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The Pólya characterization of a Gaussian measure on groups

by

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Abstract. Let X be a locally compact Abelian group and $\xi_1, \xi_2, \xi_3, \xi_4$ independent identically distributed random variables with values in X and distribution γ. The paper deals with a complete description of groups X on which the identical distribution of random variables $2\xi_1$ and $\xi_1+\xi_2+\xi_3+\xi_4$ implies that the distribution γ is invariant with respect to a compact subgroup $K \subset X$ such that $2K = K$, and by means of the natural homomorphism $X \to X/K$ induces a Gaussian measure on the factor group X/K.

A characterization theorem for Gaussian distributions on the real line was proved by Pólya in 1923 [8]. The theorem results in the following:

THEOREM A (Pólya [8]). *Let $\xi_1, \xi_2, \xi_3, \xi_4$ be independent identically distributed random variables with distribution γ. If $2\xi_1$ and $\xi_1+\xi_2+\xi_3+\xi_4$ are identically distributed, then γ is a symmetric Gaussian distribution.*

In terms of characteristic functions, the condition of $2\xi_1$ and $\xi_1+\xi_2+\xi_3+\xi_4$ being identically distributed is evidently of the form

$$\hat{\gamma}(2y) = (\hat{\gamma}(y))^4, \tag{1}$$

and Theorem A is equivalent to the statement that the only solutions of (1) in the set of characteristic functions are $\hat{\gamma}(y) = \exp\{-\alpha y^2\}$, $\alpha \geqslant 0$.

The Pólya theorem was the first result in a series of investigations made by J. Marcinkiewicz, Yu. V. Linnik, A. M. Kagan, S. R. Rao, A. A. Zinger and others who studied identically distributed linear statistics of resampling (see [5]). In the list of unsolved problems given in [5, Ch. 2] there is a problem of constructing a theory of equidistribution of forms on algebraic structures. The generalization of Theorem A to groups which is considered in the present paper may be regarded as a step in that direction.

Let X be a locally compact separable Abelian metric group, let $Y = X^*$ be its group of characters and (x, y) the value of the character $y \in Y$ on the element $x \in X$. The convolution of two distributions μ and ν, the characteristic function of a distribution μ and the distribution $\bar{\mu}$ are given by

$$(\mu * \nu)(E) = \int_X \mu(E-x)\, d\nu(x), \quad \hat{\mu}(y) = \int_X (x, y)\, d\mu(x), \quad \bar{\mu}(E) = \mu(-E).$$

Let us denote the degenerate distribution concentrated at a point $x \in X$ by E_x. The convolution $\mu * E_x$ will be called a *shift* of the distribution μ. A distribution μ is said to be *idempotent* if $\mu^{*2} = \mu * E_x$ for some $x \in X$. As is known, a distribution μ is idempotent if and only if it is a shift of the Haar distribution of a compact subgroup of X ([6]). A distribution μ_1 is called a *factor* of a distribution μ if there exists a distribution μ_2 such that $\mu = \mu_1 * \mu_2$. We denote the support of a distribution μ by $\sigma(\mu)$, and the groups of reals, integers and rotations of a circle by $\boldsymbol{R}$, $\boldsymbol{Z}$ and $\boldsymbol{T}$, respectively. In solving the problem, standard facts will be used concerning the structure of locally compact Abelian groups and the Pontryagin duality theory (see [2]).

Definition 1 ([6]. A distribution γ on X is called *Gaussian* if its characteristic function admits the representation

$$\hat{\gamma}(y) = (x, y)\exp\{-\varphi(y)\}, \tag{2}$$

where x is a fixed element of X and $\varphi(y)$ is a continuous nonnegative function on Y which satisfies the equation

$$\varphi(y_1+y_2)+\varphi(y_1-y_2) = 2[\varphi(y_1)+\varphi(y_2)] \tag{3}$$

for any $y_1, y_2 \in Y$.

A Gaussian distribution γ will be called *symmetric* if $x = 0$ in (2). The set of Gaussian distributions on X will be denoted by $\Gamma(X)$ and that of symmetric Gaussian distributions by $\Gamma^s(X)$. (It is evident that if $\mu \in \Gamma^s(X)$, then $\mu = \bar{\mu}$. Conversely, if $\mu = \bar{\mu}$, then clearly $2x = 0$ in the representation (2).) As was proved in [6], the support $\sigma(\gamma)$ of a distribution $\gamma \in \Gamma(X)$ is a coset of some connected subgroup of X.

Definition 2. A distribution γ on X is called *Gaussian in Pólya sense* if there exist independent random variables $\xi_1, \xi_2, \xi_3, \xi_4$ with values in X and with distribution γ such that $2\xi_1$ and $\xi_1+\xi_2+\xi_3+\xi_4$ are identically distributed.

Let us denote by $\Gamma_P(X)$ the set of distributions on X Gaussian in Pólya sense. Then Theorem A implies that $\Gamma^s(\boldsymbol{R}) = \Gamma_P(\boldsymbol{R})$. For an arbitrary group X, as in the case $X = \boldsymbol{R}$, the condition $\gamma \in \Gamma_P(X)$ is equivalent to (1). Note that the inclusion

$$\Gamma^s(X) \subset \Gamma_P(X)$$

follows from (1)–(3). Unlike the case $X = \boldsymbol{R}$, however, in general there may exist non-Gaussian distributions which belong to $\Gamma_P(X)$.

