A fixed point theorem for a class of multi-valued continuously differentiable maps

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Abstract. In this paper, we give a new concept of differential for a multivalued map, a problem which has already been considered by several authors from different points of view (see [1], [2], [6], [8]). Our new concept allows us to obtain some results which are well-known in the single valued case. In particular, we prove a mean-value theorem and a remainder theorem and we find as a special case these same theorems which are known in the single valued context. Finally, we will show that the fixed point index for compositions of acyclic multivalued maps in Banach space, given by G. Fournier and D. Violette [4], is defined for a multivalued map which is continuously differentiable, eventually condensing and has an acyclic decomposition.

1. Notation and preliminaries. Let E and F be two real Banach spaces. In this paper all the multivalued maps $T: E \rightarrow F$ are assumed to satisfy $T(x)$ is compact $\forall x \in E$. T is *upper semi-continuous* (u.s.c.) at $x \in E$ if for any neighbourhood V of $T(x)$, there exists a neighbourhood U of x such that $T(y) \subset V$ for any $y \in U$. T is u.s.c. on E if for each open set $V \subset F$, the set $T^{-1}(V) = \{x \in E \mid T(x) \subset V\}$ is an open subset of E . T is *lower semi-continuous* (l.s.c.) on E if for each open set $V \subset F$, the set $\{x \in E \mid T(x) \cap V \neq \emptyset\}$ is an open subset of E . If T is both u.s.c. and l.s.c., then T is continuous.

PROPOSITION 1.1. *If $T: E \rightarrow F$ is u.s.c. and K a compact subset of E , then the image $T(K) = \bigcup_{x \in K} T(x)$ of K under T is compact.*

Proof. (See [5], Proposition 1.2.) Let $T: E \rightarrow F$ and $T': F \rightarrow F'$ be two multivalued maps. Then the composition of T and T' is a map $T' \circ T: E \rightarrow F'$ defined by $T' \circ T(x) = T'(T(x))$ for each $x \in E$. The composition of u.s.c. is again u.s.c. For a multivalued map $T: E \rightarrow E$, we denote by T^n , $n \geq 1$, the n th iteration of T (i.e., $T^n = \underbrace{T \circ T \circ \dots \circ T}_{n \text{ times}}$); by T^0 , we denote the identity map

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L_E . An u.s.c. map $T: E \rightarrow F$ is said to be acyclic provided that for each $x \in E$, $T(x)$ is acyclic with respect to the Čech homology functor H with rational coefficients, i.e., $T(x)$ is non-empty, $H_0(T(x)) \approx \mathcal{Q}$ and $H_q(T(x)) = 0$ for all $q \geq 1$.

A multivalued map $T: E \rightarrow F$ is called *homogeneous* if $T(tx) = tT(x)$ for any $x \in E$ and $t \in \mathbf{R}$. T is called *semi-linear positive* if

$$T\left(\sum_{i=1}^n t_i x_i\right) \subset \sum_{i=1}^n t_i T(x_i)$$

for all $x_i \in E$, $t_i \geq 0$ and $\sum_{i=1}^n t_i \leq 1$. In this case, $T(0) = 0$ and $T(\text{co } A) \subset \text{co } T(A)$, where $\text{co } A$ is the convex hull of $A \subset E$.

Let $\mathcal{F}(E)$ be the family of all non-empty bounded and closed subsets of E . Let $A, B \in \mathcal{F}(E)$ and put $\varrho(A, B) = \sup\{d(x, B) \mid x \in A\}$, where $d(x, B) = \inf\{\|x - y\| \mid y \in B\}$. Define

$$d_H(A, B) = \sup\{\varrho(A, B), \varrho(B, A)\};$$

d_H is called the *Hausdorff metric* in $\mathcal{F}(E)$. We write $\|A\|$ in place of $d_H(A, 0) = \sup\{\|a\| \mid a \in A\}$. Let $\varepsilon > 0$, $d_H(A, B) < \varepsilon$ if and only if $A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(A)$, where $N_\varepsilon(A) = \{x \mid d(x, A) < \varepsilon\}$.

The following lemma will be used often.

LEMMA 1.2. *Let A, B be subsets of E . If for each $\varepsilon > 0$, $d_H(A, B) < \varepsilon + k$, then $d_H(A, B) \leq k$, where $k > 0$.*

We have the following properties. Let A_i, B_i, C be bounded and closed subsets of E .

$$(1.3) \quad d_H(tA_1, tB_1) = |t| d_H(A_1, B_1), \quad t \in \mathbf{R}.$$

$$(1.4) \quad \text{Let } \varepsilon, \delta > 0, d_H(A, B) < \varepsilon \text{ and } d_H(B, C) < \delta \text{ imply } d_H(A, C) < \varepsilon + \delta.$$

$$(1.5) \quad \text{If } d_H(A_1, B_1) \leq k, \text{ then } d_H(x + A_1, x + B_1) \leq k \text{ for all } x \in E.$$

$$(1.6) \quad \text{If } d_H(A_i, B_i) \leq k \text{ for all } i = 1, \dots, n, \text{ then } d_H\left(\bigcup_{i=1}^n A_i, \bigcup_{i=1}^n B_i\right) \leq k.$$

$$(1.7) \quad \text{If } d_H(A_1, B_1) \leq k, \text{ then } d_H(A_1 + C, B_1 + C) \leq k.$$

Furthermore, for any $\varepsilon, \delta > 0$ we have that

$$(1.8) \quad N_\varepsilon(N_\delta(A)) \subset N_{\varepsilon+\delta}(A).$$

$$(1.9) \quad N_\varepsilon(\text{co } A) = \text{co } N_\varepsilon(A), \text{ where } A \subset E.$$

Throughout this paper, we shall use the notation $A \approx B$ to represent $d_H(A, B) \leq \varepsilon$.

We shall need the following Kuratowski (see [7]) measure of noncompactness: for any $A \subset X$, let $\gamma(A) = \inf\{r > 0 \mid \text{there exists a finite covering of } A \text{ by subsets of diameter at most } r\}$. We have the following properties:

- (1.10) $0 \leq \gamma(A) \leq \delta(A)$, where $\delta(A)$ is the diameter of A .
- (1.11) if $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
- (1.12) $\gamma(A \cup B) = \max \{ \gamma(A), \gamma(B) \}$.
- (1.13) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$.
- (1.14) $\gamma(N_\varepsilon(A)) \leq \gamma(A) + 2\varepsilon$, where $N_\varepsilon(A) = \{x \in X \mid d(x, A) < \varepsilon\}$.
- (1.15) $\gamma(\text{co } A) = \gamma(A)$, where $\text{co } A$ is the closed convex hull of A .
- (1.16) $\gamma(A) = 0$ if and only if A is relatively compact.
- (1.17) If $K_1 \supset K_2 \supset \dots \supset K_n \supset \dots$, where K_n is closed for any n and $\lim_{n \rightarrow \infty} \gamma(K_n) = 0$, then $K_\infty = \bigcap_{n=1}^{\infty} K_n$ is compact and for any neighbourhood V of K_∞ there exists an integer n_V such that $K_n \subset V$ for all $n > n_V$.

DEFINITION 1.18. A multivalued map $F: X \rightarrow Y$, between two metric spaces X and Y , is a k -set contraction if there exists $0 < k < 1$ such that $\forall A \subset X$ we have that

$$\gamma(F(A)) \leq k\gamma(A).$$

If $X \subset Y$, then F is eventually a k -set-contraction if there exists an integer m such that $F^m: F^{m+1}(X) \rightarrow Y$ is a k -set contraction.

DEFINITION 1.19. A selection G of a multivalued map $F: X \rightarrow Y$ is a multivalued map $G: X \rightarrow Y$ such that $\forall x \in X$ we have that $G(x) \subset F(x)$.

2. The fixed point index for compositions of acyclic multivalued maps in Banach spaces. The results of this section are proved in [4] using the results of Sieberg-Skordev [10] and the technics of R. D. Nussbaum in [9].

Let $f = (F_n, \dots, F_0)$ be a sequence of acyclic maps $F_i: X_i \rightarrow X_{i+1}$ such that X_i is a Hausdorff space for any i and $X_0 = U$ is an open subset of the $X_{n+1} = X$. Call f admissible if $\text{Fix}(F) = \{x \in U \mid x \in F(x)\}$ is a compact subset of U , where $F = F_n \circ \dots \circ F_0: U \rightarrow X$. We call f an *acyclic decomposition* for the map F .

Let $\mathcal{F}$ denote the class of locally finite unions $X = \bigcup_{j \in J} C_j$ of closed convex subsets C_j of a Banach space E . Let $\mathcal{F}_0$ be the subclass of $\mathcal{F}$ for which the union is finite, i.e., J is finite.

We may now define our class of maps in Banach spaces for which the fixed point index is defined.

DEFINITION 2.1. Let $f = (F_n, \dots, F_0)$ be an admissible sequence of acyclic maps; then f is compacting if and only if there exists an open set W and a sequence $\{K_n\} \subset \mathcal{F}_0$ such that

$$(2.1.1) \quad \text{Fix}(F) \subset W \subset \bar{W} \subset U, \quad \text{where } F = F_n \circ \dots \circ F_0: U \rightarrow X \text{ and } X \in \mathcal{F};$$

$$(2.1.2) \quad W \subset K_1 \subset X;$$

$$(2.1.3) \quad F(W \cap K_n) \subset K_{n+1} \subset K_n \quad \text{for any } n \geq 1;$$

$$(2.1.4) \quad \lim_{n \rightarrow \infty} \gamma(K_n) = 0.$$

Clearly, if $g = (G_m, \dots, G_0)$ and $G = G_m \circ \dots \circ G_0$ is a selection of F , then if f is compacting so is g . The following for single valued maps is due to R. D. Nussbaum [9] and for compact maps to G. Skordev [12].

