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On a ratio ergodic theorem

by

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Abstract. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and Q the positive isometry on $L_1(\mu)$ induced by a measurable transformation T and a positive function h; thus $Qf(x) = h(x)f(Tx)$. Clearly Q acts on functions on X. In this paper the ratio ergodic theorem is discussed for Q but the underlying space $L_1(\mu)$ is replaced by $L_1(wd\mu)$, where w is a positive function. The results obtained generalize some recent ones due to Martín-Reyes and de la Torre; they considered the case where T is measure-preserving and $h = 1$.

Introduction and results. Let T be a measurable transformation from X into itself such that $\mu(A) = 0$ if and only if $\mu(T^{-1}A) = 0$, and h a positive measurable function on X such that $\int_{T^{-1}A} h\,d\mu = \mu(A)$ for all $A \in \mathscr{F}$. If we define

$$Qf(x) = h(x)f(Tx)$$

for measurable functions f on X, then Q is positive and satisfies $\|Qf\|_1 = \|f\|_1$ for all $f \in L_1(\mu)$. (In view of the Banach–Lamperti work [3], Q represents a wide class of positive L_1-isometries.) It follows from the Chacon–Ornstein ergodic theorem [2] that for all f and e in $L_1(\mu)$ with $e \geqslant 0$, the ratios

$$R_0^n(f, e)(x) = \sum_{k=0}^{n} Q^k f(x) \Big/ \sum_{k=0}^{n} Q^k e(x)$$

converge almost everywhere on $\{x\colon \sum_0^\infty Q^k e(x) > 0\}$. But if we replace the measure μ by a measure $wd\mu$, where w is a positive measurable function, and if we assume that f and e are in $L_1(wd\mu)$, then the a.e. convergence of $R_0^n(f, e)(x)$ is unknown. (In case T is measure-preserving and $h = 1$ on X, this has recently been solved by Martín-Reyes and de la Torre [4], [5].) This is the starting point for the study in this paper. For this purpose we may and do fix an e with $0 < e \in L_1(\mu)$, as is easily seen from the Chacon identification theorem [1]. We characterize the positive functions w with the property that if $f \in L_1(wd\mu)$ then the ratios $R_0^n(f, e)(x)$ converge almost everywhere.

THEOREM 1 (cf. [5]). *Let Q, w and e be as above. Assume that Q is conservative (i.e. $\sum_0^\infty Q^k e(x) = \infty$ a.e. on X). Then the following conditions are equivalent*:

(a) *For any $f \in L_1(w\,d\mu)$, $\lim_n R_0^n(f, e)(x)$ exists and is finite a.e. on X.*

(b) $\inf_{n\geq 0} w(T^n x) > 0$ *a.e. on X.*

(c) *There exists a positive measurable function u on X such that for all $n \geq 0$,*

$$\sum_{k=0}^{n} Q^k u(x) / \sum_{k=0}^{n} Q^k e(x) \leq w(T^n x) \quad \text{a.e. on } X.$$

(d) *There exists a positive measurable function u on X such that for all $\lambda > 0$ and $0 \leq f \in L_1(w\,d\mu)$,*

$$\int_{\{x:\ M(f,e)(x) > \lambda\}} u\,d\mu \leq \lambda^{-1} \int f w\,d\mu$$

where $M(f, e)(x) = \sup_{n\geq 0} R_0^n(f, e)(x)$.

Proof. (a) ⇒ (b). Let $E = \{x:\ \inf_{n\geq 0} w(T^n x) = 0\}$. Then $T^{-1}E = E$, and thus if $\mu(E) > 0$, we may and do assume without loss of generality that $E = X$. Then, since the measure μ_e defined by

$$\mu_e(A) = \int_A e\,d\mu \quad \text{for } A \in \mathscr{F}$$

is finite and equivalent to μ, there exist disjoint sets $A_1, A_2, \ldots$ in $\mathscr{F}$ such that

$$w < 2^{-n} \text{ on } A_n, \quad \mu_e(D_n) > (1-2^{-n})\,\mu_e(X)$$

where $D_n = \bigcup_{i=0}^{\infty} T^{-i} A_n$. Since $T^{-1} D_n \subset D_n$ and Q is conservative, $\mu(D_n \setminus T^{-1} D_n) = 0$, and so we may assume without loss of generality that $T^{-1} D_n = D_n$.

