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ON FUNCTIONS WHICH SUM TO ZERO ON SEMICYCLES

Abstract. For a digraph $D = (V, A)$ we consider functions $F: A \rightarrow R$ that sum to zero on every semicycle of D . We generalize the characterization of these functions in terms of fundamental cycle sets to arbitrary cycle bases by giving a new algebraic proof of the result. Then, in terms of such functions, we present some characterizations of gradable digraphs and a class of graphs for which the windy postman problem can be solved by a polynomial-time algorithm.

1. Introduction. A digraph D consists of the vertex set $V(D)$ and the arc set $A(D) \subset V(D) \times V(D)$ of ordered pairs of vertices; we write $n = |V(D)|$ and $m = |A(D)|$. A *semipath* P joining a vertex $v = v_0$ to a vertex $u = v_k$ is an alternate sequence

$$(v_0, a_1, v_1, \dots, v_{k-1}, a_k, v_k)$$

of vertices and arcs in which:

- (1) no vertex appears twice, except possibly the first and the last one,
- (2) no arc appears twice and each arc a_i is either (v_{i-1}, v_i) (in this case, the orientation of a_i coincides with the ordering $v_0, v_1, \dots, v_k$ of vertices along the semipath) or (v_i, v_{i-1}) (in this case, the orientation of a_i is opposite to the ordering $v_0, v_1, \dots, v_k$ of vertices along the semipath).

A semipath $(v_0, a_1, v_1, \dots, a_k, v_k)$ is a *semicycle* if $v_k = v_0$. A semipath (semicycle, resp.) in which the orientation of each arc coincides with the ordering of its vertices is called a *path* (*circuit*, resp.).

With every digraph D we can associate the cycle space of D which is the vector space over the two-element field $GF(2)$ generated by the subgraphs corresponding to the semicycles of D with addition defined as the symmetric difference. A *cycle basis* of D is defined as a basis for the cycle space of D which consists entirely of semicycles. If T is a spanning tree of D , then the set $\mathcal{C}(T)$ of semicycles obtained by adding to T , one at a time, each arc not in T forms a cycle basis of D called a *fundamental cycle set* of D with respect to T (see [11] for some characterizations of fundamental cycle sets). Fundamental cycle sets appear in many practical and theoretical applications

of graphs, especially in the analysis of electrical networks [1] (see also [2]). In many other situations there appear other or arbitrary cycle bases (see, e.g., [1], p. 83, [3], and [5], ex. 4.11). The dimension of the cycle space of D is equal to the *cyclomatic number* $\gamma = m - n + k$, where k is the number of connected components of D .

With a digraph D there are associated two cycle matrices

$$B(D) = [b_{ij}] \quad \text{and} \quad B'(D) = [b'_{ij}]$$

of order $p \times m$ (where p is the number of semicycles in D) over the real field R and $GF(2)$, respectively, where $b_{ij} = 1$ if the arc a_j is in the semicycle C_i and the orientation of a_j coincides with the ordering of vertices in C_i ; $b_{ij} = -1$ if a_j is in C_i and the orientation of a_j is opposite to the ordering of vertices in C_i ; $b_{ij} = b'_{ij} = 0$ if a_j is not in the semicycle C_i ; $b'_{ij} = 1$ if a_j is in C_i . It is known that the rank of the matrix $B(D)$ is equal to the cyclomatic number γ of D ([1], p. 44). Since the vector space spanned by the rows of the matrix $B'(D)$ over $GF(2)$ is isomorphic to the cycle space of D , the rank of $B'(D)$ is also equal to γ .

