(c) *There exists a positive measurable function u on X such that for all* m, $n \geq 0$,

$$\sum_{k=-m}^{n} Q^k u(x) / \sum_{k=-m}^{n} Q^k e(x) \leq w(x) \quad \text{a.e. on } X.$$

(d) *There exists a positive measurable function u on X such that for all* $\lambda > 0$ *and* $0 \leq f \in L_1(w\,d\mu)$

$$\int_{\{x:\ M^*(f,e)(x) > \lambda\}} u\,d\mu \leq \lambda^{-1} \int fw\,d\mu$$

where $M^*(f, e)(x) = \sup_{m,n \geq 0} R_{-m}^{n}(f, e)(x)$.

Sketch of proof. Since Q is an invertible positive isometry on $L_1(\mu)$, if $f \in L_1(\mu)$ then the limit $R_{-\infty}^{\infty}(f, e)(x)$ exists a.e. on X and satisfies

$$R_{-\infty}^{\infty}(f, e)(x) = E\{f/e \mid (X, \mathscr{I}, \mu_e)\}(x) \quad \text{a.e. on } X \text{ (cf. [6])}$$

where the right-hand side stands for the conditional expectation of f/e ($\in L_1(\mu_e)$) with respect to the σ-field $\mathscr{I} = \{A \in \mathscr{F}:\ A = TA\}$ and the measure μ_e. By using this, the implication (a)$\Rightarrow$(b) follows as in Theorem 1. The proofs of the other implications are also similar to those of the corresponding parts of Theorem 1. We omit the details.

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An inversion problem for singular integral operators on homogeneous groups

by

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Abstract. We prove the following inversion theorem. If K is an invertible singular integral operator on $L^2(N)$, where N is a homogeneous group, then the inverse K^{-1} is also a singular integral operator. Moreover, a family of "Riesz transforms" on an arbitrary homogeneous group is constructed.

Introduction. Let N be a homogeneous group, and let

$$(*) \qquad Kf(x) = cf(x) + \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} f(xy)\,k(y)\,dy$$

be a singular integral operator on N with a homogeneous kernel $k \in C^\infty(N \setminus \{0\})$ of critical degree satisfying the usual mean value zero condition. According to the Knapp–Stein theorem (cf. e.g. Folland–Stein [3]), K is bounded on $L^2(N)$. If we assume, moreover, that K has a bounded inverse K^{-1} on $L^2(N)$, the question arises whether also K^{-1} can be realized as a singular integral operator of the form $(*)$. In the case of $N = \boldsymbol{R}^n$, the answer is positive and well known. In fact, this is an exercise in Fourier transform.

Recently, M. Christ and D. Geller [1] have proved that it is still so for a large class of homogeneous groups, namely, for graded homogeneous groups. Their main idea was to look at the problem as a regularity problem, and apply a technique of *a priori* estimates by introducing a scale of Sobolev spaces associated to a Rockland operator on N. Recall that a left-invariant differential operator on N is said to be *Rockland* if it is homogeneous and hypoelliptic.

The aim of this note is to propose a generalization of the theorem of Christ–Geller for arbitrary homogeneous groups. The idea is very much the same, but the technique applied is rather different. As there are no Rockland operators on nongraded groups, we use instead certain nondifferential convolution operators which are homogeneous and hypoelliptic. Such operators exist on any homogeneous group, as was shown in [5]. The required *a priori* estimates for K are obtained in terms of Sobolev norms defined by means of such a convolution operator. The technique of extending estimates by continuity (cf. Lemma (2.6) below) due to Helffer–Nourrigat [7], which was an essential tool in [5], still plays a crucial role here.

In [1], Christ and Geller have also obtained, among other things, a sufficient condition for a finite collection of singular integral operators to characterize the Hardy space $H^1(N)$. A collection of singular integral operators $K_1, \ldots, K_m$ is said to *characterize* $H^1(N)$ if for every $f\in L^1(N)$, f belongs to $H^1(N)$ if and only if $K_j f\in L^1(N)$, $j=1,\ldots,m$, and, moreover,

$$\|f\|_{H^1}\approx\|f\|_{L^1}+\sum_{j=1}^{m}\|K_j f\|_{L^1}.$$

Then, by using their inversion theorem, they were able to give an explicit construction of singular integral operators that characterize $H^1(N)$ for N being a stratified homogeneous group.

As an application of our main result we show that a similar construction can be carried out on an arbitrary homogeneous group.

We wish to thank Professor Elias M. Stein for bringing the problem considered here to our attention.

1. Preliminaries. A family of *dilations* on a nilpotent Lie algebra N is a one-parameter group $\{\delta_t\}_{t>0}$ of automorphisms of N determined by

$$\delta_t e_j = t^{d_j} e_j,\quad j=1,\ldots,n, \tag{1.1}$$

where $e_1,\ldots,e_n$ is a linear basis for N and

$$1=d_1\leqslant d_2\leqslant\ldots\leqslant d_n \tag{1.2}$$

are fixed exponents of homogeneity. If we regard N also as a Lie group with the multiplication given by the Campbell–Hausdorff formula, then the dilations δ_t are automorphisms of the group structure of N, and the nilpotent group N equipped with these dilations is said to be a *homogeneous* group. If all the exponents d_j are rational, N is said to be *graded*, and if the eigenvectors e_j such that $d_j=1$ generate N as an algebra, it is said to be *stratified*.

The *homogeneous dimension* of N is a number Q defined by

$$d(\delta_t x)=t^Q dx,\quad t>0,$$

where dx is a left-invariant Haar measure on N. Obviously $Q=d_1+\ldots+d_n$.

Let

$$X_j f(x)=d/dt|_{t=0}f(x\cdot te_j),\qquad Y_j f(x)=d/dt|_{t=0}f(te_j\cdot x), \tag{1.3}$$

$1\leqslant j\leqslant n$, be, respectively, left- and right-invariant basic vector fields. Let also

$$\langle D_j, f\rangle = X_j f(0)=Y_j f(0),\quad 1\leqslant j\leqslant n. \tag{1.4}$$

If $I=(i_1,\ldots,i_n)$ is a multi-index, we set

$$X^I f = X_1^{i_1}\ldots X_n^{i_n} f,\qquad Y^I f = Y_1^{i_1}\ldots Y_n^{i_n} f, \tag{1.5}$$

$$\|I\| = i_1+\ldots+i_n,\qquad |I| = i_1 d_1+\ldots+i_n d_n. \tag{1.6}$$

A distribution T on N which is regular, i.e., smooth away from the origin and satisfies

$$\langle T, f\circ\delta_t\rangle = t^r\langle T, f\rangle,\quad f\in C_c^\infty(N),\ t>0, \tag{1.7}$$

for some real r, will be called a *kernel of order* r.

