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CHORD GRAPHS – NEW MEANS FOR REPRESENTING GRAPHS

1. Introduction. A *circle graph* G (also called an *overlap graph*) is the intersection graph of a set of chords of a circle in the plane (cf. [4]). Assume without loss of generality that the end points of the chords are mutually different. Then, after labeling the n vertices of G and correspondingly labeling the end points of the chords representing them, G can also be represented by the cyclic sequence of the $2n$ labels on the circle.

Not every graph can be represented as a circle graph. We shall generalize the concept by showing that every graph can be represented in a similar way by a cyclic sequence of $2m$ labels (on a circle C) which are subdivided into n families of equal (even) order and can be interpreted as the end points of n families of chords of C . The minimal number m of pairs of equal labels (chords) necessary to represent a graph G in this way is called its *chord number* $ch(G)$. We investigate this new invariant and a derivate and, moreover, the questions of finding a chord graph representation of a given graph and of recognizing it from such a representation. For that we essentially use the knowledge of families of circle graphs.

Generally, we use the terminology of Harary [5], but $v(G)$, $e(G)$ and $c(G)$ are the numbers of vertices, edges and components, respectively, of the graph G .

The results of this paper are summarized in [6].

2. Circle graphs.

LEMMA 1. *A graph is a circle graph if and only if each of its induced subgraphs is a circle graph.*

THEOREM 1. *A graph is a circle graph if and only if each of its blocks is a circle graph.*

Every cycle and every complete multipartite graph (especially every complete graph) is a circle graph, and so every graph whose blocks are such graphs (e.g., trees, cacti) are also circle graphs. Other families of circle graphs are given by the following theorems (cf. [7] and [8]):

THEOREM 2. *Every power of a cycle is a circle graph.*

THEOREM 3. *A graph which consists of three paths joining two vertices and being disjoint except for their end vertices is a circle graph if and only if at least one of these paths has length ≤ 2 .*

THEOREM 4. *A graph which consists of a cycle, a vertex c and some edges ("spokes") each joining c and a cycle vertex is a circle graph if and only if it has at most four spokes and in the case of three or four spokes one pair or two disjoint pairs, respectively, of their end vertices on the cycle are adjacent.*

THEOREM 5. *A homeomorph of K_n is a circle graph if and only if its vertices of degree 2 lie on a Hamiltonian cycle when n is odd and lie on a Hamiltonian path with other end vertices when n is even.*

THEOREM 6. *Every outerplanar graph is a circle graph.*

Circle graphs have been characterized by Even and Itai [1] and Fournier [2], but only recently Gabor et al. [3] settled the recognition problem for them by giving a polynomial-time algorithm.

3. Chord graphs.

MAIN LEMMA. *Let be given $2k$ red and $2l$ blue points on a (geometric) circle (k, l integers). Then the number c of crossings of k red chords with l blue chords, drawn between mutually disjoint pairs of points with the same color in any way, is given modulo 2 by the distribution of the colored points on the circle. The parity of this distribution says if c is odd or even.*

A straightforward proof is left to the reader (or cf. [6]).

DEFINITION. Let be given a family of n sets of points on a (geometric) circle, each of even cardinality. Then the graph, whose vertices correspond to the sets and whose two vertices are joined by an edge if and only if parity of the distribution of its corresponding sets is odd, is called a *chord graph*.

Remarks. 1. By choosing "odd" in the definition, its restriction to sets of cardinality 2 is a definition of circle graphs.

2. By counting the crossings modulo 2, every realization of chords joining points of fixed sets represents the same graph, although the corresponding intersection graphs may be nonisomorphic.

A *chord graph representation* of a graph G is given by a cyclic sequence of labels of the n sets from the definition in accordance with the distribution of their elements on the circle, or by that circle and any set of chords with mutually different end points each joining two points of equal sets and together covering all distinguished points. A chord joining two points of such a set v is said to be a *v-chord* and to *represent* this set v or the vertex v of G given by this set, respectively. Some chords of a circle C which together with C constitute a chord graph representation of a graph G are shortly said to *represent* G .

THEOREM 7. *Every graph is (representable as) a chord graph.*

Proof. Let G be a graph and C be a geometric circle. Draw $c(G)$ mutually nonintersecting chords of C , representing one vertex of each component of G . Then draw successively a chord for every edge, one of whose end vertices, say e , is already represented, crossing precisely one e -chord and representing the other end vertex.