For K a subgroup of X, let $K^\perp = \{y \in Y\colon (x, y) = 1 \text{ for any } x \in K\}$ be its annihilator. If K is compact, the Haar distribution on K will be denoted by m_K. From (1) it is easy to derive necessary and sufficient conditions which must be satisfied by a compact subgroup K in order that $m_K \in \Gamma_P(X)$.

A group G is called a *Corwin group* if the mapping $G \to G$ given by $x \to 2x$ is an epimorphism, i.e. $2G = G$ (see [4, Def. 5.3.6]).

Lemma 1. *Let G be a closed subgroup of X. Then the following statements are equivalent*:

1° $\overline{2G} = G$.

2° *If* $2y \in G^\perp$, *then* $y \in G^\perp$.

Proof. 1° ⇒ 2°. Let $2y \in G^\perp$, i.e. $(x, 2y) = 1$ for all $x \in G$. Then $(2x, y) = 1$ for all $x \in G$. Since $\overline{2G} = G$, we have $(x, y) = 1$ for all $x \in G$, i.e. $y \in G^\perp$.

2° ⇒ 1°. Note that $\overline{2G}$ is a closed subgroup of G and $(\overline{2G})^\perp \supset G^\perp$. Let $y \in (\overline{2G})^\perp$. Then $(2x, y) = 1$ for all $x \in G$. Hence $(x, 2y) = 1$ for all $x \in G$, i.e. $2y \in G^\perp$. It follows from 2° that $y \in G^\perp$, i.e. $(\overline{2G})^\perp \subset G^\perp$. Therefore $(\overline{2G})^\perp = G^\perp$ and $\overline{2G} = G$.

Proposition 1. *Let K be a compact subgroup of X. Then the following statements are equivalent*:

1. $2K = K$.
2. *If* $2y \in K^\perp$, *then* $y \in K^\perp$.
3. $m_K \in \Gamma_P(X)$.

Proof. The equivalence of 1 and 2 follows from Lemma 1 since $\overline{2K} = 2K$.

2 ⇒ 3. Note that $\hat{m}_K(y) = 1$ for $y \in K^\perp$ and $\hat{m}_K(y) = 0$ for $y \notin K^\perp$. Let us now verify that the characteristic function $\hat{m}_K(y)$ satisfies (1). If $y \in K^\perp$, then $2y \in K^\perp$ and $1 = \hat{m}_K(y) = \hat{m}_K(2y)$ and (1) is fulfilled. If $y \notin K^\perp$, then it follows from (2) that $2y \notin K^\perp$. Therefore $0 = \hat{m}_K(y) = \hat{m}_K(2y)$ and (1) is also fulfilled.

3 ⇒ 2. Since $m_K \in \Gamma_P(X)$, the characteristic function $\hat{m}_K(y)$ satisfies (1). If $2y \in K^\perp$, then $\hat{m}_K(2y) = 1$ and it follows from (1) that $\hat{m}_K(y) = 1$, i.e. $y \in K^\perp$.

Denote the set of idempotent distributions belonging to $\Gamma_P(X)$ by $I_P(X)$. It follows from (1) that $\Gamma_P(X)$ is a semigroup with respect to convolution. Hence we always have

$$I_P(X) * \Gamma^s(X) \subset \Gamma_P(X).$$

The main result obtained is a complete description of groups X for which

$$I_P(X) * \Gamma^s(X) = \Gamma_P(X). \tag{4}$$

Theorem 1. *Equality* (4) *is equivalent to the following condition*:

(α) *For any compact Corwin subgroup K of X the factor group X/K contains no subgroup isomorphic to $\boldsymbol{T}$.*

It should be noted that equality (4) signifies that any distribution $\gamma \in \Gamma_P(X)$ is invariant with respect to some compact Corwin subgroup K and induces a Gaussian distribution on the factor group X/K under the natural homomorphism $X \to X/K$.

A number of lemmas are required to prove Theorem 1.

LEMMA 2. *Let X be such that Y is a connected compact group. Then $\gamma = E_0$ if $\gamma \in \Gamma_P(X)$.*

Proof. Let us consider two cases.

1. $Y \not\approx T$. Since Y is a connected compact group not isomorphic to T, there exists a monomorphism p: $R \to Y$ with image dense in Y. Consider the restriction of the characteristic function $\hat{\gamma}(y)$ to $p(R)$. It is evident that $\hat{\gamma}(p(t))$, $t \in R$, is a characteristic function on R which satisfies (1). By Theorem A, $\hat{\gamma}(p(t)) = \exp\{-\alpha t^2\}$, $\alpha \geqslant 0$. Let V be a neighborhood of zero in Y. Since p is a monomorphism and $\overline{p(R)} = Y$, we can choose a sequence $t_n \to \infty$ such that $p(t_n) \in V$ for all n. If $\alpha > 0$, then $\hat{\gamma}(p(t_n)) = \exp\{-\alpha t_n^2\} \to 0$ as $t_n \to \infty$. But this contradicts the continuity of $\hat{\gamma}(y)$ since V is arbitrary. So $\alpha = 0$. Hence $\hat{\gamma}(p(t)) \equiv 1$, $t \in R$, and $\hat{\gamma}(y) \equiv 1$, $y \in Y$, since $p(R)$ is dense in Y, i.e. $\gamma = E_0$.