THEOREM 2.2. *Assume that $f = (F_n, \dots, F_0)$ is compacting, then the number $\text{ind}(X, f, U)$, the index fixed point of f , is defined.*

Note that the fixed point index is defined for f , not for F : in fact, if $f' = (F'_m, \dots, F'_0)$ is admissible and $F' = F'_m \circ \dots \circ F'_0 = F$ we may still have that $\text{ind}(X, f, U) \neq \text{ind}(X, f', U)$; thus a map F can have two different acyclic decompositions with different indices.

That number has the following properties:

$$(2.3) \quad \text{ind}(X, f, U) = \text{ind}(X(F_n, \dots, F_{i+1}, F_i \circ F_{i-1}, F_{i-2}, \dots, F_0), U) \text{ provided that } F_i \circ F_{i-1} \text{ is acyclic.}$$

The proof of this observation is similar to the proof of Lemma 3.8 in [10].

$$(2.4) \quad (\text{ADDITIVITY}). \text{ Let } U = U_1 \cup U_2, \text{ where } U_1, U_2 \text{ are open subsets of } U, \text{ be such that } \text{Fix}(F) \cap U_1 \cap U_2 = \emptyset; \text{ then}$$

$$\text{ind}(X, f, U) = \text{ind}(X, f, U_1) + \text{ind}(X, f, U_2).$$

$$(2.5) \quad (\text{EXCISION}). \text{ Let } U_1 \text{ be an open subset of } U \text{ such that } \text{Fix}(F) \subset U_1; \text{ then } \text{ind}(X, f, U) = \text{ind}(X, f, U_1).$$

$$(2.6) \quad (\text{FIXED POINT}). \text{ If } \text{ind}(X, f, U) \neq 0, \text{ then } \text{Fix}(F) \neq \emptyset.$$

$$(2.7) \quad (\text{HOMOTOPY}). \text{ Let } h = (H_n, \dots, H_0) \text{ be a sequence of acyclic maps } H_i: X_i \rightarrow X_{i+1}, \text{ where } X_0 = U \text{ an open subset of } X \times I \text{ and } X_{n+1} = X. \text{ If } \bar{h} = (\bar{H}_n, \dots, \bar{H}_0) \text{ is compacting, then } \text{ind}(X \times I, \bar{h}, U) = \text{ind}(X, h_t, U_t) \text{ for all } t \in I, \text{ where } \bar{H}_0: U \rightarrow X_1 \times I \text{ is given by } \bar{H}_0(x, t) = (H_0(x, t), t), \text{ where } \bar{H}_i: X_i \times I \rightarrow X_{i+1} \times I \text{ is given by } \bar{H}_i(x, t) = (H_i(x, t), t), \text{ where } U_t = \{x \in X \mid (x, t) \in U\} \text{ and } h_t = (H_n, \dots, H_1, H_0).$$

$$(2.8) \quad (\text{COMMUTATIVITY}). \text{ Let } f = (F_n, \dots, F_0) \text{ and } g = (G_n, \dots, G_0) \text{ be sequences of acyclic maps such that } F = F_n \circ \dots \circ F_0: U \rightarrow Y \text{ and } G = G_n \circ \dots \circ G_0: V \rightarrow X, \text{ where } U \text{ is an open subset of } X \in \mathcal{F} \text{ and } V \text{ is an open subset of } Y \in \mathcal{F}. \text{ Assume that for all } x \in U \text{ and all } y \in V,$$

$$(2.8.1) \quad \text{if } y \in F(x) \text{ and } x \in G(y) \text{ and } (y \in G^{-1}(U) \text{ or } x \in F^{-1}(V)) \text{ then } y \in G^{-1}(U) \text{ and } x \in F^{-1}(V).$$

Then $\text{Fix}(G \circ F|_{F^{-1}(V)})$ is compact if and only if $\text{Fix}(F \circ G|_{G^{-1}(U)})$ is. If

one is compact and there exists U_1 open subset of U , V_1 open subset of V and a sequence $\{K_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_0$ such that

$$(2.8.2) \quad G(\text{Fix}(F \circ G|_{G^{-1}(U)})) \subset U_1 \subset K_1 \subset X \quad \text{and} \quad F(\text{Fix}(G \circ F|_{F^{-1}(V)})) \subset V_1 \subset K_2 \subset Y;$$

$$(2.8.3) \quad F(U_1 \cap K_{2n+1}) \subset K_{2n+2} \subset K_{2n} \subset Y \text{ for all } n \geq 0 \text{ and } G(V_1 \cap K_{2n}) \subset K_{2n+1} \subset K_{2n-1} \subset X \text{ for all } n \geq 1, \text{ where } K_0 = Y;$$

$$(2.8.4) \quad \lim_{n \rightarrow \infty} \gamma(K_n) = 0; \text{ then } \text{ind}(X, gf, F^{-1}(V)) = \text{ind}(Y, fg, G^{-1}(U)).$$

(2.9) (MODULO- p). Let p be a prime number and let $f = (F_n, \dots, F_0)$ be compacting. Let $V \subset F^{-p+1}(U)$, where $F = F_n \circ \dots \circ F_0$, be such that:

$$(2.9.1) \quad \text{if } y \in F^i(x), x \in F^{p-i}(y), 1 \leq i < p \text{ and } x \in V, \text{ then } y \in V. \text{ Assume that } \text{Fix}(F^p|_V) \text{ is compact and that } F^{p-1}(\text{Fix}(F^p|_W)) \subset W. \text{ Then } \text{ind}(X, f, V) \equiv \text{ind}(X, f^p, V) \pmod{p}.$$

Note that in these last two properties and in the following, we use the notations: $fg = (F_n, \dots, F_0, G_m, \dots, G_0)$, where $f = (F_n, \dots, F_0)$ and $g = (G_m, \dots, G_0)$; $f^p = \underbrace{f \circ \dots \circ f}_{p \text{ times}}$ and $F = (F)$ if F is acyclic, in particular if F is

single valued. Furthermore, notice that (2.8.1) is trivially satisfied if F is single valued and $G(V) \subset U$, or if both F and G are single valued.

Before giving our normalization property, we need to recall the definition and properties of the Lefschetz number of a composition of acyclic maps and also of such maps of compact attraction.

For our purposes, we shall need the Čech homology functor with compact carriers with coefficients in the rational numbers, defined by L. Górniewicz [5], which we shall still denote by H .

As in (1.4), for a sequence $f = (F_n, \dots, F_0)$ of acyclic maps $F_i: X_i \rightarrow X_{i+1}$ between Hausdorff spaces, we define $f_* = F_{n*} \circ \dots \circ F_{0*}$, where $F_{i*} = q_{i*} \circ p_{i*}^{-1}$ with notations $q_i: \Gamma(F_i) \rightarrow X_{i+1}$ and $p_i: \Gamma(F_i) \rightarrow X_i$ for the projections of the graph $\Gamma(F_i)$ of the map F_i . This is possible since p_i being a Vietoris map is invertible (see Theorem 1.2 [5]).

For such a sequence f , if $X = X_0 = X_{n+1}$, denote $N(f_{*q}) = \{a \in H_q(X) \mid \text{there exists } n > 0 \text{ such that } f_{*q}^n(a) = 0\}$ and $\tilde{H}_q(X) = H_q(X)/N(f_{*q})$. Since $f_{*q}^{-1}(N(f_{*q})) = N(f_{*q})$, f_{*q} induces $\tilde{f}_{*q}: \tilde{H}_q(X) \rightarrow \tilde{H}_q(X)$. Let us define $\text{Tr}(f_{*q}) = \text{tr}(\tilde{f}_{*q})$ provided that $\tilde{H}_q(X)$ is finite dimensional. Notice that if $H_q(X)$ is finite dimensional, $\text{Tr}(f_{*q}) = \text{tr}(f_{*q})$. Finally if $X = X_0 = X_{n+1}$, all $\tilde{H}_q(x)$ are finite dimensional and all but a finite number are zero, then the Lefschetz number of f is defined to be

$$\Lambda(f) = \Lambda(f_*: H(X) \rightarrow H(X)) = \sum_{q=0}^{\infty} (-1)^q \text{Tr}(f_{*q}: H_q(X) \rightarrow H_q(X)).$$

If f is homotopic to g in the sense of (2.7) with $U = X$, then $\Lambda(f)$

$= \Lambda(g)$. Furthermore, if $f = hg$, then by the commutativity property of the trace, we have that $\Lambda(f) = \Lambda(gh)$.

DEFINITION 2.10. Let $F: X \rightarrow X$ be u.s.c., where X is Hausdorff. A set $A \subset X$ is called *F-invariant* if $F(A) \subset A$. The map F is of compact attraction if and if F has a compact attractor $M \subset X$ if and only if there exists M a compact F -invariant subset of X such that for any open neighbourhood V of M , $V \subset X$ and any compact subset K of X , there exists n_K such that $F^n(K) \subset V$ for any $n \geq n_K$.

(2.11) (NORMALIZATION) Let $f = (F_n, \dots, F_0)$ be compacting. Assume that $F(U) \subset U$ and that $F = F_n \circ \dots \circ F_0: U \rightarrow U$ has a compact attractor M such that $M \subset W$. Then $\inf(X, f, U) = \Lambda(f_U)$, where $f_U = (G_n, \dots, G_0)$ with $G_0: U \rightarrow (F_n \circ \dots \circ F_1)^{-1}(U)$ and $G_i: (F_n, \dots, F_i)^{-1}(U) \rightarrow (F_n \circ \dots \circ F_{i+1})^{-1}(U)$, $i = 1, \dots, n-1$ and $G_n = F_n^{-1}(U) \rightarrow U$ which are contractions of the F_i , $i = 0, \dots, n$.