Now let us fix an $n \geq 1$. Define

$$\mathscr{I} = \{A \in \mathscr{F}:\ A \subset D_n,\ T^{-1}A = A\},$$

$$\lambda(A \cap A_n) = \mu_e(A) \quad \text{for } A \cap A_n \in \mathscr{I} \cap A_n$$

where $\mathscr{I} \cap A_n = \{A \cap A_n:\ A \in \mathscr{I}\}$. Since D_n is the smallest invariant set containing A_n, λ is well defined and moreover equivalent to the restriction of μ_e to the σ-field $\mathscr{I} \cap A_n$. Using the Radon–Nikodym theorem, let

$$g = d\lambda / d(\mu_e|\ \mathscr{I} \cap A_n).$$

Clearly, $0 < g \in L_1(A_n, \mathscr{I} \cap A_n, \mu_e)$ and for any $A \in \mathscr{I}$

$$\int_{A\cap A_n} g e\,d\mu = \int_{A\cap A_n} g\,d\mu_e = \lambda(A \cap A_n) = \mu_e(A) = \int_A e\,d\mu.$$

(In particular, $ge \in L_1(A_n, \mu)$ and $\|ge\|_1 \leq \|e\|_1$.) Since Q is conservative, the Chacon identification theorem yields

$$\lim_m \sum_{k=0}^{m} Q^k(ge)(x) / \sum_{k=0}^{m} Q^k e(x) = 1 \quad \text{a.e. on } D_n.$$

On the other hand, since $w < 2^{-n}$ on A_n, the function $f_n = nge$ ($\in L_1(A_n, \mu)$) satisfies

$$\int_{A_n} f_n w\,d\mu = n \int_{A_n} gew\,d\mu \leq n2^{-n} \int_{A_n} ge\,d\mu \leq n2^{-n} \|e\|_1.$$

Consequently, if we put $f = \sum_{n=1}^{\infty} f_n$, then $f \in L_1(w\,d\mu)$, but

$$\lim_m \sum_{k=0}^{m} Q^k f(x) / \sum_{k=0}^{m} Q^k e(x) = \infty \quad \text{a.e.}$$

on $\liminf_n D_n = X$ (mod 0), a contradiction.

(b) ⇒ (c). (b) implies that the function $v(x) = \inf_{n\geq 0} w(T^n x)$ is positive a.e. on X. Since $v(Tx) \geq v(x)$, the function $u = ev$ satisfies, for all $n \geq 0$ and $x \in X$,

$$\frac{\sum_{k=0}^{n} Q^k u(x)}{\sum_{k=0}^{n} Q^k e(x)} = \frac{\sum_{k=0}^{n} Q^k e(x)\, v(T^k x)}{\sum_{k=0}^{n} Q^k e(x)} \leq v(T^n x) \leq w(T^n x).$$

To prove (c) ⇒ (d), we need a lemma.

LEMMA (cf. [5]). *Let $\{e(n)\}_{n=0}^{\infty}$, $\{u(n)\}_{n=0}^{\infty}$ and $\{w(n)\}_{n=0}^{\infty}$ be positive sequences such that*

$$\sum_{k=j}^{i} u(k) / \sum_{k=j}^{i} e(k) \leq w(i) \quad \text{for all } i \geq j \geq 0. \tag{1}$$

Then for any nonnegative sequence $\{f(n)\}_{n=0}^{\infty}$ and any $\lambda > 0$ we have

$$\sum_{\{n:\ Mf(n) > \lambda\}} u(n) \leq \lambda^{-1} \sum_{\{n:\ Mf(n) > \lambda\}} f(n)\, w(n),$$

where $Mf(n) = \sup_{n\leq i<\infty} \sum_{k=n}^{i} f(k) / \sum_{k=n}^{i} e(k)$ for $n \geq 0$.

Proof. An approximation argument shows that to prove the lemma it suffices to consider the case where $f(n) = 0$ for all sufficiently large n. Then the set $\{n:\ Mf(n) > \lambda\}$ is finite. Let $[m, n]$ be a maximal interval (in the nonnegative integers) contained in $\{n\colon Mf(n) > \lambda\}$. Letting $v(j) = \min\{w(i)\colon j \leqslant i \leqslant n\}$ for $m \leqslant j \leqslant n$, we have from (1)

$$\sum_{k=m}^{n} u(k) \leqslant \sum_{k=m}^{n} e(k)v(k). \tag{2}$$

On the other hand, since $Mf(n+1) \leqslant \lambda$, it follows that

$$\sum_{k=j}^{n} f(k) \Big/ \sum_{k=j}^{n} e(k) > \lambda \quad \text{for all } j \text{ with } m \leqslant j \leqslant n.$$