2. $Y \approx T$. It suffices to prove the lemma for $Y = T$. The elements of T can be written as $\exp\{it\}$, $t \in [0, 2\pi[$. Let $|\hat{\gamma}(y)| < 1$ for some $y \in T$. It follows from (1) that

$$\hat{\gamma}(2^k y) = (\hat{\gamma}(y))^{4^k}.$$

Take a sequence $k_j \to +\infty$ so that $2^{k_j} \to y_0$. Then $\hat{\gamma}(y_0) = 0$, $y_0 = \exp\{it_0\}$ and by (1) we obtain $\hat{\gamma}(\exp\{it_0/2^k\}) = 0$, $k = 1, 2, \ldots$ But this contradicts the continuity of $\hat{\gamma}(y)$ since the sequence $\exp\{it_0/2^k\}$ converges to the zero of the group T. Hence $|\hat{\gamma}(y)| \equiv 1$, $y \in T$. But then $\hat{\gamma}(y)$ is a character of the group T, i.e. $\hat{\gamma}(y) = \exp\{int\}$, $y = \exp\{it\}$, for some fixed $n \in Z$. Then it follows from (1) that $n = 0$, i.e. $\hat{\gamma}(y) \equiv 1$ and $\gamma = E_0$.

LEMMA 3. *Let $\gamma \in \Gamma_P(X)$ and $\hat{\gamma}(y) = 1$ only for $y = 0$. If Y_1 is a subgroup in Y on which $|\hat{\gamma}(y)| \equiv 1$, then either $Y_1 = \{0\}$ or $Y_1 \approx Z_2$ (Z_2 being the group of residue classes modulo 2).*

Proof. Since $|\hat{\gamma}(y)| \equiv 1$ on Y_1, $\hat{\gamma}(y) = (x, y)$ on Y_1 where $x \in Y_1^*$. Substitution of this expression in (1) gives $2x = 0$. Consider the homomorphism p: $Y_1 \to Z_2 \subset T$ given by $p(y) = (x, y)$. By assumption, p is a monomorphism. Thus Y_1 is isomorphic to a subgroup of Z_2, i.e. either $Y_1 = \{0\}$ or $Y_1 \approx Z_2$.

By the structure theorem for locally compact Abelian groups, the group X is isomorphic to a group of the type $R^n + G$ where $n \geqslant 0$ and the group G contains a compact open subgroup K. The zero component of X will be denoted by C_X.

PROPOSITION 2. *Let $X = R^n + G$, where the group G contains a compact open subgroup, and $\gamma \in \Gamma_P(X)$. Then there exists an element $x \in X$, $2x = 0$, such that $\sigma(\gamma * E_x) \subset R^n + K$, where K is a compact Corwin group.*

Proof. Let $E = \{y \in Y: \hat{\gamma}(y) = 1\}$. Then $\sigma(\gamma) \subset E^\perp$, and γ can be considered as a distribution on $E^\perp$, i.e. one can assume that X itself is such that $\hat{\gamma}(y) = 1$ for $y = 0$ only. It follows from the form of X that $Y = R^n + H$ where $H \approx G^*$. Since, by Lemma 2, $\hat{\gamma}(y) = 1$ on C_H, we have $C_H = \{0\}$, i.e. H is totally disconnected. We shall first prove that H is discrete, i.e. G is compact.

Consider any compact open subgroup L of H. Let V be a neighborhood of zero in L such that $|\hat{\gamma}(y)| > 0$ for all $y \in V$. As is known (see [2, Th. (24.7)]), for any neighborhood V of zero in L there exists a compact subgroup $M \subset V$ such that the factor group $L/M \approx T^l + F$, where $l \geqslant 0$ and F is a finite group. Consider the restriction of $\hat{\gamma}(y)$ to M. Suppose that there exists an element $y_0 \in M$ such that $0 < |\hat{\gamma}(y_0)| < 1$. The sequence $\{2^n y_0\}$ has a limit point $2^{n_j} y_0 \to y_1 \in M$. Since it follows from (1) that

$$\hat{\gamma}(2^n y) = (\hat{\gamma}(y))^{4^n},$$

we obtain $\hat{\gamma}(y_1) = 0$, which is impossible since $y_1 \in M \subset V$. Hence $|\hat{\gamma}(y)| \equiv 1$ on M. By Lemma 3, either $M = \{0\}$ or $M \approx Z_2$. Since H is totally disconnected, L is also totally disconnected, and therefore it follows from the isomorphism $L/M \approx T^l + F$ that $l = 0$. If we now take into account a possible form of M, we can conclude that L is a discrete group. Hence H is also discrete and so G is compact.

There are the following possibilities for the group H:

1. *H contains no elements of order two.* In this case the annihilator $G^\perp = R^n$, $G^\perp \subset Y$ and satisfies condition 2° of Lemma 1. Hence $\overline{2G} = G$ and since G is compact, $\overline{2G} = 2G = G$, i.e. G is a Corwin group.

2. *H contains an element ζ of order two.* In this case $\hat{\gamma}(\zeta) = -1$. Note that if $2y = 0$, then, as follows from (1), $|\hat{\gamma}(y)| = 1$. Therefore, by Lemma 3, H can contain only one element of order two.

We prove that $\zeta \notin 2H$. Indeed, if $\zeta = 2h$, $h \in H$, then it follows from (1) that $|\hat{\gamma}(y)| \equiv 1$ on the subgroup generated by h, which is impossible by Lemma 3.