For every unitary representation ϱ of N on a Hilbert space H and every kernel T of order $r>0$, the operator ϱ_T defined on the space $C^\infty(\varrho)$ of smooth vectors for ϱ by

$$(\varrho_T f, g) = \langle T, \varphi_{f,g}\rangle,\quad f\in C^\infty(\varrho),\ g\in H, \tag{1.8}$$

where $\varphi_{f,g}(x)=(\varrho_x f, g)$, preserves $C^\infty(\varrho)$ and is closable. Also if T is a compactly supported distribution, then (1.8) defines a closable operator ϱ_T on $C^\infty(\varrho)$. The closure of ϱ_T in H will be denoted by $\overline{\varrho_T}$.

Now, let T be a kernel of order 0, and let $\varphi\in C_c^\infty(N)$ be equal to 1 in a neighbourhood of the origin. For $m\in N$ set $T_m=(\varphi\circ\delta_{1/m})T$. Then, by Goodman [6], $\overline{\varrho_{T_m}}$ are bounded on H and converge in the strong operator sense to a bounded operator which we shall denote by $\overline{\varrho_T}$. Its restriction to $C^\infty(\varrho)$ will be denoted by ϱ_T. The definition of ϱ_T does not depend on the choice of φ.

It can easily be shown that if T is a kernel of order 0 and $f\in C^\infty(\varrho)$, then for every $1\leqslant j\leqslant n$, $\varrho_T f$ belongs to the domain of $\overline{\varrho_{D_j}}$, and

$$\overline{\varrho_{D_j}}\varrho_T f = -\varrho_{Y_j T} f, \tag{1.9}$$

which shows that $C^\infty(\varrho)$ is invariant under ϱ_T also for T of order 0.

In particular, if π denotes the right-regular representation of N on $L^2(N)$, then $C^\infty(\pi)$ coincides with the space $S^\infty(N)$ of all C^∞-functions f on N such that f and all its left-invariant derivatives belong to $L^2(N)$. The family of norms

$$f\to\sum_{|I|\leqslant k}\|X^I f\|,\quad k=0,1,\ldots, \tag{1.10}$$

where $\|\cdot\|$ denotes, here and in the sequel, the L^2-norm, makes $S^\infty(N)$ into a Fréchet space whose dual $S^\infty(N)^*$ is contained in the space of tempered distributions on N. We endow $S^\infty(N)^*$ with the *-weak topology. Note that $S^\infty(N)$ is then dense in it.

Let A be a continuous endomorphism of $S^\infty(N)$, and suppose that there exists another continuous endomorphism A^+ of $S^\infty(N)$ such that

$$\langle Af, g\rangle = \langle f, A^+ g\rangle$$

for $f, g \in S^{\infty}(N)$. Then A has a unique continuous extension mapping $S^{\infty}(N)^*$ into itself. In fact, $(A^+)^*$ extends A, and, since $S^{\infty}(N)$ is dense in $S^{\infty}(N)^*$, the extension is unique.

In particular, this is true for $A = \pi_T$, where T is a kernel of order $r \geqslant 0$. Therefore, for every kernel T like that, we shall consider π_T as being defined on $S^{\infty}(N)^*$, by identifying it with its extension.

If M is a left-invariant bounded linear operator on $L^2(N)$, then there exists a distribution F on N such that

$$Mf = f * \check{F} \tag{1.11}$$

for $f \in C_c^{\infty}(N)$, where

$$\langle \check{F}, f \rangle = \langle F, \check{f} \rangle$$

and $\check{f}(x) = f(x^{-1})$ for $f \in C_c^{\infty}(N)$ and $x \in N$. A distribution F with the property that (1.11) extends to a bounded operator on $L^2(N)$ will be called a *convolver*, and the extension will be denoted by π_F. It follows, by a standard argument using a Sobolev inequality, that if F is a convolver, then $\check{F}$ belongs to $S^{\infty}(N)^*$.

A distribution P on N is *dissipative* if, by definition, it is real and

$$\langle P, f \rangle \leqslant 0 \tag{1.12}$$

for every real f in $C_c^{\infty}(N)$ such that $f(x) \leqslant f(0)$ for $x \in N$. P is dissipative if and only if there exists a continuous semigroup of contractive positive measures $\{\mu_t\}$ for which P is the generating functional, i.e.,

$$\langle P, f \rangle = \lim_{t \to 0} \frac{1}{t} \langle \mu_t - \delta, f \rangle$$

for $f \in C_c^{\infty}(N)$, where δ denotes the Dirac measure at the origin (cf. e.g. Duflo [2]). A dissipative distribution P is a kernel of order $r > 0$ if and only if the semigroup $\{\mu_t\}$ is stable with exponent r, i.e.,

$$\langle \mu_t, f \rangle = \langle \mu_1, f \circ \delta_{t^r} \rangle, \tag{1.13}$$

$f \in C_c^{\infty}(N)$, $t > 0$, and the Lévy measure of $\{\mu_t\}$ is smooth (cf. e.g. [4], [5]).

Denote by V the one-parameter central subgroup of N generated by X_n (cf. (1.3)) and choose a linear complement $\tilde{N}$ to V invariant under the action of dilations. Then the corresponding projections

$$\sigma\colon N \to \tilde{N}, \qquad v\colon N \to V \tag{1.14}$$

commute with dilations, and $\tilde{N}$ with the multiplication $x \circ y = \sigma(xy)$ is a group isomorphic to N/V with σ being the canonical homomorphism. Of course, N is a homogeneous group with the dilations δ_t restricted to it.