(2.12) (LEFSCHETZ THEOREM) Let $f = (F_n, \dots, F_0)$ be compacting. Assume that $F(U) \subset U$ and that $F: U \rightarrow U$ has a compact attractor M such that $M \subset W$, then $\Lambda(f_U)$ is defined and $\Lambda(f_U) \neq 0$ implies that $F = F_n \circ \dots \circ F_0$ has a fixed point in U .

3. Differentiable multivalued maps. Let E, F be real Banach spaces and U an open subset of E .

DEFINITION 3.1. Let $T: U \rightarrow F$ be a multivalued map. Assume that $T(x)$ is compact for every $x \in U$. We say that T is *differentiable at* $x \in U$ if there exists an u.s.c. multivalued map $S_x: T(x) \times E \rightarrow F$ such that the map

$S_{x,z}: E \rightarrow F$ is homogeneous and semi-linear positive and such that for any $\varepsilon > 0$, there exists $\delta(\varepsilon, x) > 0$ such that if $\|h\| < \delta$, then

$$T(x+h) \sim \bigcup_{\varepsilon \|h\| \leq z \in T(x)} (z + S_x(z, h)).$$

The map S_x is called a *differential of T at x*. If T is differentiable at every point of U , then T is said to be differentiable on U .

We give now some examples of differentiable multivalued maps.

(i) A constant multivalued map is differentiable at every point and one of its differentials is the zero map.

(ii) Let $T: \mathbb{R} \times \mathbb{R}$ be defined by $T(x) = [0, 1]$ for all $x \in \mathbb{R}$. T is differentiable at 0 and the maps $S_0, S'_0: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ defined respectively by $S_0(z, h) = 0$ and $S'_0(z, h) = z(z-1)h$, $z \in [0, 1]$, $h \in \mathbb{R}$ are both differentials of T at 0.

This example shows that the differential at a point is not necessarily unique.

(iii) Let $T: E \rightarrow F$ be defined by $T(x) = \text{co}(T_1(x), \dots, T_n(x))$, where, for each $i = 1, \dots, n$, the map $T_i: E \rightarrow F$ is a single valued map differentiable in U , an open subset of E . Then T is differentiable in U and the map $S_x: T(x) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$S_x(z, h) = \left\{ \sum_{i=1}^n \alpha_i T_i'(x)(h) \mid \sum_{i=1}^n \alpha_i T_i(x) = z \text{ and } \sum_{i=1}^n \alpha_i = 1 \right. \\ \left. \text{and } \alpha_i \geq 0, i = 1, \dots, n \right\}$$

is a differential of T at x in U .

Remark 3.2. If T_i in Definition 3.1, is single valued, then T is differentiable at $x \in U$ if there exists a continuous linear map $DT(x): E \rightarrow F$ such that for any $\varepsilon > 0$ there exists $\delta(\varepsilon, x) > 0$ such that if $\|h\| < \delta$, then

$$\|T(x+h) - T(x) - DT(x)(h)\| \leq \varepsilon \|h\|.$$

This is exactly the definition of a differentiable single valued map.

Let $T, L: E \rightarrow F$ be two u.s.c. homogeneous multivalued maps. Put $\bar{d}(T, L) = \sup_{\|h\|=1} d_H(T(h), L(h))$, then d is a metric on the space of all u.s.c. homogeneous multivalued maps from E to F .

DEFINITION 3.3. Let $T: U \rightarrow F$ be a differentiable multivalued map on U . We say that T is *continuously differentiable on U* if there exists a map $(x, z) \mapsto S_{x,z}$ such that S_x is a differential of T at x and the map is continuous on the set $\bigcup_{x \in U} \{x \times T(x)\}$, that is for any $x \in U$ and $z \in T(x)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \delta$ and $\|z - z'\| < \delta$ imply that $\bar{d}(S_{x,z}, S_{y,z'}) < \varepsilon$ for all $y \in U$ and for all $z' \in T(y)$.

Remark 3.4. If T_i in Definition 3.3, is single valued, then T is continuously differentiable on U if the map $x \mapsto DT(x)$ from U into $\mathcal{L}(E, F)$ is continuous on U , where $\mathcal{L}(E, F)$ is the space of all linear maps from E to F . This is exactly the definition of a continuously differentiable single valued map.

Remark 3.5. The previous examples are examples of such maps provided that in Example 3 we ask further that the T_i are continuously differentiable and that the $T_i(x)$ remain always affinely independent.

4. Properties of differentiable multivalued maps. In this section, several properties of differentiable multivalued maps are given. For $A \subset F$ let us denote $\|A\| = \sup_{a \in A} \|a\|$.

LEMMA 4.1. Let $T: U \rightarrow F$ be a multivalued map differentiable at $x \in U$. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ such that $\|S_x(z, h)\| < \varepsilon$ for $\|h\| < \delta$ and for all $z \in T(x)$, where $S_x: T(x) \times E \rightarrow F$ is a differential of T at x .

Proof. Since S_x is u.s.c. at $(z, 0) \in T(x) \times E$, then for any $\varepsilon > 0$, there exists a neighbourhood V_z of $(z, 0)$ such that

$$S_x(z', h) \subset N_\varepsilon(S_x(z, 0)) = N_\varepsilon(0)$$

if $(z', h) \in V_z$. Let $r_z > 0$ and $\delta_z > 0$ such that $N_{r_z}(z) \times N_{\delta_z}(0) \subset V_z$. Then $S_x(z', h) \subset N_\varepsilon(0)$ if $z' \in N_{r_z}(z)$ and $\|h\| < \delta_z$. But $T(x)$ is compact for all $x \in U$ and so there exists $z_i \in T(x)$ such that

$$T(x) \subset \bigcup_{i=1}^n N_{r_{z_i}}(z_i).$$

Put $\delta = \min \{\delta_{z_1}, \dots, \delta_{z_n}\}$ to get the conclusion.

COROLLARY 4.2. *Under the same conditions of Lemma 4.1, there exists $\delta > 0$ and there exists $k > 0$ such that $\|S_x(z, h)\| \leq k\|h\|$ for $\|h\| < \delta$ and for all $z \in T(x)$.*

Proof. By Lemma 4.1, there exists $\delta > 0$ such that $\|S_x(z, h)\| \leq 1$ for $\|h\| < \delta$ and for all $z \in T(x)$. Let h be such that $0 \leq \|h\| < \delta/2$; then there exists S , $0 \leq S < 1$, such that $h = Sh'$ with $\|h'\| = \delta/2$. Thus, by the positive homogeneity of $S_{x,z}$, we get that

$$\|S_x(z, h)\| \leq \|S S_x(z, h')\| = S \frac{2\|h'\|}{\delta} = \frac{2\|h\|}{\delta} = k\|h\|$$

for all $z \in T(x)$, where $k = 2/\delta$.

THEOREM 4.3. *If $T: U \rightarrow F$ is a multivalued map differentiable at $x \in U$, then T is continuous at that point.*

Proof. Let us first show that T is u.s.c. at x . Let V an open set be such that $T(x) \subset V$. Since $T(x)$ is compact then $d = d(T(x), C^V) > 0$. Put $r = d/2$, by Lemma 4.1, there exists $\delta > 0$ such that $\|S_x(z, h)\| < r$ for all $\|h\| < \delta$ and for all $z \in T(x)$. On the other hand, there exists $\delta_r > 0$ such that $\|h\| < \delta_r$ implies

$$T(x+h) \subset N_{r\|h\|} \left(\bigcup_{z \in T(x)} (z + S_x(z, h)) \right).$$

Put $t = \min \{\delta, \delta_r, 1\}$, then if $\|h\| < t$

$$\begin{aligned} T(x+h) &\subset N_r \left(\bigcup_{z \in T(x)} (z + S_x(z, h)) \right) \\ &\subset N_r(T(x) + N_r(0)) \subset N_r(N_r(T(x))) \subset N_d(T(x)) \subset V. \end{aligned}$$

If $y \in N_t(x) = W$, then $T(y) \subset V$ and we get the conclusion.

Using a similar technique, we can show that T is l.s.c. at x and so we have the conclusion.

Let $T: U \rightarrow F$ a multivalued map, let $x \in U$ and let $S: T(x) \times E \rightarrow F$ be u.s.c. homogeneous in the last variable. Put $\|S\| = \inf \{k > 0\}$

$\|S(z, h)\| \leq k \|h\|$, $z \in T(x)$, $h \in E$ (see Corollary 4.2), then we have the following lemma.

LEMMA 4.4. $\|S\| = \sup_{z \in T(x)} \sup_{\|h\|=1} \|S(z, h)\|$, and so

$$\|S(z, h)\| \leq \|S\| \|h\| \quad \forall h \in E \quad \text{and} \quad \forall z \in T(x).$$

Proof. Put

$$k' = \sup_{z \in T(x)} \sup_{\|h\|=1} \|S(z, h)\|, \quad \left\| \frac{S(z, h)}{\|h\|} \right\| = \left\| S\left(z, \frac{h}{\|h\|}\right) \right\| \leq k',$$

that is $\|S(z, h)\| \leq k' \|h\|$ for all $z \in T(x)$ and $h \neq 0$ and then $\|S\| \leq k'$. On the other hand, for each $k > \|S\|$, $\|h\| = 1$ and $z \in T(x)$, $\|S(z, h)\| \leq k$, thus $\sup_{z \in T(x)} \sup_{\|h\|=1} \|S(z, h)\| < k$, i.e., $k' \leq k$ and hence $k' \leq \|S\|$.