There exists an element $\alpha \in (2H)^\perp$, $(2H)^\perp \subset G$, $(\alpha, \zeta) \neq 1$, i.e. $(\alpha, \zeta) = -1$. Note also that $2\alpha = 0$ since $\alpha \in (2H)^\perp \subset G$. Denote by S the subgroup of Y generated by ζ. It is easily seen that $S^\perp = R^n + G_1$, where G_1 is a compact subgroup of G. Since $\zeta \notin 2H$, we have $\zeta \notin 2Y$. Since there is only one element of order two in H, there is only one element of order two in Y too. Thus the subgroup $S \subset Y$ satisfies condition 2° of Lemma 1. Hence $\overline{2S^\perp} = S^\perp$, but in our case $\overline{2S^\perp} = 2S^\perp$. So G_1 is a compact Corwin group.

Consider now the distribution $\gamma_1 = \gamma * E_\alpha \in \Gamma_P(X)$. Since $\hat{\gamma}_1(y) = 1$ for $y = 0$ and $y = \zeta$, we have $\sigma(\gamma_1) \subset S^\perp = R^n + G_1$.

LEMMA 4. *Let $X = R^n + G$, where G is a compact Corwin group. Then condition (α) is equivalent to $2Y = Y$.*

Proof. $(\alpha) \Rightarrow 2Y = Y$. Let $H = G^*$. Since G is a compact Corwin group, the group H is discrete and, by Lemma 1, contains no elements of order two and hence of any even order. Any element of an odd order in H lies in $2H$.

Let us verify that any element $h_0 \in H$ of infinite order lies in $2H$. Assume the contrary and consider the subgroup $M = \{kh_0\}_{k=-\infty}^{\infty}$ generated by h_0. It is obvious that $h \in M$ if $2h \in M$. Thus by Lemma 1, $\overline{2M^\perp} = M^\perp$. But $M^\perp = \boldsymbol{R}^n + G_1$, where G_1 is a compact subgroup in G. Hence $\overline{2M^\perp} = 2M^\perp$ and therefore $M^\perp$ is a Corwin group and G_1 is a compact Corwin group. It is evident that $X/G_1 \approx \boldsymbol{R}^n + \boldsymbol{T}$, which contradicts condition (α). Hence $2H = H$ and $2Y = Y$.

$2Y = Y \Rightarrow (\alpha)$. We can prove even more: Let X be an arbitrary group. Then $\overline{2Y} = Y \Rightarrow (\alpha)$.

Assume the contrary, i.e. the factor group X/K where K is a compact Corwin group, contains a subgroup $\tilde{\boldsymbol{T}}$ isomorphic to $\boldsymbol{T}$. Then the subgroup $\tilde{\boldsymbol{T}}$ is a direct summand in X/K, and hence the group $K^\perp \approx (X/K)^*$ contains, as a direct summand, a subgroup $\tilde{\boldsymbol{Z}}$ isomorphic to $\boldsymbol{Z}$. Let us denote the elements of $K^\perp = H + \tilde{\boldsymbol{Z}}$ by (h, n). If $2y \in K^\perp$, then, by Lemma 1, $y \in K^\perp$. Let $2y = (h, n)$. Then $y = (h_1, n/2)$. So if $n \notin 2\tilde{\boldsymbol{Z}}$, then $(h, n) \notin 2Y$. In particular, $(0, n) \notin 2Y$ and since the subgroup $K^\perp$ is open, it is obvious that $(0, n) \notin \overline{2Y}$. So $\overline{2Y} \neq Y$, contrary to the assumption.

Remark 1. It follows from the proof of Lemma 4 that the condition $\overline{2Y} = Y$ is sufficient for (α) and hence, by Theorem 1, for equality (4).

Remark 2. If X and Y are Corwin groups, then the mappings $x \to 2x$ and $y \to 2y$ are isomorphisms. Hence both X and Y are groups with unique division by two.

In subsequent considerations we need some results on infinitely divisible distributions on groups.

A distribution μ on X is said to be *infinitely divisible* if for any natural number n there exist an element $x_n \in X$ and a distribution ν_n such that $\mu = \nu_n^{*n} * E_{x_n}$. As was proved in [6], [7], the characteristic function of an infinitely divisible distribution μ may be written in the form

(5) $$\hat{\mu}(y) = (x_0, y)\hat{\lambda}(y)\exp\{\int_X [(x, y) - 1 - ig(x, y)]\, dF(x) - \varphi(y)\},$$

where $x_0 \in X$, λ is the Haar distribution of a compact subgroup of X and $g(x, y)$ is a function on $X \times Y$ (independent of μ) with the following properties:

1) $g(x, y)$ is continuous as a function of (x, y).

2) $\sup_{x \in X} \sup_{y \in L} |g(x, y)| < \infty$ for each compact subset $L \subset Y$.

3) $g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2)$, $g(-x, y) = -g(x, y)$ for all $x \in X$, $y_1, y_2 \in Y$.

4) For any compact subset $L \subset Y$ there exists a neighborhood U_L of zero in X such that $(x, y) = \exp\{ig(x, y)\}$ for all $x \in U_L$, $y \in L$.

5) For any compact subset $L \subset Y$, $g(x, y)$ tends to zero uniformly in $y \in L$ as x tends to zero in X.

Moreover, F is a Borel measure on X which is finite on the complement of each neighborhood of zero and satisfies, for all $y \in Y$,

$$\int_X [1 - \mathrm{Re}(x, y)]\, dF(x) < \infty,$$

and $\varphi(y)$ is a function as in (2).