For $\xi \in V^*$ we define a representation π^{ξ} of N on $L^2(\tilde{N})$ by

$$\pi_a^{\xi} f(x) = e^{i\langle v(xa), \xi \rangle} f(\sigma(xa)), \tag{1.15}$$

where $f \in L^2(\tilde{N})$, $x \in \tilde{N}$, $a \in N$. We have

$$\pi_a f(b) = \int_{V^*} e^{i\langle v(b), \xi \rangle} \pi_a^{\xi} f^{\xi}(\sigma(b))\, d\xi \tag{1.16}$$

for $f \in C_c^{\infty}(N)$ and $a, b \in N$, where $f^{\xi} \in C_c^{\infty}(\tilde{N})$ is defined by

$$f^{\xi}(x) = \int_V f(xz)\, e^{-i\langle z, \xi \rangle}\, dz \tag{1.17}$$

for $\xi \in V^*$, $x \in \tilde{N}$.

By $\tilde{X}_j$ (resp. $\tilde{Y}_j$) we denote the left-invariant (resp. right-invariant) basic vector fields corresponding to the basis $e_1, \ldots, e_{n-1} \in \tilde{N}$ (cf. (1.4)).

By $\tilde{\pi}$ we denote the right-regular representation of $\tilde{N}$ on $L^2(\tilde{N})$.

(1.18) Lemma. *If T is a kernel of order $r \geqslant 0$ on N, then there exists a kernel $\tilde{T}$ on $\tilde{N}$ of order r such that*

$$\tilde{\pi}_{\tilde{T}} f = \pi_T^0 f \tag{1.19}$$

for $f \in C_c^{\infty}(\tilde{N})$.

Proof. If $r > 0$, we set

$$\langle \tilde{T}, f \rangle = \langle T, f \circ \sigma \rangle \tag{1.20}$$

for $f \in C_c^{\infty}(N)$, and it is checked directly that $\tilde{T}$ is a kernel of order r such that (1.19) holds.

If $r = 0$, then π_T^0 is a bounded left-invariant operator on $L^2(\tilde{N})$, so there exists a convolver $\tilde{T}$ such that (1.19) holds. Since $\tilde{T}$ clearly satisfies the homogeneity condition (1.7), we have only to show that $\tilde{T}$ is regular. But, by (1.9) and (1.4),

$$\tilde{Y}_j \tilde{T} = (Y_j T)^{\sim}, \quad 1 \leqslant j \leqslant n-1, \tag{1.21}$$

so that, by the above, $\tilde{Y}_j \tilde{T}$ is a kernel or order d_j, and hence $\tilde{T}$ is smooth away from the origin. ∎

(1.22) Lemma. *Let K be a kernel of order 0 on N. Then for every $\psi \in C_c^{\infty}(N)$ with $\int \psi(x)\,dx = 0$ and every $f \in L^2(\tilde{N})$, the mapping*

$$V^* \ni \xi \mapsto \pi_K^{\xi}(\pi_{\psi}^{\xi} f) \in L^2(\tilde{N}) \tag{1.23}$$

is continuous.

Proof. By Goodman [6], for all ψ, f as above,

$$\lim_{m \to \infty} \pi_{K_m}^{\xi}(\pi_{\psi}^{\xi} f) = \pi_K^{\xi}(\pi_{\psi}^{\xi} f), \tag{1.24}$$

where

$$K_m(x) = \begin{cases} K(x) & \text{if } 1/m \leqslant x \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (1.15) that if F is an integrable compactly supported function on N, then

$$V^* \ni \xi \mapsto \pi_F^\xi f \in L^2(\tilde{N})$$

is continuous in ξ for every $f \in L^2(\tilde{N})$. Therefore the mapping (1.23) is continuous in ξ if K is replaced by K_m. But, by an adaptation of the argument of Goodman [6], pp. 7, 8, the convergence in (1.24) is uniform in ξ on compact sets, which proves the lemma.

(1.25) Remark. *If* $\dim N \geqslant 2$, *then the functions of the form* $\pi_\psi^0 f$, *where* $\psi \in C_c^\infty(N)$ *with* $\int \psi = 0$ *and* $f \in L^2(\tilde{N})$, *are dense in* $L^2(\tilde{N})$.

2. The inversion theorem. The whole of our consideration here is based on the following extension of Theorem (2.5) of [5]. But first a definition.

A kernel R of order $r > 0$ is said to be *maximal* if for every kernel T of order $0 < s \leqslant r$ there exists a constant $C > 0$ such that

$$\|\pi_T f\| \leqslant C(\|\pi_R f\| + \|f\|) \tag{2.1}$$

for $f \in C_c^\infty(N)$.

(2.2) Theorem. *Let P be a dissipative symmetric kernel of order $0 < r \leqslant 1$, and let K be a kernel of order 0 such that there exists a convolver L satisfying $K * L = L * K = \delta$. Then for every $m \in N$, $R = P^m * K$ is a maximal kernel of order mr.*

Proof. We make an induction on the dimension of N. If N is abelian (in particular, if $\dim N = 1$), our assertion is obvious since then π_P^m and π_K commute, π_K is an isomorphism of $L^2(N)$, and P^m is maximal, which can be seen by using the Fourier transform. Therefore it suffices to show that our theorem is true for a homogeneous group N provided it is true for $\tilde{N} = N/V$ as defined in Section 1.

We first show that if kernels P, K on N satisfy the hypothesis of Theorem (2.2), then so do the kernels $\tilde{P}$, $\tilde{K}$ on $\tilde{N}$ (cf. Lemma (1.18)). In fact, by Lemma (1.18), $\tilde{P}$ is a kernel of order r and $\tilde{K}$ is a kernel of order 0. It is also easy to check that $\tilde{P}$ is symmetric and dissipative. It therefore remains to prove that there exists a convolver $\tilde{L}$ on $\tilde{N}$ such that

$$\tilde{K} * \tilde{L} = \tilde{L} * \tilde{K} = \delta. \tag{2.3}$$

In fact, by hypothesis and (1.16), there exists a constant $C > 0$ such that

$$\|\pi_K^\xi f\| \geqslant C\|f\|, \quad f \in L^2(\tilde{N}), \tag{2.4}$$

for almost every $\xi \in V^*$. Lemma (1.22) and Remark (1.25) imply that (2.4) holds, in fact, for all $\xi \in V^*$. In particular, by Lemma (1.18),

$$\|\tilde{\pi}_{\tilde{K}} f\| = \|\pi_K^0 f\| \geqslant C\|f\| \tag{2.5}$$

for $f \in L^2(\tilde{N})$. Since also K^* satisfies the hypothesis of Theorem (2.2), (2.5) holds for K^* as well, and thus (2.3) is proved.