4. Chord number, chord difference. The representation of G given in the last proof uses $c(G) + e(G)$ chords. So its chord number $\text{ch}(G)$ satisfies the condition

$$v(G) \leq \text{ch}(G) \leq c(G) + e(G).$$

Obviously, the lower bound is attained if and only if G is a circle graph. So in order to obtain a measure for the deviation of a given graph G from being a circle graph we introduce the *chord difference* $\text{cd}(G) = \text{ch}(G) - v(G)$.

For example, the noncircle graphs of Theorems 3 and 4 have chord difference 1, since they contain edges whose deletion makes the graphs outerplanar (cf. Theorem 6).

One easily shows

LEMMA 2. *If G' is an induced subgraph of G , then*

$$\text{cd}(G') \leq \text{cd}(G).$$

This lemma implies the "only if" part of Lemma 1. Similarly, the following theorem implies the "if" part of Theorem 1 and can be proved by the same argument. We cannot prove the equality there as stated in [6] by a mistake in writing.

THEOREM 8. *If $B_1, B_2, \dots, B_k$ are the blocks of a graph G , then*

$$\text{cd}(G) \leq \sum_{i=1}^k \text{cd}(B_i).$$

If a graph G has a cycle as a subgraph, then we can represent G by first representing this cycle as a circle graph, and then continuing as in the proof of Theorem 7, so using only $c(G) + e(G) - 1$ chords. This implies that $\text{cd}(G)$ attains its upper bound $c(G) + e(G) - v(G)$ if and only if G is a forest. But then it is a circle graph, i.e., $\text{cd}(G) = 0$ (= lower bound).

But for every integer d there are graphs G with $\text{cd}(G) = d$; namely

THEOREM 9.

$$\text{cd}(K_{m,n} - kK_2) = \begin{cases} k-2 & \text{if } m, n \geq 3, m+n \geq 7, k \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

COROLLARY.

$$\text{cd}(K_{n,n} - L) = \begin{cases} n-2 & \text{if } n \geq 4, \\ 0 & \text{otherwise.} \end{cases}$$

There $K_{m,n} - kK_2$ and $K_{m,n} - L$ denote the graphs arising from $K_{m,n}$ by deleting k mutually disjoint edges (so $k \leq m, n$) or the edges of a linear factor (then $m = n$), respectively.

Proof of Theorem 9. First assume without loss of generality that $m \geq 4$ and let

$$1, 2, \dots, m; 1', 2', \dots, n'$$

be the vertices of $K_{m,n}$ and

$$11', 22', \dots, (k-1)(k-1)', mn'$$

be the k edges deleted from $K_{m,n}$. The cyclic sequence

$$m(m-1) \dots 321n'(n-1)' \dots 3'2'1'123 \dots (m-1)m1'2'3' \dots (n-1)'n'$$

represents $K_{m,n}$. Exchanging there 1 with $1'$ and m with n' yields a representation of $K_{m,n} - \{11', mn'\}$. After adding two further labels i' in this sequence just at the left-hand and the right-hand side of one label i (i.e., adding a further chord representing i' , crossing precisely the chord representing i) for $i = 2, 3, \dots, k-1$ we have a representation of $G = K_{m,n} - kK_2$ by $m+n+k-2$ pairs of labels (chords), and so $\text{cd}(G) \leq k-2$.

$\text{cd}(G) \geq k-2$ is trivial when $k = 2$.

For $k = 3$ the assumption $\text{cd}(G) < k-2$ implies that G is a circle graph. But its vertices $1, 2, 3, m; 1', 2', n'$ induce a subgraph $K_{4,3} - 3K_2$ which is not a circle graph by Theorems 3 and 4, and so contradicts Lemma 1.

For $k \geq 4$ let $G' = K_{k,k} - kK_2$ be the subgraph of G induced by

$$1, 2, \dots, k-1, m; 1', 2', \dots, (k-1)', n'.$$

Then the assumption $\text{cd}(G) < k-2$ implies by Lemma 2 that $\text{cd}(G') \leq k-3$. So G' can be represented by $v(G') + (k-3) = 3k-3 > 2k = v(G')$ chords. If G' is represented so, then at least one vertex, say j , is multiply represented.

In the case $k = 4$, there is only one such a vertex. So its removal yields a circle graph. But this graph again is isomorphic to $K_{4,3} - 3K_2$.

If $k > 4$, then the removal of j and j' and of the chords representing them yields a graph isomorphic to $K_{k-1,k-1} - (k-1)K_2$ and its representation by at most $(3k-3)-3$ chords, respectively, showing that

$$\text{cd}(K_{k-1,k-1} - (k-1)K_2) < (k-1)-2,$$

which contradicts the statement of the theorem. Since the first line of the theorem is proved up to $k = 4$, the last conclusion yields the inductive step for a complete proof.