Lemma 5 (see [6], [9]). *Let $\{\mu_{nj}\}$, $j = 1, \ldots, j_n$, $n = 1, 2, \ldots$, be a triangular sequence of infinitesimal distributions, i.e.*

$$\lim_{n\to\infty} \max_{1 \leq j \leq j_n} |\hat{\mu}_{nj}(y) - 1| = 0$$

for any compact $L \subset Y$. Let

$$\mu_n = \mathop{*}_{j=1}^{j_n} \mu_{nj} \quad \text{and} \quad \mu_n \to \mu$$

(in the weak topology). Then μ is an infinitely divisible distribution.

It was proved in [6], [9] that (1) if μ is an infinitely divisible distribution on X and $\hat{\mu}(y_0) = 0$ for some $y_0 \in Y$, then μ has an idempotent factor; (2) the set $\{y \in Y: \hat{\mu}(y) \neq 0\}$ is a subgroup of Y. We use the scheme of proof of these statements to prove Lemmas 6 and 7 below.

Lemma 6. *Let X and Y be Corwin groups and $\gamma \in \Gamma_P(X)$. Then if $\hat{\gamma}(y_0) = 0$ for some $y_0 \in Y$, then the distribution γ has an idempotent factor.*

Proof. By Remark 2, the continuous mapping $y \to 2y$ is an isomorphism of the group Y. Therefore for any natural n the function $\hat{\gamma}(y/2^n)$ is defined, and by the Bochner–Khinchin theorem it is the characteristic function of a distribution ν_n on X. It follows from (1) that

(6) $$\hat{\gamma}(y) = (\hat{\gamma}(y/2^n))^{4^n}.$$

Hence $\gamma = \nu_n^{*4^n}$.

As is known [6], any sequence of factors of a given distribution μ is shift-compact, i.e. it contains a subsequence which is convergent after suitable shifts. Since the ν_n are factors of γ, let ν be any limit of shifts of ν_n. It is evident that any power of ν is again a factor of γ and hence the sequence $\{\nu^{*n}\}$ is also shift-compact and any limit of shifts of ν^{*n} is the desired nontrivial idempotent factor λ.

Lemma 7. *Let X and Y be Corwin groups and $\gamma \in \Gamma_P(X)$. Then the set $E = \{y \in Y: \hat{\gamma}(y) \neq 0\}$ is an open subgroup of Y.*

Proof. Denote by H the open subgroup in Y generated by E and consider the restriction of the function $\hat{\gamma}(y)$ to H. This restriction is the characteristic function of a distribution $\gamma' \in \Gamma_P(X/H^\perp)$. Since $2Y = Y$, we have

$2H = H$. Indeed, let $h \in H$. Then

$$h = k_1 y_1 + \ldots + k_n y_n, \quad k_i \in \mathbf{Z},\ y_i \in E.$$

Since $2Y = Y$, $y_i = 2z_i$ and it follows from (1) that $z_i \in E$. Hence $h = 2(k_1 z_1 + \ldots + k_n z_n) \in 2H$. By construction, the distribution γ' has no idempotent factor, because if a distribution μ on X has an idempotent factor, then the character group Y cannot be generated by the set $\{y \in Y: \hat{\mu}(y) \neq 0\}$. Since the factor group $X/H^{\perp}$ is a Corwin group and $(X/H^{\perp})^* \approx H$ is a Corwin group, it follows from Lemma 6 that $\hat{\gamma}'(y) \neq 0$ for $y \in H$. But $\hat{\gamma}(y) = \hat{\gamma}'(y)$ on H, which proves the lemma.

LEMMA 8. *Let X and Y be Corwin groups, $\gamma \in \Gamma_P(X)$ and $\hat{\gamma}(y) > 0$ for any $y \in Y$. Then γ is an infinitely divisible distribution.*

Proof. As in the proof of Lemma 6, consider the distribution ν_n with characteristic function $\hat{\nu}_n(y) = \hat{\gamma}(y/2^n)$. It follows from (6) that

$$\hat{\gamma}(y/2^n) = (\hat{\gamma}(y))^{1/4^n}.$$

Hence it is obvious that the distributions $\{\mu_{nj}\}$, $\mu_{nj} = \nu_n$, $j = 1, \ldots, 4^n$, $n = 1, 2, \ldots$, forming a triangular sequence, satisfy the conditions of Lemma 5. So

$$\gamma = \mu_n = \mathop{*}_{j=1}^{4^n} \mu_{nj}$$

is an infinitely divisible distribution.

LEMMA 9. *Let X and Y be Corwin groups, $\gamma \in \Gamma_P(X)$ and $\hat{\gamma}(y) \geq 0$ for any $y \in Y$. Then γ is an infinitely divisible distribution.*

Proof. Consider the set $H = \{y \in Y: \hat{\gamma}(y) > 0\}$. By Lemma 7, H is an open subgroup of Y. Hence $K = H^{\perp}$ is a compact group. It follows from (1) that if $2y \in H$, then $y \in H$. Therefore K is a compact Corwin group by Lemma 1. Note also that $2H = H$. The restriction of the function $\hat{\gamma}(y)$ to H is the characteristic function of a distribution $\gamma' \in \Gamma_P(X/H^{\perp})$. Since $X/H^{\perp}$ is a Corwin group and so is its character group $(X/H^{\perp})^* \approx H$, it follows from Lemma 8 that γ' is an infinitely divisible distribution. Therefore for any natural n

$$\hat{\gamma}(y) = (f_n(y))^n (x_n, y), \quad y \in H,\ x_n \in X,$$

where $f_n(y)$ is a characteristic function (when writing the character on H in the form (x_n, y), $x_n \in X$, we have used the possibility of extending any character on H to a character on Y, and the Pontryagin duality theorem). We put

$$\hat{\gamma}_n(y) = \begin{cases} f_n(y), & y \in H, \\ 0, & y \notin H. \end{cases}$$

The function $\hat{\gamma}_n(y)$ is continuous on Y, since the subgroup H is open, and positive-definite (see [3, (32.43)]). By the Bochner–Khinchin theorem, $\hat{\gamma}_n(y)$ is the characteristic function of a distribution γ_n on X with $\gamma = \gamma_n^{*n} * E_{x_n}$. Hence γ is an infinitely divisible distribution.