Moreover, $\|S\|$ is a norm on the set of all homogeneous maps $S: T(x) \times E \rightarrow F$: in fact, if $S = 0$, then, by Lemma 4.4, $\|S\| = 0$ and inversely if $\|S\| = 0$, then $S(z, h) = 0$ for all $z \in T(x)$ and for $\|h\| = 1$, thus for any $h \neq 0$ in E , $S(z, h) = \|h\| S(z, h/\|h\|) = 0$, i.e., $S = 0$. Also, $\|\lambda S\| = |\lambda| \|S\|$ for all λ . Finally, if $S = \tilde{S} + \bar{S}$, then $\|S(z, h)\| \leq \|\tilde{S}(z, h)\| + \|\bar{S}(z, h)\|$ and, by Lemma 4.4, $\|S\| \leq \|\tilde{S}\| + \|\bar{S}\|$.

5. A mean-value theorem and a remainder theorem. In this section, we give a mean-value theorem and a remainder theorem and we find, for the single valued case, that these are the classical ones.

THEOREM 5.1 (MEAN-VALUE THEOREM). Let $T: U \rightarrow F$ be a differentiable multivalued map on $J = \text{co}(x, x+h) \subset U$ when x and $x+h$ are points in E . Then

$$T(x+h) \sim_{\|h\| k'} T(x), \quad \text{where } k' = \sup_{y \in J} \|S_y\|$$

and $y \mapsto S_y$ is a map such that S_y is a differential of T at y .

Proof. The case $\|h\| = 0$ is trivial. Assume $\|h\| > 0$. Since T is differentiable at each point y of J , then for each $\varepsilon > 0$, there exists $\delta_y > 0$ such that $\|h\| < \delta_y$ implies

$$T(y+h) \sim_{\varepsilon/2 \|h\|} \bigcup_{z \in T(y)} (z + S_y(z, h)),$$

where $S_y: T(y) \times E \rightarrow F$ is a differential of T at y . By the definition of $\|S_y\|$ we have that $\|S_y(z, h)\| \leq k_y \|h\|$ for all $h \in E$ and for all $z \in T(y)$, where $k_y = \frac{1}{2} \varepsilon + \|S_y\|$. Then

$$(5.1.1) \quad T(y+h) \sim_{(\varepsilon + k') \|h\|} T(y) \quad \text{for } \|h\| < \delta_y.$$

Let

$A = \{t_n \in [0, 1] \mid \text{there exists } 0 = t_0 < t_1 < \dots < t_n \text{ such that}$

$$T(x+t_i h) \sim_{(\varepsilon + k') \|h\| (t_i - t_{i-1})} T(x+t_{i-1} h), \quad i = 1, \dots, n\}.$$

A is non-empty since $0 \in A$. Put $b = \sup A$ then $b \in A$ and $b = 1$. In fact, for $\|th\| < \delta_{x+bh}$ we have that

$$T(x + (b+t)h) \underset{(\varepsilon+k')\|t\|\|h\|}{\sim} T(x+bh).$$

Let $t_n \in A$ be such that $\|(t_n - b)h\| < \delta_{x+bh}$; then $T(x + t_n h) \underset{(\varepsilon+k')\|t_n - b\|\|h\|}{\sim} T(x+bh)$ and hence $b \in A$ since the sequence $\{t_0, \dots, t_n, b\}$ satisfies the conditions. Finally, assume that $b < 1$ and let t' be such that $b < t' < \min\{1, b + \delta_{x+bh}/2\|h\|\}$. Since $\|(t' - b)h\| < \delta_{x+bh}$, then $T(x + t'h) \underset{(\varepsilon+k')\|t' - b\|\|h\|}{\sim} T(x+bh)$ and so $t' \in A$ because the sequence $\{t_0, \dots, b, t'\}$ satisfies the conditions this contradicts the definition of b , $b = 1$ and $A = [0, 1]$. Consequently, there exists a sequence $\{t_0, t_1, \dots, t_n\}$ such that $0 = t_0 < t_1 < \dots < t_n = 1$ and

$$T(x) \underset{(\varepsilon+k')t_1\|h\|}{\sim} T(x+t_1h) \underset{(\varepsilon+k')(t_2-t_1)\|h\|}{\sim} T(x+t_2h) \sim \dots \underset{(\varepsilon+k')(t_n-t_{n-1})\|h\|}{\sim} T(x+h).$$

By (2.4), we get that $T(x+h) \underset{(\varepsilon+k')\|h\|}{\sim} T(x)$ since $\sum_{i=1}^n (t_i - t_{i-1}) = 1$. But because ε is arbitrary, $T(x+h) \underset{k'\|h\|}{\sim} T(x)$ by (2.2).

The following property is an immediate consequence of the mean-value theorem.

COROLLARY 5.2. *If $T: U \rightarrow F$ is a differentiable multivalued map on $J = \text{co}(0, b) \subset U$ such that $S_x = 0$ for all $x \in J$, where S_x is a differential of T at x , then T is constant on J , i.e., $T(x) = A \subset F$ for all $x \in J$.*

Remark 5.3. If T , in (5.1), is single valued, then we get the following conclusion:

$$\|T(x+h) - T(x)\| \leq \|h\| \sup_{0 \leq t \leq 1} \|T'(x+th)\|.$$

This is exactly the mean-value theorem for a single valued map (see [3]).

THEOREM 5.4 (REMAINDER THEOREM). *Let $T: U \rightarrow F$ be a differentiable multivalued map on $J = \text{co}(x, x+h) \subset U$, where x and $x+h$ are points in E . Then*

$$T(x+h) \underset{\|h\|k_{x,h}}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h)),$$

where $k_{x,h} = \inf_{r>0} k_{x,h,r}$ with $k_{x,h,r} = \sup_{(y,z,z') \in A_{J,r}} \bar{d}(S_{x,z}, S_{y,z'})$,

$$A_{J,r} = \{(y, z, z') \mid z \in T(x), z' \in T(y), y \in J, \|z - z'\| < (k_J + r)\|h\|\}$$

and $k_J = \sup_{y \in J} \|S_y\|$.

Proof. The case $\|h\| = 0$ is trivial, then assume $\|h\| > 0$. Let $\varepsilon < r$, since

T is differentiable at each point y of J , there exists $\delta_y > 0$ such that $\|y - y'\| < \delta_y$ implies

$$(5.4.1) \quad T(y') \underset{\varepsilon \|y - y'\|}{\sim} \bigcup_{z \in T(y)} (z + S_y(z, y' - y)),$$

where $S_y: T(y) \times E \rightarrow F$ is a differential of T at y . Let

$A = \{t_n \in [0, 1] \mid \text{there exists } 0 = t_0 < t_1 < \dots < t_n \text{ such that if}$

$y_i = x + t_i h, i = 0, \dots, n-1$, then

$$(a) \quad T(y_i) \underset{\varepsilon |t_i - t_{i+1}| \|h\|}{\sim} \bigcup_{z' \in T(y_{i+1})} (z' + S_{y_{i+1}}(z', (t_i - t_{i+1})h)) \quad \text{or}$$

$$(b) \quad T(y_{i+1}) \underset{\varepsilon (t_{i+1} - t_i) \|h\|}{\sim} \bigcup_{z \in T(y_i)} (z + S_{y_i}(z, (t_{i+1} - t_i)h))\}.$$

A is non-empty since $0 \in A$. Put $S = \sup A$ then $S \in A$ and $S = 1$.

This is proven in a similar manner as in (5.1). Then we can consider a sequence $\{t_0, t_1, \dots, t_n\}$ such that $0 = t_0 < t_1 < \dots < t_n = 1$ and if $y_i = x + t_i h, i = 0, \dots, n-1$, we have either (a) or either (b).

Define $f_i: T(y_i) \rightarrow T(y_{i+1})$ by

$$f_i(z) = \begin{cases} \{z' \in T(y_{i+1}) \mid d(z, z' + S_{y_{i+1}}(z', (t_i - t_{i+1})h)) \leq \varepsilon |t_i - t_{i+1}| \|h\|\} & \text{if we have case (a),} \\ \{z' \in T(y_{i+1}) \mid d(z + S_{y_i}(z, (t_{i+1} - t_i)h), z') \leq \varepsilon (t_{i+1} - t_i) \|h\|\} & \text{if we have case (b).} \end{cases}$$

Then $f_i(z)$ is non-empty. In case (a),

$$\begin{aligned} d_H(z, f_i(z)) &\leq |t_i - t_{i+1}| \|S_{y_{i+1}}\| \|h\| + \varepsilon |t_i - t_{i+1}| \|h\| \\ &= |t_i - t_{i+1}| \|h\| (\|S_{y_{i+1}}\| + \varepsilon) \\ &\leq |t_i - t_{i+1}| \|h\| (k_J + r) \quad \text{since } \varepsilon < r. \end{aligned}$$

In case (b), we obtain in the same manner that

$$d_H(z, f_i(z)) \leq (t_{i+1} - t_i) \|h\| (k_J + r).$$

It is easy to see that $\bigcup_{z \in T(y_i)} f_i(z) = T(y_{i+1})$ for any $i = 0, \dots, n-1$. So by

induction on j , we have that

$$d_H(z, f_j \circ \dots \circ f_i(z)) \leq (t_{j+1} - t_j) \|h\| (k_J + r)$$

for $0 \leq i \leq j < n$. That is if $z \in T(x)$ and $z' \in f_i \circ \dots \circ f_1(z)$, then $(y, z, z') \in A_J$, and $\|z - z'\| \leq \|h\| (k_J + r)$. Finally, we show that

$$T(x+h) = T(y_n) \underset{\|h\| (k_{x,h,r} + \varepsilon)}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h)).$$