Now, to follow the proof of Theorem (2.5) of [5], we need a result on extending estimates by continuity. This is Theorem (3.19) of [5] (cf. also Remark (3.24) there). For the sake of future reference (in Section 3) we give it a slightly more general form. The proof is essentially the same.

(2.6) Lemma. *Let T_j ($j = 1, \ldots, k$) be kernels of order $r > 0$ such that for every kernel T of order $0 < s \leqslant r$ there exists a constant $C > 0$ such that*

$$\|\pi_T^0 f\| \leqslant C\Big(\sum_{j=1}^k \|\pi_{T_j}^0 f\| + \|f\|\Big), \quad f \in C_c^\infty(\tilde{N}).$$

Then for every $M > 0$ and every kernel T of order $0 < s \leqslant r$ there exists a constant $C > 0$ such that for all $|\xi| \leqslant M$

$$\|\pi_T^\xi f\| \leqslant C\Big(\sum_{j=1}^k \|\pi_{T_j}^\xi f\| + \|f\|\Big), \quad f \in C_c^\infty(\tilde{N}).$$

Now, our induction assumption implies that $\tilde{R} = \tilde{P}^m * \tilde{K}$ is a maximal kernel of order mr on $\tilde{N}$. Therefore, by Lemma (2.6) with $k = 1$ and $T_1 = R$, for every $M > 0$ and every kernel T of order $0 < s \leqslant mr$ on N there exists a constant $C > 0$ such that for all $|\xi| \leqslant M$

$$\|\pi_T^\xi f\| \leqslant C(\|\pi_R^\xi f\| + \|f\|), \quad f \in C_c^\infty(\tilde{N}). \tag{2.7}$$

Once we prove that (2.7) holds for a.e. $\xi \in V^*$ with a constant C independent of ξ, we are done. In fact, it is then sufficient to integrate both sides of (2.7) over V^* with f replaced by f^ξ for $f \in C_c^\infty(N)$ and apply (1.16) to obtain (2.1).

To prove that (2.7) is valid for a.e. $\xi \in V^*$, it is sufficient to show the following:

(2.8) *There exist convolvers S, U on N such that*

$$S * R = \delta + U \quad \textit{and} \quad \lim_{|\xi| \to \infty} \|\pi_U^\xi\|_{\mathrm{op}} = 0,$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm on $L^2(\tilde{N})$, and ξ ranges over a set whose complement in V^* is of Lebesgue measure zero.

In fact, (2.8) implies

$$\|f\| \leqslant \|\pi_S^\xi \pi_R^\xi f\| + \|\pi_U^\xi f\|$$

for $f \in C_c^\infty(\tilde{N})$ and a.e. ξ in V^*. Consequently, we have

$$\|f\| \leqslant C\|\pi_{\tilde{R}}^\xi f\|, \quad f \in C_c^\infty(\tilde{N}),$$

for a.e. $|\xi| \geqslant M$, M being sufficiently large. Therefore, by (2.7), there exists a constant $C > 0$ such that

$$(2.7)' \qquad \|\pi_{\tilde{T}}^\xi f\| \leqslant C\|\pi_{\tilde{R}}^\xi f\|$$

for a.e. $M \leqslant |\xi| \leqslant 2M$, where M is as above.

Since both the kernels R and T are homogeneous and the order of T is smaller than that of R, it follows that (2.7)′ is valid for a.e. $|\xi| \geqslant M$, which together with (2.7) proves the desired estimate for almost all $\xi \in V^*$.

To prove that R satisfies (2.8), recall first that since P is regular and homogeneous, the measures μ_t in the semigroup generated by P are absolutely continuous with respect to the Haar measure on N (cf. [5], Lemma (3.28)). Denote by h_t the density of μ_t, $t > 0$, and let

$$F(x) = -\int_0^\infty e^{-t} h_t(x)\,dt, \quad x \in N.$$

Then $F \in L^1(N)$, and it is easily seen that there exists an $F_m \in L^1(N)$ such that

$$F^m * P^m = \delta + F_m,$$

where P^m is the mth convolution power of F. Consequently,

$$(L*F^m)*R = \delta + L*F_m*K,$$

and so R satisfies (2.8) by the following easy to prove lemma of Riemann–Lebesgue type.

(2.9) Lemma. *If $F \in L^1(N)$, then*

$$\lim_{|\xi|\to\infty} \|\pi_{\tilde{F}}^\xi\|_{\mathrm{op}} = 0.$$

This completes the proof of Theorem (2.2). ■

From now on let P be a fixed dissipative symmetric kernel of order 1. We can take, for instance, P defined by

$$\langle P, f\rangle = \lim_{\varepsilon\to 0} \int_{|x|\geqslant\varepsilon} \frac{f(x)-f(0)}{|x|^{Q+1}}\,dx$$

for $f \in C_c^\infty(N)$ (cf. [5], Section 2). By Theorem (2.2), P^m is maximal for every $m \in \mathbf{N}$.

The following proposition was proved in [5] (cf. [5], Proposition (4.13) and Corollary (4.14)).

(2.10) Proposition. *π_P considered as an unbounded operator on $L^2(N)$ $S^\infty(N)$ for its domain is essentially selfadjoint, and the space of C^∞ vectors for $\overline{\pi_P}$ coincides with $S^\infty(N)$.*

For a positive integer m we define a Sobolev space $S^m(N)$ as the completion of $S^\infty(N)$ with respect to the norm

$$(2.11) \qquad \|f\|_{(m)} = \|\pi_{P^m} f\| + \|f\|.$$

In view of Proposition (2.10), $S^m(N)$ is equal to the domain of $\overline{\pi}_P^m$. The dual space $S^m(N)^*$ is contained in $S^\infty(N)^*$, and will be denoted by $S^{-m}(N)$. By $S^0(N)$ we shall denote $L^2(N)$.

The following propositions are easy consequences of the above definition and the properties of P.

(2.12) Proposition. *Let $m \in \mathbf{N}$. If $f \in L^2(N)$ and $\pi_{P^m} f \in L^2(N)$ in the weak sense (see Section 1), then $f \in S^m(N)$.*

(2.13) Proposition. *For every $m \in \mathbf{Z}$, $S^\infty(N)$ is dense in $S^m(N)$. Moreover,*

$$\bigcap_{m\in\mathbf{Z}} S^m(N) = S^\infty(N), \quad \bigcup_{m\in\mathbf{Z}} S^m(N) = S^\infty(N)^*.$$

(2.14) Proposition. *If K is a kernel of order 0, then for every integer m, π_K is continuous as a mapping from $S^m(N)$ to itself.*

As an immediate corollary from Theorem (2.2) we get

(2.15) Corollary. *Let K be a kernel of order 0 such that π_K is an isomorphism of $L^2(N)$. Then for every $m \in \mathbf{N}$ there exists a constant $C > 0$ such that*

$$\|f\|_{(m)} \leqslant C\|\pi_K f\|_{(m)}$$

for $f \in S^\infty(N)$.

