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SOME REMARKS ON THE INTERVALS OF STABILITY OF RUNGE-KUTTA METHODS AFTER RICHARDSON EXTRAPOLATION

1. Introduction. We consider a problem which was numerically treated in [1], where the motivation is given. We want to explain and generalize the results obtained in that paper.

For any natural number p we define the polynomial

$$(1.1) \quad w_p(z) = \sum_{i=0}^p \frac{z^i}{i!}, \quad z \in R,$$

and the set

$$I_p = \{z \in R: |w_p(z)| < 1\}.$$

For any natural number $k \geq 2$ we define the polynomials

$$(1.2) \quad w_{p,k}^*(z) = \frac{k^p w_p^k(z/k) - w_p(z)}{k^p - 1}, \quad z \in R.$$

We are interested in the sets

$$I_{p,k} = \{z \in R: |w_p^k(z/k)| < 1\}, \quad I_{p,k}^* = \{z \in R: |w_{p,k}^*(z)| < 1\}.$$

Following [1], let the set I_p be called the *interval of stability*. In Lemma 2.1 we prove that I_p is really an interval. For the sets $I_{p,k}^*$ we have another situation, for example

$$I_{1,3}^* = (-7.193430246, -6) \cup (-3, 0).$$

After stating the problem we redefine the set $I_{p,k}^*$. Let us define the number

$$b_{p,k} = \inf \{r < 0: (r, 0) \subset I_{p,k}^*\}$$

and we put

$$\hat{I}_{p,k} = (b_{p,k}, 0).$$

In this paper we give sufficient conditions under which the length of the interval $\hat{I}_{p,k}$ is less than the length of $I_{p,k}$ (see Theorem 3.1 and Corollary 3.1).

All the numerical results are rounded to five decimal digits.

2. Properties of the polynomials $w_p(z)$. The polynomial $w_p(z)$ is the p -th partial sum of the power series of $\exp(z)$. Using this fact we can prove some lemmas.

LEMMA 2.1. *For any p there exists a unique negative number a_p such that*

$$(2.1) \quad w_p(a_p) = (-1)^p.$$

Proof. It is obvious that $w_p(z) > 1$ for $z > 0$. For $z < 0$ the inequalities $e^z < w_p(z)$ for p even and $e^z > w_p(z)$ for p odd hold. Consequently, for p odd the polynomial $w_p(z)$ has exactly one zero α_p and is increasing from $-\infty$ to $+\infty$. The thesis for p odd is thus proved.

For p even the polynomial $w_p(z)$ is decreasing in the interval $(-\infty, \alpha_{p-1})$ and increasing in (α_{p-1}, ∞) . Since $w_p(0) = 1$, there exists a unique negative solution of (2.1).

Lemma 2.1 implies that $I_{p,k} = (ka_p, 0)$ for any p and k . It is possible to obtain an estimation of a_p .

LEMMA 2.2. *For any p we have*

$$(2.2) \quad a_p < -[(p+1)!]^{1/p+1}$$

and

$$-p \leq a_p \quad \text{for } p \geq 2.$$

Proof. The lower estimation follows immediately from Theorem 6.4 of [2]. In order to prove the upper estimation we consider two cases. If p is odd, then according to Lemma 2.1 for any $z < 0$ we have

$$w_{p+1}(z) = \frac{z^{p+1}}{(p+1)!} + w_p(z) > 0;$$

hence

$$w_p(z) > -\frac{z^{p+1}}{(p+1)!} \quad \text{for } z < 0.$$

We observe that for $z \geq -[(p+1)!]^{1/p+1}$ we have $w_p(z) > -1$. This means that (2.2) is valid for p odd. Now we consider the case where p is even. Following Lemma 2.1, in this case there exists a unique zero α_{p+1} of the equation $w_{p+1}(z) = 0$. It is obvious that $a_p < \alpha_{p+1}$. For $z < \alpha_{p+1}$ we have the inequality

$$w_p(z) < -\frac{z^{p+1}}{(p+1)!}.$$

If $z \geq -[(p+1)!]^{1/p+1}$, then $w_p(z) < 1$, which proves (2.2) for p even.

From (2.2) it is easy to see that

$$\lim_{p \rightarrow \infty} a_p = -\infty.$$

Moreover, we can prove that the sequences $\{a_{2l+1}\}$ and $\{a_{2l}\}$ are decreasing.

LEMMA 2.3. For any p we have $a_{p+2} < a_p$.