Therefore, using the fact that $0 < v(m) \leqslant \ldots \leqslant v(n)$, we obtain

$$\begin{aligned}\sum_{k=m}^{n} e(k)v(k) &= v(m)\sum_{k=m}^{n} e(k) + \sum_{j=m}^{n-1}\Big[\big(v(j+1)-v(j)\big)\sum_{k=j+1}^{n} e(k)\Big]\\ &\leqslant \lambda^{-1}v(m)\sum_{k=m}^{n} f(k) + \lambda^{-1}\sum_{j=m}^{n-1}\Big[\big(v(j+1)-v(j)\big)\sum_{k=j+1}^{n} f(k)\Big]\\ &= \lambda^{-1}\sum_{k=m}^{n} f(k)v(k) \leqslant \lambda^{-1}\sum_{k=m}^{n} f(k)w(k),\end{aligned}$$

and by (2),

$$\sum_{k=m}^{n} u(k) \leqslant \lambda^{-1}\sum_{k=m}^{n} f(k)w(k).$$

This completes the proof.

(c) $\Rightarrow$ (d). Let us fix an $N \geqslant 1$, and put for $0 \leqslant f \in L_1(w\,d\mu)$

$$M_N(f, e)(x) = \max_{0 \leqslant n \leqslant N} \sum_{k=0}^{n} Q^k f(x) \Big/ \sum_{k=0}^{n} Q^k e(x).$$

Let $A(\lambda, N) = \{x\colon M_N(f, e)(x) > \lambda\}$ for $\lambda > 0$. Since Q is a positive isometry on $L_1(\mu)$,

$$\begin{aligned}\int_{A(\lambda,N)} u\,d\mu &= \frac{1}{n+1}\int \sum_{k=0}^{n} Q^k(u1_{A(\lambda,N)})\,d\mu\\ &= \frac{1}{n+1}\int \sum_{k=0}^{n} Q^k u(x)1_{A(\lambda,N)}(T^k x)\,d\mu.\end{aligned} \tag{3}$$

Now, (c) implies that for a.e. $x \in X$ and all $i \geqslant j \geqslant 0$

$$\frac{\sum_{k=j}^{i} Q^k u(x)}{\sum_{k=j}^{i} Q^k e(x)} = \frac{Q^j\big(\sum_{k=0}^{i-j} Q^k u\big)(x)}{Q^j\big(\sum_{k=0}^{i-j} Q^k e\big)(x)} = \frac{\big(\sum_{k=0}^{i-j} Q^k u\big)(T^j x)}{\big(\sum_{k=0}^{i-j} Q^k e\big)(T^j x)} \leqslant w(T^i x).$$

Hence by the preceding lemma, for a.e. $x \in X$,

$$\sum_{k=0}^{n} Q^k u(x)1_{A(\lambda,N)}(T^k x) \leqslant \lambda^{-1}\sum_{k=0}^{n+N} Q^k f(x)\,w(T^k x)1_{A(\lambda)}(T^k x)$$

where $A(\lambda) = \{x\colon M(f, e)(x) > \lambda\}$, and the last member of (3) is less than or equal to

$$\frac{1}{\lambda}\frac{1}{n+1}\int \sum_{k=0}^{n+N} Q^k(fw1_{A(\lambda)})\,d\mu = \frac{1}{\lambda}\frac{n+N+1}{n+1}\int_{A(\lambda)} fw\,d\mu;$$

letting n and then N tend to infinity, (d) follows.

(d) $\Rightarrow$ (a). Since the ratio ergodic theorem holds on $L_1(\mu)$ by the Chacon–Ornstein theorem and $L_1(\mu)$ is dense in $L_1(w\,d\mu)$, a standard approximation argument together with (d) implies (a). We omit the details.

Example. We give an example which shows that if Q is not conservative then (a) does not necessarily imply (b).

Let $X = \{(m, n)\colon m, n \geqslant 1\}$, $\mathscr{F}$ the subsets of X, μ the measure determined by $\mu\{(m, n)\} = 1$ if $n \geqslant 2$ and $\mu\{(m, 1)\} = m$, T the transformation given by

$$T(m, n) = (m, n-1) \quad \text{if } n \geqslant 2,$$

$$T(m, 1) = (m+1, 1),$$

and w the positive function defined by $w(m, n) = 1/m$.

Then T is measure-preserving and $\lim_{k\to\infty} w\big(T^k(m, n)\big) = 0$ for all (m, n), but, as is easily seen, for any $0 \leqslant f \in L_1(w\,d\mu)$ and all $(m, n) \in X$ we have

$$\sum_{k=0}^{\infty} f\big(T^k(m, n)\big) < \infty,$$

whence (a) holds with $Qf(x) = f(Tx)$.