If $z_n \in T(y_n)$, then $z_n \in \bigcup_{z_{n-1} \in T(y_{n-1})} f_{n-1}(z_{n-1})$ and there exists $z_{n-1} \in T(y_{n-1})$

such that $z_n \in f_{n-1}(z_{n-1})$. Consequently,

$$d(z_{n-1}, z_n + S_{y_n}(z_n, (t_{n-1} - t_n)h)) \leq \varepsilon |t_{n-1} - t_n| \|h\|$$

in case (a) or

$$d(z_n, z_{n-1} + S_{y_{n-1}}(z_{n-1}, (t_n - t_{n-1})h)) \leq \varepsilon (t_n - t_{n-1}) \|h\|$$

in case (b). We repeat this process n times to obtain a sequence $\{z_0, z_1, \dots, z_{n-1}\}$ such that $z_{i+1} \in T(y_{i+1}) \cap f_i(z_i)$ and such that

$$d(z_i, z_{i+1} + S_{y_{i+1}}(z_{i+1}, (t_i - t_{i+1})h)) \leq \varepsilon |t_i - t_{i+1}| \|h\|$$

in case (a) or

$$d(z_{i+1}, z_i + S_{y_i}(z_i, (t_{i+1} - t_i)h)) \leq \varepsilon (t_{i+1} - t_i) \|h\|$$

in case (b) for $i = 0, \dots, n-1$, and $z_0 \in T(x) = T(y_0)$. But,

$$\begin{aligned} d_H(S_{y_{i+1}}(z_{i+1}, (t_i - t_{i+1})h), S_x(z_0, (t_i - t_{i+1})h)) \\ = |t_i - t_{i+1}| \|h\| d_H(S_{y_{i+1}}(z_{i+1}, h/\|h\|), S_x(z_0, h/\|h\|)) \end{aligned}$$

and

$$\begin{aligned} d_H(S_{y_i}(z_{i+1}, (t_{i+1} - t_i)h), S_x(z_0, (t_{i+1} - t_i)h)) \\ = |t_{i+1} - t_i| \|h\| d_H(S_{y_i}(z_{i+1}, h/\|h\|), S_x(z_0, h/\|h\|)), \end{aligned}$$

since the differential is homogeneous in the last variable and by (2.3).

On the other hand, since $z_{i+1} \in f_i(z_i)$ for any $i = 0, \dots, n-1$, then $z_{i+1} \in f_i \circ \dots \circ f_0(z_0)$ and so $(y_{i+1}, z_0, z_{i+1}) \in A_{J_r}$ and thus

$$d_H(S_{y_{i+1}}(z_{i+1}, h/\|h\|), S_x(z_0, h/\|h\|)) \leq k_{x,h,r}.$$

Then,

$$d_H(S_{y_{i+1}}(z_{i+1}, (t_i - t_{i+1})h), S_x(z_0, (t_i - t_{i+1})h)) \leq |t_i - t_{i+1}| \|h\| k_{x,h,r}$$

and

$$d_H(z_i, z_{i+1} + S_x(z_0, (t_i - t_{i+1})h)) \leq (\varepsilon + k_{x,h,r})(|t_i - t_{i+1}| \|h\|)$$

in case (a).

In case (b), we obtain in the same manner that

$$d_H(z_{i+1}, z_i + S_x(z_0, (t_{i+1} - t_i)h)) \leq (\varepsilon + k_{x,h,r})(t_{i+1} - t_i) \|h\|.$$

But, by the homogeneity of the differential, we get that, in case (a),

$$d_H(z_{i+1}, z_i + S_x(z_0, (t_{i+1} - t_i)h)) \leq (\varepsilon + k_{x,h,r})(t_{i+1} - t_i) \|h\|.$$

Consequently, by (2.4) and (2.7),

$$(5.3.1) \quad d_H(z_n, z_0 + \sum_{i=0}^{n-1} (t_{i+1} - t_i) S_x(z_0, h)) \leq (\varepsilon + k_{x,h,r}) \|h\| \sum_{i=0}^{n-1} (t_{i+1} - t_i)$$

and since $\sum_{i=0}^{n-1} (t_{i+1} - t_i) = 1$, and z_n is a single point, we get that

$$(5.3.2) \quad d_H(z_n, z_0 + S_x(z_0, h)) \leq (\varepsilon + k_{x,h,r}) \|h\|.$$

Thus,

$$T(x+h) \subset N_{(\varepsilon + k_{x,h,r}) \|h\|} \left(\bigcup_{z \in T(x)} (z + S_x(z, h)) \right).$$

For the inclusion $\bigcup_{z \in T(x)} (z + S_x(z, h)) \subset N_{(\varepsilon + k_{x,h,r}) \|h\|} (T(x+h))$, take $z_0 \in T(x)$ and choose $z_1, \dots, z_n$ satisfying the same conditions as above, so we still have (5.3.2), that is our inclusion. Thus

$$T(x+h) \underset{(\varepsilon + k_{x,h,r}) \|h\|}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h)).$$

By (2.2), we get the conclusion.

Remark 5.5. If T , in (5.4), is single valued, then we have the following conclusion:

$$\|T(x+h) - T(x) - DT(x)(h)\| \leq \|h\| \sup_{y \in J} \|DT(x) - DT(y)\|.$$

This is exactly the remainder theorem for a single valued map.

Remark 5.6. It is possible to delete the property of homogeneity on the definition of the differential and find all the previous theorems including a remainder theorem. But in this case, the term $k_{x,h}$ does not necessarily approach zero if h does; however, this property is required later for continuously differentiable map. In order to prove this result, we need the following lemmas.

LEMMA 5.7. Let $T: U \rightarrow F$ be a continuously differentiable multivalued map on U . Let $S_{x,z}$ satisfy the condition rendering T continuously differentiable on U . Then, for any $x \in U$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\bar{d}(S_{x,z}, S_{y,z'}) < \varepsilon$ for $\|x - y\| < \delta$, $\|z - z'\| < \delta$ for all $y \in U$, for all $z \in T(x)$ and all $z' \in T(y)$.

Proof. Let $x \in U$ and $z \in T(x)$; since T is continuously differentiable on U ; for any $\varepsilon > 0$ there exists $\delta_z > 0$ such that $\|x - y\| < 2\delta_z$ and $\|z - z'\| < 2\delta_z$ imply $\bar{d}(S_{x,z}, S_{y,z'}) < \varepsilon/2$ for all $y \in U$ and all $z' \in T(y)$. But $T(x)$ is compact and $T(x) \subset \bigcup_{z \in T(x)} N_{\delta_z}(z)$, so there exists $z_i \in T(x)$ such that $T(x) \subset \bigcup_{i=1}^n N_{\delta_{z_i}}(z_i)$. Let δ' be the Lebesgue number of that converging of $T(x)$. Put $\delta = \min \{\delta_{z_1}, \dots, \delta_{z_n}, \delta'\}$, then for $\|x - y\| < \delta$ and $\|z - z'\| < \delta$, we let that $z, z' \in N_{\delta_{z_i}}(z_i)$ for some i and so

$$\bar{d}(S_{x,z}, S_{y,z'}) \leq \bar{d}(S_{x,z}, S_{x,z_i}) + \bar{d}(S_{x,z_i}, S_{y,z'}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $y \in U$, all $z \in T(x)$ and all $z' \in T(y)$.

COROLLARY 5.8. *Let $T: U \rightarrow F$ have a continuous differential $S_{x,z}$ multi-valued map on U . Then, for any $\varepsilon > 0$, any $x \in U$ and any $r > 0$, there exists $\delta > 0$ such that*

$$\sup_{(y,z,z') \in A_{x,\delta,r}} \bar{d}(S_{x,z}, S_{y,z'}) < \varepsilon,$$

where

$$A_{x,\delta,r} = \{(y, z, z') \mid \|x - y\| < \delta \text{ and } \|z - z'\| \leq k_{x,r} \|x - y\| \text{ and}$$

$$z \in T(x) \text{ and } z' \in T(y) \text{ and } y \in U \text{ and } k_{x,r} = \|S_x\| + r\}.$$

Proof. By (5.7), for any $\varepsilon > 0$, any $x \in U$ there exists $\delta' > 0$ such that if $\|x - y\| < \delta'$ and $\|z - z'\| < \delta$, then $\bar{d}(S_{x,z}, S_{y,z'}) < \varepsilon/2$, for all $y \in U$, all $z \in T(x)$ and all $z' \in T(y)$. Put $\delta = \min\{\delta', \delta'/k_{x,r}\}$ to get the conclusion.

PROPOSITION 5.9. *Let $T: U \rightarrow F$ be a continuously differentiable multi-valued map on U with $S_{x,z}$ as a continuous differential. Then the map $x \mapsto \|S_x\|$ is continuous on U .*

Proof. Let $x \in U$; since T is differentiable on U , there exists $\delta > 0$ such that

$$T(x+h) \sim \bigcup_{\|h\| \leq \delta} (z + S_x(z, h)) \quad \text{for } \|h\| \leq \delta.$$

But

$$\bigcup_{z \in T(x)} (z + S_x(z, h)) \sim \bigcup_{\|h\| \leq \sup_{z \in T(x)} \bar{d}(S_x, z^0)} T(x).$$

Then,

$$(5.9.1) \quad T(x+h) \sim_{L\|h\|} T(x) \quad \text{for } \|h\| < \delta, \text{ where } L = 1 + \sup_{z \in T(x)} \bar{d}(S_{x,z}, 0).$$

Let $\varepsilon > 0$, by (5.7), there exists $\delta' > 0$ such that if $\|h\| < \delta'$ and $\|z - z'\| < \delta'$, then $\bar{d}(S_{x,z}, S_{x+h,z'}) < \varepsilon$ for all $z \in T(x)$ and all $z' \in T(x+h)$. Put $s = \min(\delta, \delta', \delta'/L)$ by (5.9.1), for all $z \in T(x)$ there exists $z' \in T(x+h)$ such that $\|z - z'\| \leq L\|h\| < \delta'$ if $\|h\| < s$ and then

$$\sup_{\|h'\|=1} \|S_x(z, h')\| \leq \sup_{\|h'\|=1} \|S_{x+h}(z', h')\| + \varepsilon.$$

Thus,

$$\|S_x\| = \sup_{z \in T(x)} \sup_{\|h'\|=1} \|S_x(z, h')\| \leq \sup_{z' \in T(x+h)} \sup_{\|h'\|=1} \|S_{x+h}(z', h')\| + \varepsilon$$

$$= \|S_{x+h}\| + \varepsilon \quad \text{if } \|h\| < s.$$

In the same manner, we can show that $\|S_{x+h}\| \leq \|S_x\| + \varepsilon$ and we get the conclusion.