Proof. By Lemma 2.2, for any $z \in (-(p+2), 0)$ we have

$$\operatorname{sgn}[w_{p+2}(z) - w_p(z)] = \begin{cases} -1 & \text{for } p \text{ even,} \\ 1 & \text{for } p \text{ odd.} \end{cases}$$

This means that for $z \in (-(p+2), 0)$

$$(2.3) \quad w_{p+2}(z) < w_p(z) \quad \text{for } p \text{ even}$$

and

$$(2.4) \quad w_{p+2}(z) > w_p(z) \quad \text{for } p \text{ odd.}$$

Putting $z = a_p$ in (2.3) and (2.4) and using (2.1) we obtain the thesis.

Finally, we obtain a weaker upper estimation of a_p . This result is used in the next section.

LEMMA 2.4. For any $p \geq 5$

$$(2.5) \quad |a_p| > [2p!]^{1/p}.$$

Proof. It is easy to verify by induction that for $p \geq 5$ the inequality

$$\left(\frac{p+1}{2}\right)^p > 2p!$$

holds. Hence we can see that for $p \geq 5$

$$[(p+1)!]^{1/p+1} = [2p!]^{1/p} \left\{ \frac{p+1}{2[2p!]^{1/p}} \right\}^{1/p+1} > [2p!]^{1/p}$$

and (2.5) is proved.

In Table 2.1 we give the values of a_p , $-[(p+1)!]^{1/p+1}$ and $|a_p|^p/p!$ for $1 \leq p \leq 10$.

TABLE 2.1

p	a_p	$-[(p+1)!]^{1/p+1}$	$ a_p ^p/p!$
1	-2.00000	-1.41421	2.00000
2	-2.00000	-1.81712	2.00000
3	-2.51275	-2.21336	2.64420
4	-2.78529	-2.60517	2.50768
5	-3.21705	-2.99380	2.87148
6	-3.55344	-3.38002	2.79616
7	-3.95413	-3.76435	2.99865
8	-4.31363	-4.14717	2.97317
9	-4.70083	-4.52873	3.08891
10	-5.06952	-4.90924	3.08961

3. Behaviour of the polynomials $w_{p,k}^*(z)$. The polynomials $w_{p,k}^*(z)$ in (1.2) have a more complicated form than $w_p(z)$ in (1.1). In our investigations we use the following notice. Namely, for the interval $\hat{I}_{p,k}$ to be shorter than $I_{p,k}$ it is sufficient that

$$(3.1) \quad |w_{p,k}^*(ka_p)| > 1.$$

Transforming this condition we obtain

THEOREM 3.1. *If for p even or p odd and k even we have the inequality*

$$(3.2) \quad 2k^p - 1 < w_p(ka_p)$$

or if for p odd and k odd we have the inequality

$$(3.3) \quad -2k^p + 1 > w_p(ka_p),$$

then (3.1) holds.

Proof. Condition (3.1) means that $w_{p,k}^*(ka_p) > 1$ or $w_{p,k}^*(ka_p) < -1$. Using the form of $w_{p,k}^*(z)$, Lemma 2.1, and excluding the cases which are always or never satisfied, we get (3.2) or (3.3).

TABLE 3.1

$p \backslash k$	2	3	4	5	6	7
1	*	*	*	*	*	*
2					*	*
3	*	*	*	*	*	*
4			*	*	*	*
5	*	*	*	*	*	*
6			*	*	*	*
7	*		*	*	*	*
8			*	*	*	*
9	*		*	*	*	*
10			*	*	*	*

Since $w_p(ka_p)$ may be treated as a polynomial of variable k , Theorem 3.1 implies

COROLLARY 3.1. *Assume that, for some p the inequality $|a_p|^p/p! > 2$ holds. Then there exists an integer k_p such that the interval $\hat{I}_{p,k}$ is shorter than $I_{p,k}$ for any $k > k_p$.*

In Table 3.1 we denote by a star the indices p and k ($1 \leq p \leq 10$, $2 \leq k \leq 7$) for which the interval $\hat{I}_{p,k}$ is really shorter than $I_{p,k}$. The circled stars denote the indices for which the criterion (3.1) gives the true answer to our problem.

It is easy to verify that for any $k \geq 2$ we have $b_{1,k} = -k$. Thus, the inequality $|b_{1,k}| < 2k$ always holds. By Lemma 2.4 and Table 2.1 we see that for any $p \geq 3$ the assumptions of Corollary 3.1 are satisfied. The criterion (3.1) does not work in the case $p = 2$, because

$$w_{2,k}^*(-2k) = -\frac{k-1}{k+1} > -1 \quad \text{for any } k \geq 2.$$

References

- [1] M. Szyszkowicz, *The intervals of stability of Runge-Kutta methods after Richardson extrapolation*, Zastos. Mat. 18 (1985), pp. 657-661.
- [2] A. Turowicz, *Geometria zer wielomianów*, PWN, Warszawa 1967.

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