A}_\nu;\ \pi_{\mu\nu}\}$. Thus a complete LMC algebra can be identified with

$$\lim\operatorname{proj}\{\mathfrak{A}_\nu;\ \pi_{\mu\nu}\} \equiv \{(a_\nu) \in \prod \mathfrak{A}_\nu\colon \pi_{\mu\nu}(a_\nu) = a_\mu\ (\mu \leqslant \nu)\}.$$

We write $\pi_\nu\colon a \mapsto a_\nu$, $\prod \mathfrak{A}_\mu \to \mathfrak{A}_\nu$, for the natural projection.

A complete LMC algebra is a *Fréchet algebra* if its topology is metrizable. This condition is equivalent to the requirement that the topology be given by a countable family of algebra seminorms.

Let $\mathfrak{A}$ be a complete, commutative LMC algebra, say $\mathfrak{A} = \lim\operatorname{proj}\{\mathfrak{A}_\nu;\ \pi_{\mu\nu}\}$, and let $a = (a_\nu) \in \mathfrak{A}$. By [7, 5.2], $a \in \operatorname{Inv}\mathfrak{A}$ if and only if $a_\nu \in \operatorname{Inv}\mathfrak{A}_\nu$ for each ν, and, by [7, 7.3], $a \in \operatorname{rad}\mathfrak{A}$ if and only if $a_\nu \in \operatorname{rad}\mathfrak{A}_\nu$ for each ν.

3.5. THEOREM. *Let $\mathfrak{A}$ be a complete, commutative, unital LMC algebra. Then $\mathfrak{A}$ satisfies condition $\mathscr{P}_n$ for each $n \in \mathbf{N}$.*

Proof. Let $\boldsymbol{p}_\nu \in (\mathfrak{A}_\nu[X])^{(n)}$ be obtained by applying π_ν to each coefficient of each component of $\boldsymbol{p}$. Then $\boldsymbol{p}_\nu(0) \in (\operatorname{rad}\mathfrak{A}_\nu)^{(n)}$ and $\det \boldsymbol{p}_\nu'(0) \in \operatorname{Inv}\mathfrak{A}_\nu$, so that $\boldsymbol{p}_\nu'(0)$ is non-singular at $\boldsymbol{0}$.

By 3.4, there exists $x_\nu \in (\operatorname{rad}\mathfrak{A}_\nu)^{(n)}$ with $\boldsymbol{p}_\nu(x_\nu) = \boldsymbol{0}$, and, by 2.5, x_ν is uniquely specified by these conditions.

Take μ, ν with $\mu \leqslant \nu$. Then $\pi_{\mu\nu}(x_\nu) \in (\operatorname{rad}\mathfrak{A}_\nu)^{(n)}$ and $\boldsymbol{p}_\mu(\pi_{\mu\nu}(x_\nu)) = \boldsymbol{0}$, and so, by the uniqueness of x_μ, $\pi_{\mu\nu}(x_\nu) = x_\mu$. Thus $x = (x_\nu) \in (\operatorname{rad}\mathfrak{A})^{(n)}$, and $\boldsymbol{p}(x) = \boldsymbol{0}$. ∎

A second generalization of the class of Banach algebras is the class of pseudo-Banach algebras, which was introduced in [1]. In [1], pseudo-Banach algebras were defined in terms of certain "bound structures", but it is now more convenient to give an equivalent definition in terms of inductive limits: a unital algebra $\mathfrak{A}$ is a *pseudo-Banach algebra* if $\mathfrak{A}$ is isomorphic to the inductive limit of a system $\{\mathfrak{A}_\mu;\ \pi_{\nu\mu}\}$ of unital Banach algebras $\mathfrak{A}_\mu$ and of continuous unital monomorphisms $\pi_{\nu\mu}$: $\mathfrak{A}_\mu \to \mathfrak{A}_\nu$ $(\mu \leqslant \nu)$. (The inductive limit, $\operatorname{lim\,ind}\{\mathfrak{A}_\nu;\ \pi_{\nu\mu}\}$, is the algebra $\bigcup \mathfrak{A}_\mu/\sim$, where, for $a \in \mathfrak{A}_\mu$ and $b \in \mathfrak{A}_\nu$, $a \sim b$ if there exists λ with $\lambda \geqslant \mu$, $\lambda \geqslant \nu$, and $\pi_{\lambda\mu}(a) = \pi_{\lambda\nu}(b)$.)