LEMMA 5.10. *Let $T: U \rightarrow F$ have a continuously differential $S_{x,z}$ on U and let $x_0 \in U$. Then, for any $\varepsilon > 0$ and for any $r > 0$, there exists $s > 0$ such that*

$$\sup_{(y,z,z') \in A_{x,s,r}} \bar{d}(S_{x,z}, S_{y,z'}) < \varepsilon$$

for all $x \in N_s(x_0)$, where $A_{x,s,r}$ is as in 5.8.

Proof. By (5.7), for any $\varepsilon > 0$ and any $x_0 \in U$, there exists $\delta > 0$ such that $\|x_0 - y\| < \delta$ and $\|z_0 - z'\| < \delta$ imply $\bar{d}(S_{x_0,z_0}, S_{y,z'}) < \varepsilon/2$ for all $x, y \in U$, all $z_0 \in T(x_0)$ and all $z' \in T(y)$. Let $r > 0$, by (5.9), there exists $t > 0$ such that if $\|x - x_0\| < t$, then $\|S_x\| < \|S_{x_0}\| + r = k_{x_0,r}$ and hence $k_{x,r} < k_{x_0,2r}$. Put $s = \min\{\delta/2, \delta/2k_{x_0,2r}, t\}$. For $\|x - x_0\| < s$ and $\|x - y\| < s$ we have $\|x_0 - y\| < \delta$. If $z \in T(x)$ and $z' \in T(y)$ such that $\|z - z'\| < k_{x,r}\|x - y\|$, then $\|z - z'\| < \delta/2$. By (5.1), we can take $z_0 \in T(x_0)$ such that $\|z_0 - z\| \leq \|x_0 - x\| \sup_{y' \in \text{co}(x_0, x)} \|S_{y'}\|$ and so $\|z_0 - z\| < k_{x_0,r}s < \delta/2$. Thus, $\|z_0 - z'\| \leq \|z_0 - z\| + \|z - z'\| < \delta$ and we get that $\bar{d}(S_{x_0,z_0}, S_{y,z'}) < \varepsilon/2$. Since $\|x - x_0\| < \delta$ and $\|z - z_0\| < \delta$ then $\bar{d}(S_{x,z}, S_{x_0,z_0}) < \varepsilon/2$. Consequently, $\bar{d}(S_{x,z}, S_{y,z'}) < \varepsilon$ and we get the conclusion.

We may now state and prove the property previously mentioned in (5.6).

COROLLARY 5.11. *Under the same conditions of Lemma 5.10, for any $\varepsilon > 0$ and any $r > 0$, there exists $s > 0$ such that if $N_{2s}(x_0) \subset U$ then $k_{x,h} < \varepsilon$ for $\|h\| < s$ and for all $x \in N_s(x_0)$.*

Proof. By (5.10) let us to show that if A_{J_r} is as in (5.2), then $A_{J_r} \subset A_{x,s,r}$: but if $y \in J$ then $\|x - y\| < s$ for $\|h\| < s$, so the inclusion is true. Hence by (5.10) there exists s such that

$$k_{x,h} = \inf_{r > 0} \sup_{(y,z,z') \in A_{J_r}} \bar{d}(S_{x,z}, S_{y,z'}) \leq \inf_{r > 0} \sup_{(y,z,z') \in A_{x,s,r}} \bar{d}(S_{x,z}, S_{y,z'}) \leq \varepsilon$$

for all $x \in N_s(x_0)$ (p.s. here put $r = 1$).

PROPOSITION 5.12. *Let $T: U \rightarrow F$ have a continuous differential $S_{x,z}$ on U and let K be compact in U . Then for any $\varepsilon > 0$, there exists $\alpha > 0$ and there exists an open neighbourhood V of K such that*

$$T(x+h) \underset{\varepsilon\|h\|}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h)) \quad \text{for } \|h\| < \alpha \text{ and for all } x \in V.$$

Proof. Let $x_0 \in U$, there exists $r > 0$ such that $N_{2r}(x_0) \subset U$. By (5.4), $T(x+h) \underset{\|h\|k_{x,h}}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h))$ for all $x \in N_r(x_0)$ and $\|h\| < r$. Let $\varepsilon > 0$; by (5.11), there exists $s > 0$ such that if $N_{2s}(x_0) \subset U$ then $k_{x,h} < \varepsilon$ for $\|h\| < s$ and for all $x \in N_s(x_0)$. Put $\delta'_{x_0} = \min\{s, r\}$, then $T(x+h) \underset{\varepsilon\|h\|}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h))$ for $\|h\| < \delta'_{x_0}$ and for all $x \in N_{\delta'_{x_0}}(x_0)$. But K is compact in

U , so there exists $x_i \in K$ such that $K \subset \bigcup_{i=1}^n N_{\delta'_{x_i}}(x_i) = \emptyset$. Put $t = d(K, \mathcal{C}\mathcal{O})$, $\alpha = \min\{\delta'_{x_1}, \dots, \delta'_{x_n}, t\}$ and $V = N_\alpha(K)$ to get the conclusion.

We are now in position to state and prove a very important property to be used in Section 7.

THEOREM 5.13. *Under the same conditions of Proposition (5.12), for any $\varepsilon > 0$, there exists $\alpha > 0$ and there exists an open neighbourhood V of K such that if $\text{co } A \subset V$ and $\delta(A) < \alpha$, there exist sets $B_i \subset T(A)$, $i = 1, \dots, n$, such that $\delta(B_i) < (1 + k'')\delta(A)$ and $T(\text{co } A) \subset \bigcup_{i=1}^n N_{\varepsilon\delta(A)}(\text{co } B_i)$, where $k'' = \sup_{x \in V} \|S_x\|$.*

Proof. By (5.12), for any $\frac{1}{2} > \varepsilon > 0$, there exists $\alpha > 0$ and there exists an open neighbourhood V of K such that

$$(5.13.1) \quad T(x+h) \underset{\varepsilon\|h\|}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h)) \quad \text{for } \|h\| < \alpha \text{ and for all } x \in V.$$

Let A be a set such that $\text{co } A \subset V$ and $\delta(A) < \alpha$. Take $x \in A$, since $T(\text{co } A) = \bigcup_{x+h \in \text{co } A} T(x+h) = \bigcup_{h \in \text{co}(A-x)} T(x+h)$, by (5.13.1), we have that

$$T(\text{co } A) \underset{\varepsilon\|h\|}{\sim} \bigcup_{h \in \text{co}(A-x)} \bigcup_{z \in T(x)} (z + S_x(z, h)) = \bigcup_{z \in T(x)} (z + S_x(z, \text{co}(A-x)))$$

for $\|h\| < \delta(A)$. Since $S_{x,z}$ is semi-linear positive, we get that $T(\text{co } A) \subset \bigcup_{z \in T(x)} N_{\varepsilon\delta(A)}(\text{co}(z + S_x(z, A-x)))$. In the same manner, we get that $T(A) \subset \bigcup_{z \in T(x)} N_{\varepsilon\delta(A)}(z + S_x(z, A-x))$. Put $B_z = N_{\varepsilon\delta(A)}(z + S_x(z, A-x)) \cap T(A)$ then $\text{co}(z + S_x(z, A-x)) \subset N_{\varepsilon\delta(A)}(\text{co } B_z)$.

By (5.7) there exists $\alpha_z < \varepsilon\delta(A)/2$ such that $\bar{d}(S_{x,z}, S_{x,z'}) < \varepsilon\delta(A)/2$ for all $z, z' \in T(x)$ and $z' \in N_{\alpha_z}(z)$. Thus, if $z' \in N_{\alpha_z}(z)$, then

$$z' + S_x(z', A-x) \subset N_{\varepsilon\delta(A)}(z + S_x(z, A-x))$$

and so

$$\text{co}(z' + S_x(z', A-x)) \subset N_{\varepsilon\delta(A)}(\text{co}(z + S_x(z, A-x))) \subset N_{2\varepsilon\delta(A)}(\text{co } B_z).$$

Since $T(x)$ is compact, there exists $z_i \in T(x)$ such that $T(x) \subset \bigcup_{i=1}^n N_{\alpha_{z_i}}(z_i)$. Put $B_i = B_{z_i}$, if $z \in T(x)$, then $z \in N_{\alpha_i}(z_i)$ for some i and so $\text{co}(z + S_x(z, A-x)) \subset N_{2\varepsilon\delta(A)}(\text{co } B_{z_i})$. Thus, $T(\text{co } A) \subset \bigcup_{i=1}^n N_{3\varepsilon\delta(A)}(\text{co } B_i)$, where $B_i \subset T(A)$. Finally,

$$\delta(B_i) = \delta(N_{\varepsilon\delta(A)}(z_i + S_x(z_i, A-x))) < 2\varepsilon\delta(A) + \|S_x\| \delta(A) < (1 + k'')\delta(A).$$

6. Composition of differentiable maps. We can immediately prove the only theorem of this section.

THEOREM 6.1. *Let E, E' and F be Banach spaces and let U, U' be open in E and E' respectively. If $T: U \rightarrow U'$ is multivalued differentiable at x and $T': U' \rightarrow F$ has a continuous differential $S'_{z,z'}$ in U' , then $T' \circ T: U \rightarrow F$ is a differentiable multivalued map at x with differential $\tilde{S}_x: T'(T(x)) \times E \rightarrow F$ defined by $\tilde{S}_x(z', h) = \bigcup_{z \in A_{z'}} S'_z(z', S_x(z, h))$, where $A_{z'} = \{z \in T(x) \mid z' \in T'(z)\}$.*

Proof. (a) $\tilde{S}_x(z', -)$ is homogeneous:

$$\begin{aligned} \tilde{S}_x(z', \lambda h) &= \bigcup_{z \in A_{z'}} S'_z(z', S_x(z, \lambda h)) = \bigcup_{z \in A_{z'}} S'_z(z', \lambda S_x(z, h)) \\ &= \bigcup_{z \in A_{z'}} \lambda S'_z(z', S_x(z, h)) = \lambda \tilde{S}_x(z', h) \end{aligned}$$

since both $S'_z(z', -)$ and $S_x(z, -)$ are homogeneous.