In the cases $k = 0$ and $k = 1$, the cyclic sequence given above and the one obtained from it by exchanging 1 with $1'$, respectively, are circle graph representations of G .

Finally, in the case $m = n = k = 3$, G is a cycle, and so it is also a circle graph.

By Theorem 9 and its Corollary we also know graphs which show that generally there is no relation between $cd(G)$ and $c(G)$. But

$$cd(G) \leq c(G) + e(G) - v(G) \quad \text{and} \quad e(G) \leq \binom{v(G) - c(G) + 1}{2}$$

imply

$$(1) \quad \begin{aligned} cd(G) &\leq e(G) - \frac{1}{2}(\sqrt{1 + 8e(G)} - 1), \\ cd(G) &\leq \binom{v(G) - c(G)}{2} \leq \binom{v(G) - 1}{2}. \end{aligned}$$

A generally better upper bound for bipartite graphs is given by Theorem 11 derived in the next section. But first we prove another inequality concerning chord differences.

THEOREM 10. *If $G(k)$ is a graph arising from a graph G by subdividing an edge by k vertices, then*

$$cd(G) \leq cd(G(k)) \leq cd(G) + 1.$$

Proof. Let $G(1)$ arise from G by subdividing the edge uv of G by the vertex a .

If there is a chord graph representation of $G(1)$ by a circle C and a set of $ch(G(1))$ chords containing only one a -chord, then we can draw the other chords in such a way that the a -chord is crossed by precisely one u -chord and one v -chord (and no other chord). Then canceling both these chords, cutting the interior face of C along the a -chord, reflecting one of the parts obtained, again gluing these parts together along the cutline, joining the equally labeled end points of the canceled chords by new chords, and canceling the a -chord yield a chord graph representation of G by $ch(G(1)) - 1$ chords, i.e.,

$$\begin{aligned} cd(G) &= ch(G) - v(G) \\ &\leq (ch(G(1)) - 1) - (v(G(1)) - 1) = cd(G(1)). \end{aligned}$$

If there is a chord graph representation of $G(1)$ by $ch(G(1))$ chords containing two a -chords, then

$$cd(G(1) - a) \leq cd(G(1)) - 1,$$

and so by

$$cd(G) - 1 \leq cd(G - uv) = cd(G(1) - a)$$

again

$$\text{cd}(G) \leq \text{cd}(G(1)).$$

This proves the first inequality.

Now let us take a chord graph representation of G by a cyclic sequence of $2\text{ch}(G)$ labels and replace one " v " by the sequence $kuv1u2132\dots k(k-1)$ (cf. Fig. 1). Then a chord graph representation of $G(k)$ by a cyclic sequence of $2(\text{ch}(G)+k+1)$ labels is obtained, proving the second inequality.

Both the inequalities in Theorem 10 become sharp for examples given in Theorems 3–5.

Fig. 1

5. Constructions of chord graph representations. First we consider an arbitrary bipartite graph $G_{m,n}$ whose vertex set consists of two disjoint independent sets V_m and V_n with m and n vertices, respectively.

We shall construct a chord graph representation of $G_{m,n}$ using exactly n chords for representing the vertices of V_n and 2^{m-1} chords for representing those of V_m . For that we partition V_n into 2^m subsets (possibly empty) in accordance with their neighbouring sets (in V_m), and after representing the vertices of V_n each by exactly one of n "parallel" chords, subset by subset in a special order, we show inductively that the vertices of V_m can be represented by 2^{m-1} further chords (each crossing some of the parallel ones drawn before) in such a manner that $G_{m,n}$ is represented.

Let $V_m = \{1, 2, \dots, m\}$. The order of the 2^m subsets of V_m we shall use is determined by the following algorithm:

1. Take the empty set and the one-element subset $\{1\}$.
2. Take the sequence of subsets derived so far and extend it by attaching another its copy having the opposite direction. Add the vertex with the smallest number which did not appear so far to each subset belonging to the second copy.
3. Repeat step 2 until m appears.

Obviously, the algorithm stops after m steps and yields a sequence containing each subset of V_m precisely once.

Fig. 2 shows the sequence of the 2^m parallel "bundles" of parallel chords obtained so up to the case $m = 4$. Their labels show to which vertices of V_m the vertices of V_n , represented by the bundle in question, are neighbouring in $G_{m,n}$. To obtain the corresponding sequences for $m = k$ ($k = 1, 2, \dots$)

cut the figure just behind the bundle "k" and take the outline as a part of the circle.