LEMMA 10. *Let ν be an infinitely divisible distribution on X with characteristic function $\hat{\nu}(y) \neq 0$ for all $y \in Y$, and let $\mu = \nu * \bar{\nu}$. Then*

$$\hat{\mu}(2y) \geq (\hat{\mu}(y))^4 \tag{7}$$

and equality occurs only for Gaussian distributions μ.

Proof. Since $\hat{\nu}(y) \neq 0$ for all $y \in Y$, the distribution ν has no idempotent factors. The representation (5) for $\hat{\nu}(y)$ is then of the form

$$\hat{\nu}(y) = (x_0, y) \exp\{\int_X [(x, y) - 1 - ig(x, y)]\, dF(x) - \varphi(y)\}.$$

According to property 3) of the function $g(x, y)$, we obtain for $\hat{\mu}(y) = \hat{\nu}(y)\hat{\nu}(-y)$ the expression

$$\hat{\mu}(y) = \exp\{\int_X [2\operatorname{Re}(x, y) - 2 - 2ig(x, 0)]\, dF(x) - 2\varphi(y)\}.$$

From properties 1), 4) and 5) of $g(x, y)$ it follows that $g(x, 0) = 0$ in a neighborhood of zero $U \subset X$. If we take into account that the measure F is finite on the complement of U and the function $g(x, 0)$ is bounded according to property 2), we obtain

$$\exp\{-i\int_X 2g(x, 0)\, dF(x)\} = C.$$

Since $\hat{\mu}(0) = 1$, we have $C = 1$. Therefore we may assume that

$$\hat{\mu}(y) = \exp\{\int_X [\operatorname{Re}(x, y) - 1]\, dF(x) - \varphi(y)\}.$$

Consider the trivial inequality

$$\operatorname{Re}(x, 2y) - 1 \geq 4(\operatorname{Re}(x, y) - 1). \tag{8}$$

Note that equality occurs if and only if $(x, y) = 1$. It follows from (8) that

$$\int_X [\operatorname{Re}(x, 2y) - 1]\, dF(x) \geq 4\int_X [\operatorname{Re}(x, y) - 1]\, dF(x). \tag{9}$$

Since $\varphi(2y) = 4\varphi(y)$, inequality (7) follows from (9). Equality in (7) for any $y \in Y$ means that the measure F is concentrated on a set where $\operatorname{Re}(x, 2y) - 1 = 4(\operatorname{Re}(x, y) - 1)$ for all $y \in Y$, i.e. where $(x, y) = 1$ for all $y \in Y$. So F is degenerate at zero, which proves the lemma.

Remark 3. Let $X = \mathbf{R}$ and $\gamma \in \Gamma_P(\mathbf{R})$. It can easily be seen that $\hat{\gamma}(y) \neq 0$ for all $y \in \mathbf{R}^* \approx \mathbf{R}$. Let $\nu = \gamma * \bar{\gamma} \in \Gamma_P(\mathbf{R})$. By Lemma 8, ν is an infinitely divisible distribution and by Lemma 10, $\mu = \nu * \bar{\nu} \in \Gamma(\mathbf{R})$. By the

Cramér theorem on the decomposition of a Gaussian distribution, $\gamma\in\Gamma(\boldsymbol{R})$, and hence $\gamma\in\Gamma^s(\boldsymbol{R})$. Thus we have proved the equality $\Gamma^s(\boldsymbol{R})=\Gamma_P(\boldsymbol{R})$ (Theorem A) which was used in the proof of Lemma 2.

LEMMA 11 ([1]). *Let X contain no subgroup isomorphic to $\boldsymbol{T}$, let $\gamma\in\Gamma(X)$ and let γ_1 be a factor of γ. Then $\gamma_1\in\Gamma(X)$.*

Proof of Theorem 1. Sufficiency. Let X satisfy condition (α) and $\gamma\in\Gamma_P(X)$. It follows from Proposition 2 that the distribution γ may, if necessary, be replaced by its shift $\gamma'=\gamma*E_{x_0}$, $2x_0=0$, so that $\sigma(\gamma')\subset G$, where the group G is isomorphic to $\boldsymbol{R}^n+K$, K being a compact Corwin group. It is obvious that $\gamma'\in\Gamma_P(G)$. Consider the distribution $\nu=\gamma'*\bar{\gamma}'\in\Gamma_P(G)$. Since condition (α) is fulfilled for any subgroup of X, it is, in particular, fulfilled for G. Let $H=G^*$. By Lemma 4, $2H=H$. By Lemma 7, the set $E=\{h\in H:\ \hat{\nu}(h)\neq 0\}$ is then an open subgroup of H. Therefore the subgroup $E^\perp\subset G$ is compact. Notice that if $2h\in E$, then, as follows from (1), $h\in E$. So, by Lemma 1, $E^\perp$ is a Corwin group.