(b) $\tilde{S}_x(z', -)$ is semi-linear positive; take $h = \sum \alpha_i h_i$, where $\sum \alpha_i \leq 1$ and $\alpha_i \geq 0 \forall i$ then

$$\begin{aligned} \tilde{S}_x(z', \sum \alpha_i h_i) &= \bigcup_{z \in A_{z'}} S'_z(z', S_x(z, \sum \alpha_i h_i)) \\ &\subset \bigcup_{z \in A_{z'}} S'_z(z', \sum \alpha_i S_x(z, h_i)) \\ &\subset \bigcup_{z \in A_{z'}} \sum \alpha_i S'_z(z', S_x(z, h_i)) = \sum \alpha_i \tilde{S}_x(z, h_i) \end{aligned}$$

since both $S'_z(z', -)$ and $S_x(z, -)$ are semi-linear positive;

(c) it remains to show that $\forall \varepsilon > 0 \exists \delta > 0$ such that

$$(6.1.1) \quad T'(T(x+h)) \underset{\varepsilon \|h\|}{\sim} \bigcup_{z' \in T'(T(x))} (z' + \tilde{S}_x(z', h)).$$

First, by (5.9), $z \rightarrow \|S'_z\|$ is continuous, so $k = \sup_{z \in T(x)} \|S'_z\| + 1$ exists, since $T(x)$ is compact; and if $U'' = \{z \in U' \mid \|S'_z\| < k\}$, then U'' is open. But since T is differentiable at x for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$(6.1.2) \quad T(x+h) \underset{\varepsilon \|h\|}{\sim} \bigcup_{z \in T(x)} (z + S_x(z, h)) \quad \text{if } \|h\| < \delta_1.$$

Since $T(x)$ is compact and so $d = d(T(x), \mathcal{C}U'') > 0$ there exist $0 < \delta_2 < \delta_1$ such that $T(x+h) \subset N_{d/2}(T(x))$, because T is u.s.c., and $\varepsilon \|h\| < \frac{1}{2}d$ provided that $\|h\| < \delta_2$. That is if $\|h\| < \delta_2$ we may assume (6.1.2) and also that both members of (6.1.2) are contained in $N_d(T(x)) \subset U''$.

Now, by the mean value theorem, we have that

$$(6.1.3) \quad T' \circ T(x+h) \underset{\varepsilon \|h\|}{\sim} \bigcup_{z \in T(x)} T'(z + S_x(z, h));$$

since if a and b belong to respectively the first and second members of (6.1.2) such that $\|a-b\| < \varepsilon \|h\|$, then $a \in N_{d/2}(T(x))$ and J the segment between a and b lies in $N_d(T(x)) \subset U''$, so $T'(a) \underset{k \|a-b\|}{\sim} T'(b)$. Finally by (5.12) since $T': U' \rightarrow F$ has a continuous differential $S'_{z,z'}$ and $T(x)$ is compact, there exists

$\delta_3 > 0$ such that

$$(6.1.4) \quad T'(z+g) \underset{\varepsilon \|g\|}{\sim} \bigcup_{z' \in T'(z)} (z + S_z(z', g))$$

for all $z \in T(x)$ and for $\|g\| < \delta_3$. Thus if $\delta = \inf \{\delta_2, \delta_3 \|S_x\|^{-1}\}$, then $\|h\| < \delta$ implies (6.1.3) and $\|S_x(z, h)\| \leq \|S_x\| \|h\| < \delta_3$ for all $z \in T(x)$. Hence by (6.1.3)

$$(6.1.5) \quad T'(z + S_x(z, h)) \underset{\varepsilon \|S_x\| \|h\|}{\sim} \bigcup_{z' \in T'(z)} (z + S_z(z', S_x(z, h))).$$

Taking the union on each member of (6.1.4) we get

$$(6.1.6) \quad T' \left(\bigcup_{z \in T(x)} (z + S_z(z', h)) \right) \underset{\varepsilon \|S_x\| \|h\|}{\sim} \bigcup_{z \in T(x)} \bigcup_{z' \in T'(z)} (z + S_z(z', S_x(z, h))).$$

Putting together (6.1.2) and (6.1.5), commuting the 2 unions in the last member of (6.1.5) gives us (6.1.1) but with $\varepsilon(\|S_x\| + k)$ replacing ε ; since both can be arbitrary small and so it remains to show that $\bar{S}_x$ is u.s.c. on $T'(T(x)) \times E$.

Let V be an open set such that $\tilde{S}_x(z', h) \subset V$ and denote $K_{x,z'}$ the set $\{z \in T(x) \mid z' \in T'(z)\}$. We have that $K_{x,z'}$ is compact. The set $V_\delta = \{z \in T(x) \mid T'(z) \subset \mathcal{C}N_\delta(z')\}$ is open for any $\delta > 0$ and $\bigcup_{\delta > 0} V_\delta = T(x) \setminus K_{x,z'}$.

$\supset T(x) \setminus N_{\varepsilon_1}(K_{x,z'})$ for any $\varepsilon_1 > 0$ and so $\{V_\delta \mid \delta > 0\}$ is an open cover of the compact $T(x) \setminus N_{\varepsilon_1}(K_{x,z'})$. Then, for any $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that $T(x) \setminus N_{\varepsilon_1}(K_{x,z'}) \subset V_{\delta_1}$ and so, for $\|a\| < \delta_1$, $\{z \in T(x) \mid z' + a \in T'(z)\} \subset N_{\varepsilon_1}(K_{x,z'})$. Since $\bar{S}_x$ is u.s.c. in $T(x) \times E$, S_x is u.s.c. at the point $(z, h) \in T(x) \times E$ and then for any $\varepsilon_2 > 0$, there exists $\delta_2 > 0$ such that if $\|\gamma\| < \delta_2$ and $\|\beta\| < \delta_2$, then $S_x(z + \gamma, h + \beta) \subset N_{\varepsilon_2}(S_x(z, h))$. Consequently, if we take $\delta_2 < \varepsilon_2 < \varepsilon_1 < \varepsilon$, then for each $z \in \{z \in T(x) \mid z' + a \in T'(z)\}$, for each $g \in S_x(z, h + \beta)$, for $\|\beta\| < \delta_2$ and for $\|a\| < \delta_2$, there exists $z_1 \in K_{x,z'}$ and there exists $g_1 \in S_x(z, h)$ such that $\|z - z_1\| < \varepsilon$, $\|g - g_1\| < \varepsilon$ and $\|z' + a - z'\| = \|a\| < \varepsilon$. Let $H = \{(z, z', S_x(z, h)) \mid z \in K_{x,z'}\}$ and consider the map $G: E' \rightarrow E' \times F \times E'$ defined by $G(z) = (z, z', S_x(z, h))$. Since S_x is u.s.c. then so is G and $G(K_{x,z'}) = H$ is compact. Since $S': \bigcup_{z \in T(x)} \{z \times T'(z)\} \times E' \rightarrow F$ defined by $S'(z, z', g) = S'_z(z', g)$ is u.s.c. then $S'^{-1}(V)$ is an open set which contains H . Put $d = d(H, \mathcal{C}S'^{-1}(V))$ and $\varepsilon = \delta/3$ then $\|z - z'\| < \varepsilon$, $\|g - g_1\| < \varepsilon$ and $\|a\| < \varepsilon$ imply that $(z, z' + a, g) \in N_d(H) \subset S'^{-1}(V)$, that is $S'(z, z' + a, g) \subset V$, and so

$$\tilde{S}_x(z' + a, h + \beta) = \bigcup_{z \in \{z \in T(x) \mid z' + a \in T'(z)\}} \bigcup_{g \in S_x(z, h + \beta)} S'_z(z' + a, g) \subset V.$$

Thus, we get the conclusion.

7. The fixed point theorem. Before proving our fixed point theorems let us show that if $f = (F_n, \dots, F_0)$ is an admissible sequence of acyclic multi-valued maps, if $F = F_n \circ \dots \circ F_0$ is continuously differentiable eventually a

k -set-contraction with $k < 1$, then f is compacting and $\text{ind}(U, f, X)$ is defined. Our fixed point theorems shall be obtained from this result as immediate corollaries.