3.6. THEOREM. *Let $\mathfrak{A}$ be a commutative pseudo-Banach algebra. Then $\mathfrak{A}$ satisfies condition $\mathscr{P}_n$ for each $n \in N$.*

Proof. Let $\mathfrak{A} = \operatorname{lim\,ind}\{\mathfrak{A}_\nu;\ \pi_{\nu\mu}\}$, and take p as in 2.4. Choose μ so that all the coefficients of all the components of p have a representative in $\mathfrak{A}_\mu$, and regard p as a member of $(\mathfrak{A}_\mu[X])^{(n)}$. Let M, N, and S be associated with p as before, set

$$C = n^2 + nNM \max\{\nu(a)+2\colon\ a \in S\},$$

and take $\varepsilon > 0$ so that $4\varepsilon C < 1$.

By [1, 1.6], $\operatorname{rad}\mathfrak{A} = \{a \in \mathfrak{A}\colon\ \varphi(a) = 0\ (\varphi \in \Phi_{\mathfrak{A}})\}$, and so, by [1, 3.2(ii)], for each $a \in \operatorname{rad}\mathfrak{A}$ and each $\eta > 0$, there exists λ such that $\nu_\lambda(a) < \eta$, where ν_λ denotes the spectral radius of a in $\mathfrak{A}_\lambda$.

Let $p(0) = (a_1, \ldots, a_n)$. By increasing μ if necessary, we can suppose that $\nu_\mu(a_j) < \varepsilon$ $(j = 1, \ldots, n)$. This change of μ does not increase C. By 3.1, there is an algebra norm on $\mathfrak{A}_\mu$, equivalent to the given norm, with $\|p(0)\|_\infty < \varepsilon$ and $\|a\| < \nu(a)+1$ $(a \in S)$. By 3.2, there exists $x \in \mathfrak{A}_\mu^{(n)}$ with $p(x) = 0$, and so $\mathfrak{A}$ satisfies $\mathscr{P}_n$. ∎

The main result now follows from 2.6 and either 3.4 or 3.5 or 3.6.

3.7. THEOREM. *Let $\mathfrak{A}$ be a local algebra which is either a homomorphic image of a Banach algebra or a complete LMC algebra or a pseudo-Banach algebra. Then $\mathfrak{A}$ is Henselian.* ∎

4. In this final section, we give some examples of algebras which can be shown to be Henselian by an application of Theorem 3.7.

Let $\mathfrak{A}$ be a local algebra with maximal ideal M. The set $\{a + M^n\colon\ a \in \mathfrak{A},\ n \in N\}$ is a base for the open sets of a topology on $\mathfrak{A}$; this topology is the *M-adic*, or *Krull, topology*. The M-adic topology is Hausdorff if and only if $\bigcap\{M^n\colon\ n \in N\} = \{0\}$. In this case, set

$$v(a) = \max\{n \in N\colon\ a \in M^n\} \qquad (a \in M \setminus \{0\}),$$

and, for $a, b \in M$, set $d(a, b) = e^{-v(a-b)}$ (with $d(a, b) = 0$ if $a = b$). Then d is a metric on M defining the M-adic topology. The algebra is a *complete local algebra* if the M-adic topology is Hausdorff and if the corresponding metric space is complete.

Nagata's theorem ([11, (30.4)]) asserts that a complete local algebra is

Henselian. The M-adic topology is necessarily Hausdorff if $\mathfrak{A}$ is a Noetherian algebra, and so this result subsumes the version of Hensel's lemma given by Zariski and Samuel ([13, VIII, Theorem 17]).

The M-adic topology in a local Banach algebra is certainly not Hausdorff in general. For example, let

$$R = \{f\colon\ \|f\| = \int_0^\infty |f(t)|\, e^{-t^2}\, dt < \infty\}.$$

Then R is a radical Banach algebra with respect to convolution multiplication given by

$$(f * g)(t) = \int_0^t f(t-s)\, g(s)\, ds \qquad (t \geqslant 0).$$

By Titchmarsh's convolution theorem, R is an integral domain. Let χ_n be the characteristic function of the interval $[0, 1/n]$. Then the sequence $(n\chi_n\colon\ n \in N)$ is a bounded approximate identity for R, and so, by Cohen's factorization theorem ([2, 11.12]), $R^2 = R$. Let $A = R^\#$, the algebra formed by adjoining an identity to R. Then A is a local Banach algebra, $M_A = R$, and $\bigcap\{M_A^n\colon\ n \in N\} = R$.

Let Ω be a compact Hausdorff space, and let P be a prime ideal in $C(\Omega)$, the Banach algebra of all continuous complex-valued functions on Ω. Then P is contained in a unique maximal ideal of $C(\Omega)$. Let $A = C(\Omega)/P$. Then A is a local algebra which is an integral domain, and $M_A^2 = M_A$. Theorem 3.7 shows that A is Henselian.

