COLLOQUIUM MATHEMATICUM

VOL. LIII 1987 FASC. 2

ON DIFFERENTIAL SPACES WHOSE CARTESIAN PRODUCT IS A DIFFERENTIABLE MANIFOLD

BY

WŁODZIMIERZ WALISZEWSKI (ŁÓDŹ)

1. Introduction. In the category of Sikorski's differential spaces [1] a *differentiable C^{∞}-manifold* is considered as a differential space locally diffeomorphic to a Euclidean space. The Cartesian product of two differential spaces is in general a differential space. If these differential spaces are differentiable manifolds, so their Cartesian product is. There is a question concerning the inverse statement. The main aim of the present paper is to prove the following

THEOREM. *If the Cartesian product of two differential spaces is a differentiable manifold, then these differential spaces are also differentiable manifolds.*

The following lemma plays the essential role in the proof of the Theorem and it seems to be interesting independently of the theory of Sikorski's differential spaces.

LEMMA. *If M and N are subsets of an open set Q of points of $\boldsymbol{R}^n$ such that $M \times N$ is diffeomorphic to Q, then M and N are C^{∞}-submanifolds of $\boldsymbol{R}^n$.*

Here the mapping

$$f\colon P \to Q, \tag{1.1}$$

where $P \subset \boldsymbol{R}^m$ and $Q \subset \boldsymbol{R}^n$, is regarded as *smooth* if there exist an open subset of $\boldsymbol{R}^m$ containing P and a C^{∞}-extension of f to this subset. The one-one mapping being smooth together with its inverse mapping is meant as a *diffeomorphism of sets*. The mapping (1.1) is smooth if and only if f maps smoothly the differential subspace (P, E_P) of $(\boldsymbol{R}^m, E)$ (see [1] and [3]) into (Q, E_Q), where E stands for the set of all real C^{∞}-functions on $\boldsymbol{R}^m$. The subsequent two theorems (see [3]) will be useful in the proof of the Lemma. To formulate them we recall the concept of tangent hyperplane M_p to the set $M \subset \boldsymbol{R}^n$ at the point p which is defined as follows.

We consider the identity mapping id_M: $(M, E_M) \to (\boldsymbol{R}^n, E)$ and the induced tangent mapping

$$\mathrm{id}_{M*}\colon (M, E_M)_p \to (\boldsymbol{R}^n, E)_p. \tag{1.2}$$

If w in $(\boldsymbol{R}^n, E)_p$ is of the form $w^i \partial_{ip}$, where ∂_{ip} denotes the partial derivative with respect to the i-th variable in $\boldsymbol{R}^n$ at the point p of the functions in E, then

$$w \mapsto (w^1, \ldots, w^n) \tag{1.3}$$

establishes the isomorphism between $(\boldsymbol{R}^n, E)_p$ and the vector space $\boldsymbol{R}^n$. The hyperplane M_p derived from the tangent space $(M, E_M)_p$ (see [2] and [3]) to the differential space (M, E_M) at the point p by taking the mappings (1.2), (1.3) and the translation $u \mapsto p+u$ is called (cf. [3]) the *tangent hyperplane* to the set M at the point p.

THEOREM A. *If for any $p \in M \subset \boldsymbol{R}^n$* $\dim M_p \geqslant 1$, *then for any p there exists a mapping f satisfying the following conditions*:

(i) *the domain of the mapping f is contained in M_p and dense in itself*;

(ii) *there exists an $r > 0$ such that the set of all values of f coincides with $M \cap B(p, r)$, where $B(p, r)$ denotes the ball in $\boldsymbol{R}^n$ with centre p and radius r*;

(iii) *for every x of the domain of f the orthogonal projection of $f(x)$ onto the tangent hyperplane M_p coincides with x*;

(iv) *the mapping f has the derivatives of all orders at each point of its domain with respect to any direction of the domain.*

THEOREM B. *For any subset M of $\boldsymbol{R}^n$ the following conditions are equivalent*:

(v) *M is an m-dimensional C^∞-submanifold in $\boldsymbol{R}^n$*;

(vi) *for any $p \in M$ the orthogonal projection π_p^M of M onto the tangent hyperplane M_p to M at p is a locally open mapping at the point p, i.e., for any neighbourhood V of p open in M there exists a neighbourhood U of p, $U \subset V$, such that the image $\pi_p^M[U]$ is open in M_p*; $\dim M_p = m$.