THEOREM 7.1. *Let $g = (G_n, \dots, G_0)$ be an admissible sequence of acyclic maps $G_i: X_i \rightarrow X_{i+1}$, where each X_i is Hausdorff and $X_0 = U$ is an open subset of $X_{n+1} = X \in \mathcal{F}$. If $G = G_n \circ \dots \circ G_0: U \rightarrow X$ is a selection of $F: U \rightarrow X$ which is continuously differentiable in U and eventually a k -set contraction, then g is compacting and $\text{ind}(U, g, X)$ is defined. Moreover, for any K compact subset of U , F -invariant and containing $\text{Fix}(F)$ we may choose W and $\{K_n\}$ satisfying the conditions of definition (2.1) and such that $K \subset W$.*

Proof. By (5.9), we get that the map $x \mapsto \|S_x\|$ is continuous in U , hence in K so $L = \sup_{x \in K} \|S_x\| + 1$ exists. The set $\theta = \{x \in U: \|S_x\| < L\}$ is open and $K \subset \theta$ since if $x \in K$, then $\|S_x\| < L$. Since F is eventually a k -set-contraction, then $\gamma(F^n(D)) \leq r\gamma(D)$ for any bounded subset D of X for some integer n and $r < 1$. Take $0 < \varepsilon < (1-r)/4nL$, by (5.13), there exist $\alpha > 0$ and $V \subset \theta$ an open neighbourhood of K such that if $\text{co } A \subset V$ and $\delta(A) < \alpha$, then

(7.1.1) there exist sets $B_i \subset F(A)$, $i = 1, \dots, b_A$ such that $\delta(B_i) < L\delta(A)$ and

$$F(\text{co } A) \subset \bigcup_{i=1}^{b_A} N_{\frac{\varepsilon\delta(A)}{2}}(\text{co } B_i).$$

Take $\alpha' > 0$ such that $L^n \alpha' < \alpha$ and $N_{2L^n \alpha'}(K) \subset \bigcap_{i=0}^n F^{-i}(V) \subset V$. Since K is compact, there exist $K_{11}, K_{12}, \dots, K_{1m_1}$ closed convex subsets of X such that $K \subset \mathring{K}_1$, where $K_1 = \bigcup_{j=1}^{m_1} K_{1j}$ and $\delta(K_{1j}) < 2\gamma(K_1)$. Again, since K is compact, there exist $G_1, \dots, G_m$ closed convex such that $K \subset \bigcup_{i=1}^m G_i$ $= W \subset \bar{W} \subset U \cap \mathring{K}_1 \cap F^{-1}(\mathring{K}_1)$ and $\delta(G_i) < \alpha'$. Hence $\text{Fix}(F) \subset K \subset W \subset \bar{W} \subset U$ and $W \subset K_1 \subset X$ and $F(W \cap K_1) = F(W) \subset K_1$. Let us assume that $K \cap G_i \neq \emptyset$ for any $i = 1, \dots, m$.

$A = G_l \cap K_{1k}$ alternately there exist sets $B_{21}, \dots, B_{2m_2}$ such that $\delta(B_{2i}) < L\delta(A) < L\alpha' < \alpha$, $\delta(B_{2i}) < 2L\gamma(K_1)$ and for all i , there exist integers l, k such that $B_{2i} \subset F(G_l \cap K_{1k})$ and

$$F(\text{co}(G_l \cap K_{1k})) \subset \bigcup_{i=1}^{m_2} N_{\varepsilon\gamma(K_1)}(\text{co } B_{2i}) \quad \text{for all } l, k.$$

So,

$$F(\bar{W} \cap K_1) \subset \bigcup_{i=1}^{m_2} \overline{N_{\varepsilon\gamma(K_1)}(\text{co } B_{2i})} \cap K_1.$$

Define $K_2 = \bigcup_{k=1}^{m_1} \bigcup_{i=1}^{m_2} \overline{N_{\varepsilon\gamma(K_1)}(\text{co } B_{2i})} \cap K_{1k} = \bigcup_{i=1}^{m'_2} K_{2i}$, then $K_2 \subset K_1$ and

$F(\bar{W} \cap K_1) \subset K_2$. Let $z \in \overline{N_{\varepsilon\gamma(K_1)}(\text{co } B_{2i})}$; then there exists $x \in \text{co } B_{2i}$ such that $z \in \overline{N_{\varepsilon\gamma(K_1)}(x)}$ and so $F(z) \subset F(\overline{N_{\varepsilon\gamma(K_1)}(x)}) \subset \overline{N_{\varepsilon L\gamma(K_1)}(F(x))}$ by the mean value theorem (5.1). Thus

$$F(\bar{W} \cap K_2) \subset \bigcup_{i=1}^{m_2} \overline{N_{\varepsilon L\gamma(K_1)}(F(\text{co } B_{2i}))}.$$

Note that $\text{co } B_{2i} \subset V$. In fact, there exists l such that $B_{2i} \subset F(G_l) \subset F(N_{\alpha'}(K))$ since $\delta(G_l) < \alpha'$, and $K \cap G_i \neq \emptyset$ by the mean value theorem, $B_{2i} \subset N_{\alpha'L}(F(K)) \subset N_{\alpha'L}(K)$ and so $\text{co } B_{2i} \subset N_{2\alpha'L}(K) \subset V$. By (7.1.1), there exist sets $B_{31}, \dots, B_{3m_3}$ such that $\delta(B_{3i}) < L^2 \alpha' < \alpha$ and $\delta(B_{3i}) < 2L^2 \gamma(K_1)$ and for all i , there exists an integer l such that $B_{3i} \subset F(B_{2l})$ and $F(\text{co } B_{2l}) \subset \bigcup_{i=1}^{m_3} \overline{N_{\varepsilon L\gamma(K_1)}(\text{co } B_{3i})}$. Thus $F(\bar{W} \cap K_2) \subset \bigcup_{i=1}^{m_3} \overline{N_{\varepsilon L\gamma(K_1)}(\text{co } B_{3i})}$. Define

$$K_3 = \bigcup_{i=1}^{m_3} \bigcup_{l=1}^{m_2} \overline{N_{\varepsilon L\gamma(K_1)}(\text{co } B_{3i})} \cap K_{2l} = \bigcup_{i=1}^{m_3} K_{3i},$$

then $K_3 \subset K_2$ and $F(\bar{W} \cap K_2) \subset K_3$.

Repeat this process n times to obtain $K_1 \supset K_2 \supset \dots \supset K_{n+1}$ such that $F(\bar{W} \cap K_i) \subset K_{i+1} \subset K_i \in \mathcal{F}_0$ and $K_{n+1} \subset \bigcup_{i=1}^{m_{n+1}} \overline{N_{\varepsilon L\gamma(K_1)}(\text{co } B_{n+1,i})}$ and for all i , there exist $i_1, \dots, i_n$ such that

$$B_{n+1,i} \subset F(B_{n,i_n}) \subset F^2(B_{n-1,i_{n-1}}) \subset \dots \subset F^n(B_{1,i_1}),$$

where $B_{1,i_1} = G_l \cap K_{1k}$ for some l and k .

Let us show that there exists $0 < r' < 1$ such that $\gamma(K_{n+1}) \leq r' \gamma(K_1)$. By (1.14) and (1.15), we get that

$$\begin{aligned} \gamma(K_{n+1}) &\leq \gamma(B_{n+1,i}) + 2n\varepsilon L\gamma(K_1) \leq \gamma(F^n(K_1)) + 2n\varepsilon L\gamma(K_1) \\ &\leq r\gamma(K_1) + \frac{(1-r)}{2}\gamma(K_1) \\ &= \frac{(1+r)}{2}\gamma(K_1) = r'\gamma(K_1), \quad \text{where } 0 < r' < 1, \end{aligned}$$

Repeat this n fold process indefinitely to obtain a sequence $\{K_n\}_{n \geq 1} \subset \mathcal{F}_0$ such that

$$\gamma(K_{in+1}) < r'\gamma(K_{(i-1)n+1}) < (r')^2\gamma(K_{(i-2)n+1}) < \dots < (r')^i\gamma(K_1)$$

and so $\lim_{i \rightarrow \infty} (K_{in+1}) = 0$. But by construction,

$$\begin{aligned} \gamma(K_{kn+j}) &\leq 2(j-1)\varepsilon L\gamma(K_{kn+1}) + \sup_a \delta(B_{kn+j,a}) \\ &\leq 2(j-1)\varepsilon L\gamma(K_{kn+1}) + 2L^{j-1}\gamma(K_{kn+1}) \\ &< (\tfrac{1}{2}(1-r) + 2L^{j-1})\gamma(K_{kn+1}) \leq M\gamma(K_{kn+1}) \quad \text{where } M = 1 + 2L^{j-1}. \end{aligned}$$

Thus $\lim_{k \rightarrow \infty} \gamma(K_{kn+j}) = 0$ and so $\lim_{n \rightarrow \infty} \gamma(K_n) = 0$. Note that W and $\{K_n\}$ satisfy the conditions of Definition (2.1) for g that is g is compacting and $\text{ind}(U, f, X)$ is defined. This ends the proof.

We may now state our fixed point theorems.

THEOREM 7.2. (LEFSCHETZ THEOREM). *Let $g = (G_n, \dots, G_0)$ be a sequence of acyclic maps $G_i: X_i \rightarrow X_{i+1}$, where each X_i is Hausdorff and $X = X_0 = X_{n+1}$ is an open subset of an element of $\mathcal{F}$. If $G = G_n \circ \dots \circ G_0: X \rightarrow X$ is a selection of $F: X \rightarrow X$ which is continuously differentiable on X and eventually a k -set-contraction, then $\Lambda(g)$ is defined and $\Lambda(g) \neq 0$ implies that G has a fixed point.*

Proof. Since $K_\infty = \bigcap_{n=1}^{\infty} K_n$ is a compact attractor for G , by (1.17). By (2.12), we get the conclusion.

THEOREM 7.3. *If in 7.2, the space X is an acyclic (in particular a convex) subset of a Banach space, then g has a fixed point.*

Proof. Then $\Lambda(g) = 1$.

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