To prove that π_K is, in fact, an isomorphism of $S^m(N)$, we need some more preparation. For $\varphi \in C_c^\infty(N)$ and $k \in \mathbf{N}$, let

$$\|\varphi\|_{C^k(N)} = \max_{\|I\|\leqslant k} \|X^I\varphi\|_\infty.$$

(2.16) Lemma. *Let T be a kernel of order $r+1$, where $r \in \mathbf{N}$. Then there exists a constant $C > 0$ such that for every $\varphi \in C_c^\infty(N)$ and every $f \in S^\infty(N)$,*

$$(2.17) \qquad \|[M_\varphi, \pi_T]f\| \leqslant C\|\varphi\|_{C^{r+2}(N)}\|f\|_{(r)}.$$

Proof. This is, essentially, Proposition (3.2) of [5] (see also the remarks following its proof). The dependence of the right-hand side of (2.17) on φ follows from the proof. ■

(2.18) Corollary. *For every $m \in \mathbf{Z}$, there exists a constant $C > 0$ such that*

$$\|M_\varphi f\|_{(m)} \leqslant C\|\varphi\|_{C^{m+1}(N)}\|f\|_{(m)}$$

for all $f \in S^\infty(N)$ and all $\varphi \in C_c^\infty(N)$.

Proof. By duality, we can assume that $m \geqslant 0$. Then, by Lemma (2.16),

$$\begin{aligned}\|M_\varphi f\|_{(m)} &= \|\pi_{pm}(\varphi f)\| + \|\varphi f\| \\ &\leqslant \|\varphi \pi_{pm} f\| + \|\varphi f\| + C\|\varphi\|_{C^{m+1}(N)}\|f\|_{(m-1)} \\ &\leqslant C\|\varphi\|_{C^{m+1}(N)}\|f\|_{(m)}. \ \blacksquare\end{aligned}$$

(2.19) Proposition. *Let K be a kernel of order 0. Then for every $m \in \mathbf{Z}$ there exists a constant $C > 0$ such that for all $\varphi \in C_c^\infty(N)$ and all $f \in S^\infty(N)$*

$$\|[\pi_K, M_\varphi] f\|_{(m+1)} \leqslant C\|\varphi\|_{C^{m+2}(N)}\|f\|_{(m)}.$$

Proof. By duality, we can assume that $m \geqslant 0$. Then for $f \in S^\infty(N)$ and $\varphi \in C_c^\infty(N)$

$$\|\pi_{pm+1}[\pi_K, M_\varphi] f\| \leqslant \|[\pi_{pm+1_{*}K}, M_\varphi] f\| + \|[\pi_{pm}, M_\varphi]\pi_K f\|,$$

whence, by Lemma (2.16) and Proposition (2.14),

$$\begin{aligned}\|\pi_{pm+1}[\pi_K, M_\varphi] f\| &\leqslant C(\|\varphi\|_{C^{m+2}(N)}\|f\|_{(m)} + \|\varphi\|_{C^{m+1}(N)}\|\pi_K f\|_{(m)}) \\ &\leqslant C\|\varphi\|_{C^{m+2}(N)}\|f\|_{(m)},\end{aligned}$$

which, by (2.11), completes the proof. $\blacksquare$

Let U be an open subset of N. We say that a distribution F on N belongs to $S^m(N)$ *locally* on U if for every $\varphi \in C_c^\infty(U)$, $\varphi F \in S^m(N)$. Then we write $F \in S^m_{\mathrm{loc}}(U)$.

Let us fix a cut-off function φ in $C_c^\infty(N)$ such that $\varphi = 1$ in a neighbourhood of the origin, and let

$$\varphi^t(x) = \varphi(\delta_t x)$$

for $x \in N$ and $t > 0$.

(2.20) Lemma. *Let $m \in \mathbf{N}$, and let $F \in S^m_{\mathrm{loc}}(N)$. If*

$$\sup_{0<t\leqslant 1} \|\varphi^t F\|_{(m)} < \infty,$$

then $F \in S^m(N)$.

Proof. For $m = 0$, the lemma follows immediately from the Lebesgue dominated convergence theorem, since

$$\lim_{t\to 0} \varphi^t(x) = 1, \quad x \in N.$$

Assume, by induction, that it is true for some $m \in \mathbf{N}$, and

$$\sup_{0<t\leqslant 1} \|\varphi^t F\|_{(m+1)} = C_{m+1} < \infty. \tag{2.21}$$

Since $\|\cdot\|_{(m)} \leqslant \|\cdot\|_{(m+1)}$, (2.21) implies, by the induction hypothesis, that $F \in S^m(N)$. But then, for every $0 < t \leqslant 1$,

$$\varphi^t \pi_{pm+1} F = \pi_{pm+1}\varphi^t F - [\pi_{pm+1}, M_{\varphi^t}] F,$$

whence, by Lemma (2.16), $\varphi^t \pi_{pm+1} F \in L^2(N)$, and

$$\begin{aligned}\|\varphi^t \pi_{pm+1} F\| &\leqslant \|\varphi^t F\|_{(m+1)} + C\|\varphi\|_{C^{m+2}(N)}\|F\|_{(m)} \\ &\leqslant C_{m+1} + C\|\varphi\|_{C^{m+2}(N)}\|F\|_{(m)}.\end{aligned}$$

Thus, by the Lebesgue theorem, $\pi_{pm+1} F \in L^2(N)$, and consequently, by Proposition (2.12), F belongs to $S^{m+1}(N)$. $\blacksquare$

(2.22) Proposition. *Let K be a kernel of order 0 such that π_K is an isomorphism of $L^2(N)$. Then for every $m \in \mathbf{Z}$, π_K is an isomorphism of $S^m(N)$.*

Proof. First, by duality, we can assume that $m \geqslant 0$. Then, by Proposition (2.14) and Corollary (2.15), it is sufficient to show that π_K maps $S^m(N)$ onto $S^m(N)$.