2. Proof of the Lemma. Let $p \in M$ and $q \in N$, where M, N and Q satisfy the hypothesis of the Lemma. Then there exists a diffeomorphism

$$\varphi\colon M \times N \to Q. \tag{2.1}$$

Hence the vector space $M_p \oplus N_q$ is isomorphic to $Q_{\varphi(p,q)}$. Thus

$$\dim M_p + \dim N_q = \dim Q_{\varphi(p,q)} = n.$$

Consequently, we have two constant functions $p \mapsto \dim M_p$ and $q \mapsto \dim N_q$. Setting $m = \dim M_p$, we get $\dim N_q = n-m$. By Theorem A there exist a mapping f fulfilling conditions (i)–(iv) and a mapping g with similar conditions obtained from (i)–(iv) by setting g, N and q instead of f, M and p, respectively. We have then the domain D_f of the function f in M_p and, similarly, the domain D_g of g in N_q. From (ii) it follows that there exists $r > 0$ such that the set $f[D_f]$ of all values of f coincides with $M \cap B(p, r)$. Similarly, there exists $s > 0$ such that $g[D_g] = N \cap B(q, s)$. Let us set

$$h(x, y) = \varphi(f(x), g(y)) \quad \text{for } (x, y) \in D_f \times D_g \tag{2.2}$$

and

$$Q_0 = \varphi\left[(M \times N) \cap (B(p, r) \times B(q, s))\right]. \tag{2.3}$$

The set $(M \times N) \cap (B(p, r) \times B(q, s))$ is open in $M \times N$. Thus the set Q_0 is open in Q, so in $\boldsymbol{R}^n$. By (2.1)–(2.3) we have the continuous mapping

$$h\colon D_f \times D_g \to Q_0. \tag{2.4}$$

It is easy to check that the mapping (2.4) is onto. Taking any $z \in Q_0$, $x \in D_f$ and $y \in D_g$ such that $h(x, y) = z$ we get in turn

$$(f(x), g(y)) = \varphi^{-1}(z), \quad f(x) = \mathrm{pr}_1\, \varphi^{-1}(z), \quad g(y) = \mathrm{pr}_2\, \varphi^{-1}(z),$$

where $\mathrm{pr}_i\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$, $\mathrm{pr}_i(u_1, u_2) = u_i$ for $(u_1, u_2) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$, $i = 1, 2$. Applying the orthogonal projections π_p and π'_q of $\boldsymbol{R}^n$ onto the hyperplanes M_p and N_q, respectively, we obtain

$$\pi_p(\mathrm{pr}_1\, \varphi^{-1}(z)) = \pi_p(f(x)) = x \quad \text{and} \quad \pi'_q(\mathrm{pr}_2\, \varphi^{-1}(z)) = \pi'_q(g(y)) = y.$$

This yields that the mapping $h^{-1}\colon Q_0 \to D_f \times D_g$ inverse to (2.4) is given by the formula

$$h^{-1}(z) = \left(\pi_p(\mathrm{pr}_1\, \varphi^{-1}(z)), \pi'_q(\mathrm{pr}_2\, \varphi^{-1}(z))\right) \quad \text{for } z \in Q_0.$$

Hence the mapping (2.4) is a homeomorphism.

Let us remark that $D_f \times D_g$ is contained in the vector space $M_p \oplus N_q$ of dimension n. From the fact that Q_0 is open in $\boldsymbol{R}^n$ and that (2.4) is a homeomorphism, by the Brouwer Theorem on openness of mappings, it follows that $D_f \times D_g$ is open in $M_p \oplus N_q$. Therefore, D_f and D_g are open in M_p and N_q, respectively.