Since $\hat{\nu}(h)\geqslant 0$, it follows from Lemma 6 that ν is an infinitely divisible distribution. Let $\mu=\nu*\bar{\nu}\in\Gamma_P(G)$ and consider the restriction $f(h)$ of the characteristic function $\hat{\mu}(h)$ to E. It follows from Lemma 10 that $f(h)$ is the characteristic function of a Gaussian distribution on the factor group $G/E^\perp$. Since $(G/E^\perp)^*\approx E$ and $2E=E$, the factor group $G/E^\perp$ contains no subgroup isomorphic to $\boldsymbol{T}$. Thus, by Lemma 11, any factor of a Gaussian distribution on $G/E^\perp$ is a Gaussian distribution. Therefore the restriction of the characteristic function $\hat{\nu}(h)$ to E is the characteristic function of a Gaussian distribution. If we again apply Lemma 11 to the restriction of the characteristic function $\hat{\nu}(h)$ to E, we conclude that

$$\hat{\gamma}'(h)=([g],h)\exp\{-\varphi_0(h)\},\quad h\in E,$$

where $[g]\in G/E^\perp$, and $\varphi_0(h)$ is a continuous function which is nonnegative on E and satisfies (3). Notice now that the function $([g],h)$ satisfies (1). This results from the fact that $\hat{\gamma}'(h)$ satisfies (1) and $\varphi_0(h)$ satisfies (3). Thus $2[g]=0$. But since $G/E^\perp$ and E are Corwin groups, the group $G/E^\perp$ contains no elements of order two. So $[g]=0$. Therefore we obtain the following representation of the characteristic function $\hat{\gamma}'(h)$ on H:

$$\hat{\gamma}'(h)=\begin{cases}\exp\{-\varphi_0(h)\}, & h\in E,\\ 0, & h\notin E.\end{cases}$$

The function $\varphi_0(h)$ can be extended from the subgroup E onto the whole group H, its properties being preserved (see, for instance, [4, Lemma 5.2.5]). Let the extended function be also denoted by $\varphi_0(h)$. Let γ_0 be the Gaussian distribution on G with characteristic function $\hat{\gamma}_0(h)=\exp\{-\varphi_0(h)\}$. The foregoing implies that

$$\gamma=m_{E^\perp}*E_{x_0}*\gamma_0\in I_P(X)*\Gamma^s(X).$$

Necessity. Let us first construct a distribution $\gamma_0\in\Gamma_P(\boldsymbol{T})$ such that $\gamma_0\notin\Gamma(\boldsymbol{T})$ and $\hat{\gamma}_0(n)\neq 0$ for all $n\in\boldsymbol{Z}$. To this end, we define the function

$$\psi_0(n)=\begin{cases}4^p a_l, & |n|=2^p(2l+1),\\ 0, & n=0,\end{cases}$$

on $\boldsymbol{Z}$, where the numbers a_l are to be chosen so that

$$\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\exp\{-4^p a_l\}<\tfrac{1}{2}$$

and $\psi_0(n)$ does not satisfy (3). By construction, $\exp\{-\psi_0(n)\}$ is the characteristic function of a distribution γ_0 on $\boldsymbol{T}$ with density

$$\varrho(t)=\sum_{n=-\infty}^{\infty}\exp\{-int-\psi_0(n)\}>0.$$

It is evident that $\gamma_0\in\Gamma_P(\boldsymbol{T})$, $\gamma_0\notin\Gamma(\boldsymbol{T})$ and $\hat{\gamma}_0(n)\neq 0$ for all $n\in\boldsymbol{Z}$.

Suppose now that condition (α) is not fulfilled for X. Thus for some compact Corwin subgroup K the factor group X/K contains a subgroup $\tilde{\boldsymbol{T}}$ isomorphic to $\boldsymbol{T}$. By using the isomorphism $\boldsymbol{T}\approx\tilde{\boldsymbol{T}}$, the distribution γ_0 constructed above can be transferred to X/K, the distribution on X/K being also denoted by γ_0. Then $\gamma_0\in\Gamma_P(X/K)$ and $\gamma_0\notin\Gamma(X/K)$. Consider the function

(10) $$f(y)=\begin{cases}\hat{\gamma}_0(y), & y\in K^\perp,\\ 0, & y\notin K^\perp,\end{cases}$$

on Y. The function $f(y)$ is continuous since the subgroup K is open, and positive-definite [3, (32.43)]. By the Bochner–Khinchin theorem, there exists a distribution λ on X with characteristic function $\hat{\lambda}(y)=f(y)$.

We first check that $f(y)$ satisfies (1), i.e. $\lambda\in\Gamma_P(X)$. If $y\in K^\perp$, then $2y\in K^\perp$ and (1) is fulfilled since the function $\hat{\gamma}_0(y)$ satisfies (1). If $y\notin K^\perp$, then by Lemma 1, $2y\notin K^\perp$, and hence $0=f(2y)=f(y)$ and (1) is also fulfilled.