Let $F \in S^m(N)$. Since π_K is an isomorphism of $L^2(N)$, there exists an $f \in L^2(N)$ such that $\pi_K f = F$. Our proposition will be proved once we show that f belongs to $S^m(N)$.

Let us consider first the case when f is compactly supported. Suppose $\varphi \in C_c^\infty(N)$ with $\int \varphi = 1$, and let

$$\varphi_t(x) = t^{-Q}\varphi(\delta_{t^{-1}} x)$$

for $x \in N$ and $t > 0$. Since f is compactly supported,

$$\varphi_t * f \in C_c^\infty(N) \subseteq S^\infty(N)$$

so that, by Corollary (2.15),

$$\|\varphi_t * f - \varphi_s * f\|_{(m)} \leqslant C\|\pi_K(\varphi_t * f - \varphi_s * f)\|_{(m)} = C\|\varphi_t * F - \varphi_s * F\|_{(m)}$$

for all $t, s > 0$. But since $F \in S^m(N)$, $\varphi_t * F$ tends to F in $S^m(N)$ as $t \to 0$, which shows that the sequence $\varphi_t * f$ is fundamental in $S^m(N)$, and, being convergent in $L^2(N)$ to f, must converge in $S^m(N)$ to the same limit. Hence $f \in S^m(N)$, and

$$\|f\|_{(m)} \leqslant C\|\pi_K f\|_{(m)}, \tag{2.23}$$

where the constant C depends neither on f nor on its support.

Now, let f be an arbitrary function in $L^2(N)$, and let φ be a cut-off function as in Lemma (2.20). We shall proceed by induction on m. If $m = 0$, the assertion is trivial. Assume, therefore, that it is true for some $m \geqslant 0$. Since $S^{m+1}(N) \subseteq S^m(N)$, by induction hypothesis, $f \in S^m(N)$. Moreover, by (2.23), for every $0 < t \leqslant 1$,

$$\|\varphi^t f\|_{(m+1)} \leqslant C\|\pi_K \varphi^t f\|_{(m+1)} \leqslant C(\|\varphi^t \pi_K f\|_{(m+1)} + \|[\pi_K, M_{\varphi^t}] f\|_{(m+1)})$$

so that, by Proposition (2.19),

$$\|\varphi^t f\|_{(m+1)} \leqslant C(\|F\|_{(m+1)} + \|f\|_{(m)}),$$

where the constant C does not depend on t, and Lemma (2.20) shows that f belongs to $S^m(N)$. ■

Now we are in a position to prove our main result. We begin with two simple lemmas.

(2.24) LEMMA. *Let F be a distribution on N such that for every pair of multi-indices I, $J \in N^n$, $X^I Y^J F$ is a convolver. Then π_F is smoothing, i.e., it extends to a continuous mapping from $S^\infty(N)^*$ to $S^\infty(N)$.*

Proof. This is a direct consequence of the definitions of the spaces $S^\infty(N)$ and $S^\infty(N)^*$. ■

(2.25) LEMMA. *Let K be a kernel of order 0, and let $F \in S^\infty(N)^*$. Then $\pi_K F$ is smooth on the complement of the support of F.*

Proof. Let U be an open set such that $\bar{U} \cap \operatorname{supp} F = \emptyset$. Then there exists a $\varphi \in C_c^\infty(N)$ equal to 1 in a neighbourhood of the origin and such that

$$\langle \pi_K F, f \rangle = \langle F * \check{k}, f \rangle \tag{2.26}$$

for $f \in C_c^\infty(U)$, where $k(x) = (1-\varphi(x))K(x)$. By Lemma (2.25), $\pi_k F = F * \check{k} \in S^\infty(N)$, so $\pi_K F$ is smooth on U. ■

(2.27) PROPOSITION. *Let K be a kernel of order 0 on N such that π_K is an isomorphism of $L^2(N)$. Then for every distribution F of finite order, F is smooth wherever $\pi_K F$ is.*

Proof. Suppose that $\pi_K F$ is smooth on an open subset U of N. We show that for every $m \in \mathbf{Z}$, $F \in S^m_{\mathrm{loc}}(U)$, since this, by Proposition (2.13), implies $F \in C^\infty(U)$. By the same proposition, there exists an $m_0 \in \mathbf{Z}$ such that $F \in S^{m_0}(N)$. Therefore it is sufficient to show that for every integer m

$$F \in S^m_{\mathrm{loc}}(U) \text{ implies } F \in S^{m+1}_{\mathrm{loc}}(U). \tag{2.28}$$

Assume $F \in S^m_{\mathrm{loc}}(U)$. We have to show that for every $\varphi \in C_c^\infty(U)$, $\varphi F \in S^{m+1}(N)$. According to Proposition (2.22), this will follow as soon as we show that $\pi_K(\varphi F) \in S^{m+1}(N)$.

Let $\psi \in C_c^\infty(U)$ be equal to 1 in a neighbourhood of $\operatorname{supp} \varphi$. Then

$$\pi_K(\varphi F) = \varphi \pi_K(\psi F) + [\pi_K, M_\varphi](\psi F).$$

By hypothesis and Proposition (2.19), $[\pi_K, M_\varphi](\psi F) \in S^{m+1}(N)$. On the other hand, it follows easily from Lemma (2.25) that $\pi_K(\psi F) - \pi_K F$ is smooth in a neighbourhood of $\operatorname{supp}\varphi$, which, by hypothesis, implies $\varphi \pi_K(\psi F) \in S^{m+1}(N)$. This proves (2.28), and thus completes the proof of the proposition. ■

Now the inversion theorem follows as a corollary.

(2.29) THEOREM. *Let K be a kernel of order 0 such that π_K is an isomorphism of $L^2(N)$. Then there exists a kernel L of order 0 such that*

$$\pi_L f = \pi_K^{-1} f \tag{2.30}$$

for $f \in L^2(N)$.