To prove that M is an m-dimensional C^∞-submanifold of $\boldsymbol{R}^n$ we shall apply Theorem B. Let us take any $\varepsilon > 0$. We may assume that $\varepsilon \leqslant \min\{r, s\}$. Setting $G = f^{-1}[M \cap B(p, \varepsilon)]$ we have the set G open in D_f. Thus G is open in M_p. Taking any u in $M \cap B(p, \varepsilon)$ we have $u \in M \cap B(p, r)$. Then there exists $x \in D_f$ such that $u = f(x)$. This yields $\pi_p(u) = \pi_p(f(x)) = x \in G$. Now, let $x \in G$ and $u = f(x)$. Then $\pi_p(u) = x$ and $f(x) \in M \cap B(p, \varepsilon)$. Thus $x \in \pi_p[M \cap B(p, \varepsilon)]$. Hence

$$G = \pi_p[M \cap B(p, \varepsilon)].$$

Therefore the mapping $\pi_p\colon M \to M_p$ is open at the point p. Condition (vi) is then fulfilled, which completes the proof.

3. Proof of the Theorem. Let (M, C) and (N, D) be differential spaces such that the Cartesian product $(M, C)\times(N, D)$ (see [2] and [4]) is a C^∞-differentiable n-dimensional manifold. Take any $p\in M$ and any $o\in N$. Then $p_0 = (p, o)\in M\times N$. There exists a chart x of the manifold $(M, C)\times(N, D)$ around the point p_0. We may assume that the domain of x is of the shape $U\times V$, where U and V are open in (M, C) and (N, D), respectively. Setting $Q = x[U\times V]$ we have the diffeomorphism

$$x\colon (U, C_U)\times(V, D_V)\to(Q, E_Q). \tag{3.1}$$

Consider the diffeomorphisms

$$a\colon (U, C_U)\to\big(U\times\{o\}, (C\times D)_{U\times\{o\}}\big) \tag{3.2}$$

and

$$b\colon (V, D_V)\to\big(\{p\}\times V, (C\times D)_{\{p\}\times V}\big), \tag{3.3}$$

where $a(t) = (t, o)$ for $t\in U$ and $b(t) = (p, t)$ for $t\in V$, and the mapping

$$x_0\colon U_0\times V_0\to Q \tag{3.4}$$

defined by the formula

$$x_0(u, v) = x\big(a^{-1}(x^{-1}(u)), b^{-1}(x^{-1}(v))\big) \quad \text{for } (u, v)\in U_0\times V_0,$$

where $U_0 = x[U\times\{o\}]$ and $V_0 = x[\{p\}\times V]$. It is easy to check that (3.4) is one-one, and

$$x_0^{-1}(t) = \Big(x\big(a(\mathrm{pr}_1 x^{-1}(t))\big), x\big(b(\mathrm{pr}_2 x^{-1}(t))\big)\Big) \quad \text{for } t\in Q, \tag{3.5}$$

where pr_1 and pr_2 are the canonical projections from $\big(U\times V, (C\times D)_{U\times V}\big)$ onto (U, C_U) and (V, D_V), respectively. Formula (3.5), by diffeomorphisms (3.1)–(3.3) and the smooth mappings pr_1 and pr_2, yields that we have the diffeomorphism

$$x_0\colon (U_0, E_{U_0})\times(V_0, E_{V_0})\to(Q, E_Q).$$

From the Lemma it follows that (U_0, E_{U_0}) is a differentiable C^∞-manifold. Taking the diffeomorphisms (3.1) and (3.2) we get the diffeomorphism of (U, C_U) onto (U_0, E_{U_0}). Then (U, C_U) is a differentiable C^∞-manifold, which completes the proof.

REFERENCES

[1] R. Sikorski, *Abstract covariant derivative*, Colloq. Math. 18 (1967), pp. 251–272.
[2] — *Introduction to Differential Geometry* (in Polish), Warszawa 1972.
[3] W. Waliszewski, *On differential subspaces of Cartesian space*, Colloq. Math. 45 (1981), pp. 257–265.

[4] — *Regular and coregular mappings of differential spaces*, Ann. Polon. Math. 30 (1975), pp. 263–281.

Reçu par la Rédaction le 15. 5. 1983