Let us now verify that $\lambda\notin I_P(X)*\Gamma(X)$. Assume to the contrary that there exist a compact Corwin subgroup $K_1\subset X$ and $\gamma\in\Gamma(X)$ such that

(11) $$\lambda=m_{K_1}*\gamma.$$

Since $\hat{\gamma}(y)\neq 0$ for all $y\in Y$, it follows from (11) that $K_1^\perp=\{y\in Y:\ \hat{\lambda}(y)\neq 0\}$. On the other hand, since $\hat{\gamma}_0(y)\neq 0$ for $y\in K^\perp$, it follows from (10) that $K^\perp=\{y\in Y;\ \hat{\lambda}(y)\neq 0\}$. Hence $K_1^\perp=K^\perp$ and therefore $K_1=K$. Then it follows from (11) that the restriction of $\hat{\lambda}(y)$ to $K^\perp$ is the characteristic function of a distribution $\gamma_1\in\Gamma(X/K)$, which is impossible because then $\gamma_0=\gamma_1\in\Gamma(X/K)$. The proof of Theorem 1 is complete.

COROLLARY 1.

(12) $$\Gamma^s(X)=\Gamma_P(X)$$

if and only if the group X contains no nontrivial compact Corwin subgroups.

Proof. The necessity is evident. We prove the sufficiency. By assumption, the only compact Corwin subgroup in X is $K = \{0\}$. Therefore no factor group X/K, where K is a compact Corwin group (in our case the factor group is unique and isomorphic to X), contains a subgroup isomorphic to $\boldsymbol{T}$ since $\boldsymbol{T}$ is a compact Corwin group. Therefore condition (α) is fulfilled. Thus, by Theorem 1, equality (4) and hence (12) are true.

Remark 4. In order that any distribution $\gamma \in \Gamma_P(X)$ be symmetric, i.e. $\gamma = \bar{\gamma}$, it is necessary and sufficient that the group X satisfies condition (α).

Proof. The sufficiency follows directly from Theorem 1 because, as can easily be seen, distributions which belong to $I_P(X) * \Gamma^s(X)$ are symmetric. Let us verify the necessity. If we apply the scheme of the proof of necessity in Theorem 1, it is obvious that it suffices to construct a nonsymmetric distribution $\gamma \in \Gamma_P(\boldsymbol{T})$.

Let $\hat{\gamma}_0(n)$ be the characteristic function of the distribution γ_0 constructed in the proof of Theorem 1. Consider the function

$$t(n) = \begin{cases} \hat{\gamma}_0(n), & |n| \neq 1, \\ i\hat{\gamma}_0(1), & n = 1, \\ -i\hat{\gamma}_0(1), & n = -1, \end{cases}$$

on $\boldsymbol{Z}$. It is evident that $t(n)$ is the characteristic function of a distribution $\gamma \in \Gamma_P(\boldsymbol{T})$ and since the characteristic function $\hat{\gamma}(n)$ is nonreal, $\gamma \neq \bar{\gamma}$.

Remark 5. Condition (α) is also necessary and sufficient for the set $E = \{y \in Y: \hat{\gamma}(y) \neq 0\}$ to be a subgroup in Y for any distribution $\gamma \in \Gamma_P(X)$.

The sufficiency follows immediately from Theorem 1. To prove the necessity, consider the function

$$S(n) = \begin{cases} \exp\{-n^2\}, & |n| \neq 2^p, \ p = 0, 1, \ldots, \\ 0, & |n| = 2^p, \ p = 0, 1, \ldots, \end{cases}$$

on $\boldsymbol{Z}$. It is evident that $S(n)$ is the characteristic function of a distribution $\gamma_1 \in \Gamma_P(\boldsymbol{T})$ for which the set $\{n \in \boldsymbol{Z}: \hat{\gamma}_1(n) \neq 0\}$ is not a subgroup in $\boldsymbol{Z}$. If we apply the scheme of the proof of necessity in Theorem 1, we can construct the desired distribution by using the distribution γ_1 on an arbitrary group X which does not satisfy condition (α).

We can now complement the characterization theorem.

PROPOSITION 3. *Let $\gamma \in \Gamma_P(X)$ be an infinitely divisible distribution. Then $\gamma \in I_P(X) * \Gamma^s(X)$.*

Proof. The representation (5) of the characteristic function of an infinitely divisible distribution implies that $E = \{y \in Y: \hat{\gamma}(y) \neq 0\}$ is an open subgroup in Y. Therefore the group $K = E^{\perp}$ is compact. It follows from (1) that if $2y \in E$ then $y \in E$. By Lemma 1, K is a compact Corwin group. Let $\mu = \gamma * \bar{\gamma}$ and consider the restriction of the characteristic function $\hat{\mu}(y)$ to E.

By Lemma 10, this restriction is the characteristic function of a Gaussian measure on the factor group X/K. Since in the class of infinitely divisible distributions a Gaussian measure has only Gaussian factors, the restriction of the characteristic function $\hat{\gamma}(y)$ to E is also the characteristic function of a Gaussian measure. So $\hat{\gamma}(y)$ can be written as follows:

$$\hat{\gamma}(y) = \begin{cases} ([x], y) \exp\{-\varphi(y)\}, & y \in E, \\ 0, & y \notin E, \end{cases}$$

where $[x] \in X/K$, $2[x] = 0$, and the function $\varphi(y)$ is as in (2). Let us extend the character $([x], y)$ from E to Y. The extended character will be denoted by (x, y). The function $\varphi(y)$ can also be extended from E to Y ([4]), its properties being preserved. We keep the notation $\varphi(y)$ for that extension. Let us now denote by γ_0 the Gaussian measure on X with characteristic function $\hat{\gamma}_0(y) = \exp\{-\varphi(y)\}$. Then it can easily be seen that

$$\gamma = m_K * E_x * \gamma_0 = \lambda * \gamma_0$$

where $\lambda = m_K * E_x \in I_P(X)$ and $\gamma_0 \in \Gamma^s(X)$.

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