The dashed lines regarded as chords of a circle represent the vertices of V_m in accordance with their labeling (also if they are ending at a cutline). Fig. 2 shows how this can be realized by 2^{m-1} chords from $m = 1$ to $m = 4$ in such a manner that $G_{m,n}$ is represented.

Fig. 2

The inductive step can now be easily derived from our construction. Notice that the (only) chord representing the vertex with the greatest label in V_m can be lengthened every time into the second copy of the sequence of chord bundles (cf., e.g., the dashed chord "3" in Fig. 2).

The more subsets of V_n according to their neighbouring sets in V_m are empty, the less chords are needed in general by the representation found by the algorithm.

Our construction proves the following

THEOREM 11. *If $G_{m,n}$ is a bipartite graph having m and n vertices in its vertex classes, respectively, then*

$$\text{cd}(G_{m,n}) \leq 2^{m-1} - m.$$

Notice that this upper bound depends only on the cardinality of one vertex class. So we can expect that the upper bound containing smaller cardinality is better than other bounds, provided the cardinality of the other class is large enough.

Indeed, it is better than that, e.g., of (1) if

$$n > \frac{1}{2}(\sqrt{2^{m+2} - 8m + 1} - (2m - 3)),$$

i.e., provided $m \leq n$, if $m \leq 7$; $m = 8$, $n \geq 10$; $m = 9$, $n \geq 15$; $m = 10$, $n \geq 24$; ...

Now we consider an arbitrary graph G . In order to get a chord graph representation of G better than that given in the proof of Theorem 7 we extend the idea used after Theorem 8 for proving that $\text{cd}(G)$ attains the upper bound derivable from Theorem 7 only if G is a forest.

Therefore, let H be a graph on the vertex set of G for which we know a chord graph representation by h chords and define $\text{sd}(G, H)$ (the symmetric difference of G and H) to be the graph consisting of the edges belonging to precisely one of G and H and their end vertices. Then compose a chord graph representation of G by taking the h chords representing H and either adding the chords of a representation of $\text{sd}(G, H)$ or (if no such representation is known or if every known one needs more than $e(\text{sd}(G, H))$ chords) adding successively a chord for every edge of $\text{sd}(G, H)$ which crosses precisely one chord representing one of its end vertices and represents the other one.

In this way we get a chord graph representation of G by at most $h + e(\text{sd}(G, H))$ chords which in the worst case is better than our first one if the number of edges in H also belonging to G exceeds the number of edges in H not belonging to G by at least $h - c(G)$, but which in general, roughly speaking, is the better the more similar to G the graph H is and the better its representation used is.

In order to find any graph H with the desired property we first can cover the edges (and isolated vertices) of G by graphs $H_1, H_2, \dots$ with "good" known chord graph representations (e.g., circle graphs) and "small" intersections with each other, and then compose their representations to the one of some graph H .

The more families of circle graphs or graphs with "small" chord differences together with their corresponding representations are known the better our method works.

6. Recognizing a graph from its chord graph representation. Finally, we give a simple algorithm *FINCH* for *F*inding the set N of vertices *N*ighbouring with a vertex u in a graph G for which is given a *CH*ord graph representation by a sequence S of s labels $a_1, a_2, \dots, a_s$ on a circle C ($s \geq 2\text{ch}(G)$ even). Let $v_1, v_2, \dots, v_{n-1}$ be the vertices of G different from u and extend S to a sequence S' by adding an auxiliary label u in front of it.

FINCH: Examine S' step by step starting with u, a_1 and simultaneously count modulo 2 in the corresponding counters:

- (a) the appearance of $v_1, v_2, \dots, v_{n-1}$ and u ;
- (b) the change of labels u and v_j in the partial sequence of S' containing its u 's and v_j 's ($j = 1, 2, \dots, n-1$);
- (c) the occurrence of the pattern such that
 - (i) $a_i \in \{u, v_j\}$,
 - (ii) both u and v_j can be found an odd number of times in the partial sequence $a_1, a_2, \dots, a_i$,
 - (iii) among these u 's and v_j 's the last one is not a_i ($j = 1, 2, \dots, n-1$).

Then the set N consists of those vertices for which the counters for (c) return 1 after checking S' .

In (c) *FINCH* counts the crossings of u - and v_j -chords ($j = 1, \dots, n-1$) appearing if every two consecutive u - and v_j -labels on C , respectively, beginning at a_1 , are joined by a chord. It can be simplified to an algorithm for deciding if two vertices u and v of G are adjacent.

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