The form of Hensel's lemma given by Bourbaki ([3, III.4.3]) applies to "linearly topologized topological rings", and no Banach algebra other than C belongs to this class.

The algebras $C[[X_1, \ldots, X_n]]$ of formal power series in n indeterminates are the standard examples of complete local algebras, and so they are Henselian algebras. These algebras are also Fréchet algebras with respect to the *simple topology* of coordinatewise convergence. Specifically, for $a = \sum a_{j_1,\ldots,j_n} X_1^{j_1} \ldots X_n^{j_n}$ in $C[[X_1, \ldots, X_n]]$, set

$$\|a\|_k = \sum\{|a_{j_1,\ldots,j_n}|\colon\ j_1 + \ldots + j_n \leqslant k\} \qquad (k \in N).$$

Then $(\|\cdot\|_k)$ is a sequence of algebra seminorms defining a complete metric on $C[[X_1, \ldots, X_n]]$, and this metric induces the simple topology. Thus, Theorem 3.7 also implies that $C[[X_1, \ldots, X_n]]$ is a Henselian algebra. The simple topology is strictly weaker than the M-adic topology on $C[[X_1, \ldots, X_n]]$.

The subalgebra $C\langle\langle X_1, \ldots, X_n\rangle\rangle$ of $C[[X_1, \ldots, X_n]]$ consisting of the formal power series which are absolutely convergent on some neighbourhood of the origin in C^n is a pseudo-Banach algebra. For let Δ_j be the polydisc with centre 0 and polyradius $1/j$ in C^n, let $H^\infty(\Delta_j)$ be the uniform algebra of bounded analytic functions on Δ_j, and let $r_{jk}\colon H^\infty(\Delta_j) \to H^\infty(\Delta_k)$ $(k \geqslant j)$ be the restriction maps. Then $C\langle\langle X_1, \ldots, X_n\rangle\rangle$ is the inductive limit of the system $\{H^\infty(\Delta_j);\ r_{jk}\}$. Thus $C\langle\langle X_1, \ldots, X_n\rangle\rangle$ is a Henselian algebra. The classical proof of this fact ([11, (45.5)]) is an application of the Weierstrass preparation theorem.

Let $C\langle\langle X_1, \ldots, X_n\rangle\rangle$ have the inductive limit topology induced by the natural injections r_j from the algebras $H^\infty(\Delta_j)$: this is the strongest locally convex topology on $C\langle\langle X_1, \ldots, X_n\rangle\rangle$ such that each map r_j is continuous. (It is the *Folgen topologie* of [6].) By [1, Example 4.6], $C\langle\langle X_1, \ldots, X_n\rangle\rangle$ is a complete LMC algebra with respect to this topology – and so this is another reason why $C\langle\langle X_1, \ldots, X_n\rangle\rangle$ is Henselian.

There is a natural generalization of the above example. Let E be a locally convex space, let U be an open subset of E, and let $H^\infty(U)$ be the uniform algebra of bounded analytic functions on U: here, we define "analytic" as in [2, 21.1]. Let $x \in E$. Then $\mathcal{O}_x$, the algebra of germs of analytic functions at x, can be identified with the inductive limit of the system $\{H^\infty(U);\ r_{VU}\}$, where U and V are open neighbourhoods of x in E, and, for $V \subset U$, $r_{VU}\colon H^\infty(U) \to H^\infty(V)$ is the restriction map. Then $\mathcal{O}_x$ is a local pseudo-Banach algebra, and so it is Henselian.

We conclude with two further examples of local Banach algebras. Let ω be a continuous function on R^+ such that $\omega(t) > 0$, $\omega(s+t) \leqslant \omega(s)\omega(t)$ $(s, t \in R^+)$, and $\omega(t)^{1/t} \to 0$ as $t \to \infty$. Set

$$l^1(Q^+, \omega) = \{a = \sum_{t \in Q^+} \alpha_t \delta_t\colon\ \|a\| = \sum |\alpha_t|\,\omega(t) < \infty\},$$

$$l^1(R^+, \omega) = \{a = \sum_{t \in R^+} \alpha_t \delta_t\colon\ \|a\| = \sum |\alpha_t|\,\omega(t) < \infty\}.$$

Here, δ_t is the characteristic function of the singleton $\{t\}$. Then $l^1(Q^+, \omega)$ and $l^1(R^+, \omega)$ are both local Banach algebras with respect to convolution multiplication, and so they are Henselian. This does not seem to be apparent from the classical algebraic theory.

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