Although the assumption of Q's being conservative is not omitted in Theorem 1, if T is invertible and the ratios

$$R^n_{-m}(f, e)(x) = \sum_{k=-m}^{n} Q^k f(x) \Big/ \sum_{k=-m}^{n} Q^k e(x) \quad \text{with } m, n \geqslant 0$$

are considered, this assumption is not necessary. (Recall that if T is invertible then Q is uniquely determined, and T^{-1} determines Q^{-1}.)

Theorem 2 (cf. [4]). *Let Q, w and e be as in Theorem 1. Assume that the transformation T is invertible. Then the following conditions are equivalent:*

(a) *For any $f \in L_1(w\,d\mu)$, the limit*

$$R^{\infty}_{-\infty}(f, e)(x) = \lim_{m,n\to\infty} R^n_{-m}(f, e)(x)$$

exists and is finite a.e. on X.

(b) $\inf_{-\infty<n<\infty} w(T^n x) > 0$ *a.e. on X.*

(c) *There exists a positive measurable function u on X such that for all m, $n \geqslant 0$,*

$$\sum_{k=-m}^{n} Q^k u(x) / \sum_{k=-m}^{n} Q^k e(x) \leqslant w(x) \quad \text{a.e. on } X.$$

(d) *There exists a positive measurable function u on X such that for all $\lambda > 0$ and $0 \leqslant f \in L_1(w\,d\mu)$*

$$\int_{\{x:\ M^*(f,e)(x) > \lambda\}} u\,d\mu \leqslant \lambda^{-1} \int fw\,d\mu$$

where $M^*(f, e)(x) = \sup_{m,n \geqslant 0} R^n_{-m}(f, e)(x)$.

Sketch of proof. Since Q is an invertible positive isometry on $L_1(\mu)$, if $f \in L_1(\mu)$ then the limit $R^{\infty}_{-\infty}(f, e)(x)$ exists a.e. on X and satisfies

$$R^{\infty}_{-\infty}(f, e)(x) = E\{f/e \mid (X, \mathscr{I}, \mu_e)\}(x) \quad \text{a.e. on } X \text{ (cf. [6])}$$

where the right-hand side stands for the conditional expectation of f/e ($\in L_1(\mu_e)$) with respect to the σ-field $\mathscr{I} = \{A \in \mathscr{F}: A = TA\}$ and the measure μ_e. By using this, the implication (a) $\Rightarrow$ (b) follows as in Theorem 1. The proofs of the other implications are also similar to those of the corresponding parts of Theorem 1. We omit the details.

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An inversion problem for singular integral operators on homogeneous groups

by

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Abstract. We prove the following inversion theorem. If K is an invertible singular integral operator on $L^2(N)$, where N is a homogeneous group, then the inverse K^{-1} is also a singular integral operator. Moreover, a family of "Riesz transforms" on an arbitrary homogeneous group is constructed.

Introduction. Let N be a homogeneous group, and let

$$(*) \qquad Kf(x) = cf(x) + \lim_{\varepsilon \to 0} \int_{|y| \geqslant \varepsilon} f(xy)\,k(y)\,dy$$

be a singular integral operator on N with a homogeneous kernel $k \in C^{\infty}(N \setminus \{0\})$ of critical degree satisfying the usual mean value zero condition. According to the Knapp–Stein theorem (cf. e.g. Folland–Stein [3]), K is bounded on $L^2(N)$. If we assume, moreover, that K has a bounded inverse K^{-1} on $L^2(N)$, the question arises whether also K^{-1} can be realized as a singular integral operator of the form $(*)$. In the case of $N = \boldsymbol{R}^n$, the answer is positive and well known. In fact, this is an exercise in Fourier transform.

Recently, M. Christ and D. Geller [1] have proved that it is still so for a large class of homogeneous groups, namely, for graded homogeneous groups. Their main idea was to look at the problem as a regularity problem, and apply a technique of *a priori* estimates by introducing a scale of Sobolev spaces associated to a Rockland operator on N. Recall that a left-invariant differential operator on N is said to be *Rockland* if it is homogeneous and hypoelliptic.

The aim of this note is to propose a generalization of the theorem of Christ–Geller for arbitrary homogeneous groups. The idea is very much the same, but the technique applied is rather different. As there are no Rockland operators on nongraded groups, we use instead certain nondifferential convolution operators which are homogeneous and hypoelliptic. Such operators exist on any homogeneous group, as was shown in [5]. The required *a priori* estimates for K are obtained in terms of Sobolev norms defined by means of such a convolution operator. The technique of extending