Proof. Since π_K^{-1} is a bounded left-invariant mapping of $L^2(N)$, there exists a convolver L such that (2.30) holds true. Note also that $\check{L}$ belongs to $S^\infty(N)^*$. Since π_K commutes with dilations, so does π_L. Consequently, L satisfies (1.7) with $r = 0$, and it remains to show that L is regular. But (2.30) implies

$$\pi_K \check{L} = \delta$$

so that, by Proposition (2.27), $\check{L}$, and hence L, is smooth away from the origin. This completes the proof. ■

3. A characterization of $H^1(N)$ in terms of singular integral operators. The Hardy space $H^1(N)$ on a homogeneous group N is defined by Folland–Stein [3] by means of the grand maximal function

$$\mathcal{M}f(x) = \sup_{|\varphi| \leq 1} \mathcal{M}_\varphi f(x),$$

where $|\cdot|$ is a sufficiently strong seminorm in the Schwartz space $\mathcal{S}(N)$, and

$$\mathcal{M}_\varphi f(x) = \sup_{t>0} |f * \varphi_t(x)|$$

for $\varphi \in \mathcal{S}(N)$, $f \in \mathcal{S}^*(N)$, and $x \in N$. A function $f \in L^1(N)$ belongs to $H^1(N)$ if and only if $\mathcal{M}f \in L^1(N)$, and the norm in $H^1(N)$ is defined by

$$\|f\|_{H^1(N)} = \|\mathcal{M}f\|_{L^1(N)}.$$

The following theorem was proved by Christ–Geller [1] in the case when the homogeneous group N is graded (cf. Section 1 for the definition of a graded homogeneous group). Their proof uses the inversion theorem for singular integrals (cf. Introduction) which they were able to prove for graded groups only. However, once Theorem (2.29) is known to be true in the general case, the proof of Christ–Geller is valid for an arbitrary homogeneous group.

(3.1) THEOREM. *Let K_j $(j = 1, \ldots, m)$ be odd kernels of order 0. If there exists a constant $C > 0$ such that*

$$\|f\| \leq C \sum_{j=1}^{m} \|\pi_{K_j} f\| \tag{3.2}$$

for $f \in C_c^\infty(N)$, then the singular integral operators π_{K_j} characterize $H^1(N)$ (cf. Introduction).

Our aim in this section is to construct, on an arbitrary homogeneous group, a family of odd kernels satisfying (3.2), and thus, by Theorem (3.1), characterizing $H^1(N)$.

Let P be a fixed symmetric kernel of order 1 such that $-P$ is dissipative. Recall from [5] that the measures μ_t in the semigroup generated by $-P$ have densities $h_t \in C^\infty(N)$ which satisfy the estimates

$$|X^I h_t(x)| \leqslant C_I t(t+|x|)^{-Q-1-|I|} \tag{3.3}$$

for all $I \in N^n$, $x \in N$, and $t > 0$ ([5], Theorem (2.3)). Recall also that π_P is an unbounded closable operator on the Hilbert space $L^2(N)$ with $S^\infty(N)$ for its invariant domain. By Proposition (2.10), π_P is essentially selfadjoint, and $S^\infty(N)$ is equal to the space of C^∞ vectors for its closure $\overline{\pi_P}$. It follows easily from (1.12) that $\overline{\pi_P}$ is positive. We denote by $E(d\lambda)$ the spectral resolution for $\overline{\pi_P}$.

(3.4) PROPOSITION. *Let $\alpha > 0$. Then for $f \in S^\infty(N)$,*

$$\overline{\pi}_P^\alpha f = f * P^\alpha, \tag{3.5}$$

where P^α is a kernel of order α.

Proof. It is sufficient to consider $0 < \alpha < 1$ only. We have

$$\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{-1+\alpha} e^{-\lambda t}\, dt \tag{3.6}$$

for all $\lambda, \alpha > 0$, and

$$\lambda^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-1-\alpha}(e^{-t\lambda}-1)\, dt \tag{3.7}$$

for $\lambda > 0$, $0 < \alpha < 1$. Consequently, by the spectral representation,

$$\overline{\pi}_P^\alpha f = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-1-\alpha}(f - f*h_t)\, dt = f*P^\alpha \tag{3.8}$$

for $f \in S^\infty(N)$, where

$$\langle P^\alpha, f\rangle = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-1-\alpha} \langle \delta - h_t, f\rangle\, dt \tag{3.9}$$

is a kernel of order α, which is readily checked by using (3.3). ∎

For $0 < \alpha < Q$ let

$$\langle P^{-\alpha}, f\rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{-1+\alpha} \langle h_t, f\rangle\, dt, \quad f \in C_c^\infty(N). \tag{3.10}$$

Since $h_t \in L^1(N) \cap L^2(N)$, it can easily be seen that $P^{-\alpha} \in L^1(N) + L^\infty(N)$, Moreover, (3.3) implies that $P^{-\alpha}$ is regular. As $P^{-\alpha}$ obviously satisfies (1.7) with $r = -\alpha$, we conclude that $P^{-\alpha}$ is a kernel of order $-\alpha$.

(3.11) LEMMA. *For every $\alpha > 0$, the space of vectors of the form $\overline{\pi}_P^\alpha f$, where $f \in S^\infty(N)$, is dense in $L^2(N)$.*

Proof. Since, by Proposition (2.10), $S^\infty(N)$ is a core for $\overline{\pi}_P^\alpha$ and $\overline{\pi_P}$ is selfadjoint, it is sufficient to show that $\overline{\pi_P}$ is injective.

Let $\overline{\pi_P} f = 0$ for some f in the domain of $\overline{\pi_P}$. Then $f*\mu_t = f$ for every $t > 0$, whence

$$\int \tilde{f}*f(x)\, \mu_t(dx) = \|f\|^2 \tag{3.12}$$

for $t > 0$. But since $\tilde{f}*f$ vanishes at infinity,

$$\lim_{t\to\infty} \int \tilde{f}*f(x)\, \mu_t(dx) = \lim_{t\to\infty} \int \tilde{f}*f(\delta_t x)\, \mu_1(dx) = 0,$$

which, by (3.12), implies $f = 0$. ∎

We now present the promised construction of kernels characterizing the Hardy space $H^1(N)$.

(3.13) THEOREM. *Let $-P$ be a symmetric dissipative kernel of order 1, and let*

$$K_j = Y_j P^{-d_j}$$

for $j = 1, \ldots, n$. Then K_j's are kernels of order 0, and there exists a constant $C > 0$ such that

$$\|f\| \leqslant C \sum_{j=1}^n \|\pi_{K_j} f\| \tag{3.14}$$

for $f \in C_c^\infty(N)$.

Proof. We make an induction on the dimension n of N. When $n = 1$, K_1 is a nonzero multiple of the kernel corresponding to the Hilbert transform so that (3.14) holds true. Thus it is sufficient to show that our assertion holds true for N provided it holds for any symmetric dissipative kernel of order 1 on $\tilde{N} = N/V$, where $\tilde{N}$, V are defined in Section 1. But first – two lemmas.

(3.15) LEMMA. *Let $1 \leqslant j \leqslant n$. If $\alpha > d_j$, then*

$$\pi_{K_j} \pi_{P^\alpha} f = \pi_{Y_j P^{\alpha - d_j}} f$$

for $f \in C_c^\infty(N)$.

Proof. This is an exercise in convolving distributions.

(3.16) LEMMA. *$-\tilde{P}$ is a dissipative symmetric kernel of order 1 on $\tilde{N}$. Moreover, for every $\alpha > 0$, $(P^\alpha)^\sim = \tilde{P}^\alpha$.*

Proof. By Lemma (1.18), $\tilde{P}$ is a symmetric kernel of order 1, and it follows immediately from (1.12) and (1.20) that $-\tilde{P}$ is dissipative.

Since for all kernels T_1, T_2 of order greater than 0, $(T_1 * T_2)^\sim = \tilde{T}_1 * \tilde{T}_2$, it is sufficient to consider the case when $0 < \alpha < 1$.

By Duflo [2], $\{\pi^0_{\mu_t}\}$ is a strongly continuous semigroup of contractions on $L^2(\tilde{N})$ whose infinitesimal generator is

$$-\overline{\pi_P^0} = -\overline{\tilde{\pi}_{\tilde{P}}}. \tag{3.17}$$

Since

$$(\widetilde{\pi_P^0})^\alpha f = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-1-\alpha}(f-\pi_{\mu_t}^0 f)\,dt$$

for $f \in C_c^\infty(N)$ (cf. Yosida [8], Chap. IX, Sec. 11), it follows, by (3.9), that $(\pi_P^0)^\alpha = \pi_{P^\alpha}^0$. This, by (3.5) and (3.17), implies the assertion of the lemma. ∎

We return to the proof of Theorem (3.13). By our inductive hypothesis and Lemma (3.16), there exists a constant $\tilde{C} > 0$ such that

$$\|f\| \leqslant \tilde{C} \sum_{j=1}^{n-1} \|\tilde{\pi}_{\bar{K}_j} f\|$$

for $f \in C_c^\infty(\tilde{N})$, where $\bar{K}_j = \tilde{Y}_j \tilde{P}^{-d_j}$. (Actually, $\bar{K}_j = \tilde{K}_j$ but we need not this.) Letting $f = \tilde{\pi}_{\tilde{P}^{d_n}} g$, where $g \in C_c^\infty(\tilde{N})$, we get, by Lemma (3.15),

$$\|\tilde{\pi}_{\tilde{P}^{d_n}} g\| \leqslant \tilde{C} \sum_{j=1}^{n-1} \|\tilde{\pi}_{\tilde{Y}^j \tilde{P}^{d_n - d_j}} g\|,$$

and so, by Lemma (3.16), Lemma (1.18), and (1.20),

$$\|\pi^0_{P^{d_n}} g\| \leqslant C \sum_{j=1}^{n-1} \|\pi^0_{Y_j P^{d_n - d_j}} g\| \tag{3.18}$$

for $g \in C_c^\infty(\tilde{N})$.

Let us remark that $\tilde{P}^{d_n}$ is a maximal kernel of order d_n (cf. Section 2 for the definition). In fact, since $-\tilde{P}$ is dissipative (Lemma (3.16)), it follows from (3.9) that so is $-\tilde{P}^\alpha$ for every $0 < \alpha < 1$. But, by Proposition (3.4),

$$\tilde{P}^{d_n} = (\tilde{P}^\alpha)^k$$

for some $0 < \alpha < 1$, $k \in N$, and hence, by Theorem (2.2), $\tilde{P}^{d_n}$ is maximal.

This, together with (3.18) and Lemma (1.18), implies that the kernels $T_j = Y_j P^{d_n - d_j}$ satisfy the hypothesis of Lemma (2.6), and so there exists a constant $C > 0$ such that

$$\|\pi^\xi_{P^{d_n}} g\| \leqslant C\Big(\sum_{j=1}^{n-1} \|\pi^\xi_{Y_j P^{d_n - d_j}} g\| + \|g\|\Big) \tag{3.19}$$

for $g \in C_c^\infty(\tilde{N})$ and $|\xi| \leqslant 1$. By the equality

$$\|\pi_T^{\delta_t \xi} g^t\| = t^{s - \tilde{Q}/2} \|\pi_T^\xi g\|,$$

valid for any kernel T of order $s \geqslant 0$ and any $\xi \in V^*$, $t > 0$, and $g \in C_c^\infty(\tilde{N})$, where $\tilde{Q}$ is the homogeneous dimension of $\tilde{N}$ and $g^t = g \circ \delta_t$, we obtain from (3.19) the estimate

$$\|\pi^\xi_{P^{d_n}} g\| \leqslant C \sum_{j=1}^{n-1} \|\pi^\xi_{Y_j P^{d_n - d_j}} g\| + C|\xi|^{d_n} \|g\|$$

for all $g \in C_c^\infty(\tilde{N})$, $\xi \in V^*$. But, obviously,

$$|\xi|^{d_n} \|g\| = \|\pi^\xi_{D_n} g\|$$

for $g \in C_c^\infty(\tilde{N})$ and $\xi \in V^*$ so that

$$\|\pi^\xi_{P^{d_n}} g\| \leqslant C \sum_{j=1}^{n} \|\pi^\xi_{Y_j P^{d_n - d_j}} g\| \tag{3.20}$$

for g and ξ as above, where $Y_n P^0 = D_n$ (cf. (1.4)).

Finally, by integration of both sides of (3.20) over V^* with g replaced by f^ξ, where $f \in C_c^\infty(N)$, and applying (1.16) and Lemma (3.15), we come to

$$\|\pi_{P^{d_n}} f\| \leqslant C \sum_{j=1}^{n} \|\pi_{K_j} \pi_{P^{d_n}} f\|$$

for $f \in C_c^\infty(N)$. Since the vectors of the form $\pi_{P^{d_n}} f$, where f ranges over $C_c^\infty(N)$, are dense in $L^2(N)$ (Lemma (3.11)), this ends the proof. ∎

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