For a function $F: A(D) \rightarrow R$ and a semipath (semicycle, resp.) P whose arcs are $a_1, \dots, a_k$, the sum of F along (around, resp.) P is defined to be

$$F[P] = \varepsilon_1 F(a_1) + \dots + \varepsilon_k F(a_k),$$

where $\varepsilon_i = 1$ if the orientation of the arc a_i coincides with the ordering of vertices along (around, resp.) the semipath (semicycle, resp.) P , and $\varepsilon_i = -1$ in the other case. In many applications of graphs there appear functions F which sum to zero on every semicycle of a digraph. For instance, in electrical networks a voltage function F of arcs sums to zero on every semicycle (see [1], [2], and [7]) and in the study of differential equations by difference equations a so-called mesh function sums to zero on every periodic mesh [3]. In Section 2, we characterize pairs (D, F) such that a function $F: A(D) \rightarrow R$ sums to zero on every semicycle of a digraph D . We prove that F sums to zero on every semicycle of D if and only if F sums to zero on every semicycle of an arbitrary cycle basis of D . This is an important result since, in general, the number of semicycles in a digraph may be exponential in n . Then we present some applications of the results from Section 2. In Section 3, some new characterizations of gradable digraphs introduced in [6] are presented, and in Section 4 we prove some results of [4] related to the windy postman problem.

We assume that all graphs considered in this paper are finite and connected. For the terms not defined here the reader is referred to [1], [2], or [5].

2. A characterization of functions which sum to zero on every semicycle.

THEOREM 1 (KIRCHHOFF'S VOLTAGE LAW [7]). *For a digraph D and a function $F: A(D) \rightarrow R$ the following conditions are equivalent:*

(1) *There exists a function $f: V(D) \rightarrow R$ such that $F((v, u)) = f(v) - f(u)$ for each arc $(v, u) \in A(D)$.*

(2) *$F[C] = 0$ for every semicycle C of D .*

(3) *$F[P] = F[Q]$ for every two vertices $v, u \in V(D)$ and every two semipaths P and Q joining v to u in D .*

An elementary proof of this theorem may be obtained by a simple generalization of the proof of Theorem 10.5 in [6].

To verify condition (2) one has to know all semicycles of D . In the following theorem, we show that the amount of work to be done can be significantly reduced.

THEOREM 2. *For any cycle basis $\mathcal{C}$ of a digraph D and a function $F: A(D) \rightarrow R$ the condition*

(4) *$F[C] = 0$ for every $C \in \mathcal{C}$*

is equivalent to each of the conditions (1)–(3) of Theorem 1.

Proof. We show that conditions (2) and (4) are equivalent. The function $F: A(D) \rightarrow R$, where $A(D) = \{a_1, \dots, a_m\}$, defines an m -vector F , where

$$F^T = [F(a_1), \dots, F(a_m)].$$

Now, let $B(D)$ and $B'(D)$ be the cycle matrices of D over the real field and $\text{GF}(2)$, respectively, in which the i -th column corresponds to the i -th arc a_i of D ($i = 1, \dots, m$), i.e., the columns of $B(D)$, $B'(D)$ and F^T are arranged in the same order of arcs. Let $B(\mathcal{C})$ ($B'(\mathcal{C})$, resp.) be the submatrix of $B(D)$ ($B'(D)$, resp.) of order $\gamma \times m$ formed by the rows corresponding to the set $\mathcal{C}$ of semicycles of D . It is obvious that in terms of matrices $B(D)$ and $B(\mathcal{C})$, conditions (2) and (4) are satisfied if and only if F is a solution of the systems of linear equations

$$(2') \quad B(D) \cdot X = 0$$

and

$$(4') \quad B(\mathcal{C}) \cdot X = 0,$$

respectively. Thus, it remains to show that every solution of (4') is also a solution of (2'). For this purpose it suffices to show that the rank of $B(\mathcal{C})$ equals the rank of $B(D)$, i.e.,

$$r(B(\mathcal{C})) = r(B(D)) = \gamma.$$

The rank of $B'(\mathcal{C})$ is γ , since $\mathcal{C}$ is a cycle basis, and by permuting the columns of $B'(\mathcal{C})$, if necessary, we may assume that the first γ columns $B'_1, \dots, B'_\gamma$ of $B'(\mathcal{C})$ are independent. Thus, the determinant of $B' = [B'_1, \dots, B'_\gamma]$ is equal to 1 over $\text{GF}(2)$ and, therefore, in the expression

$$\det B' = \sum b'_{1s_1} b'_{2s_2} \dots b'_{\gamma s_\gamma}$$

(where the summation is taken over all permutations $s_1, s_2, \dots, s_\gamma$ of integers $1, 2, \dots, \gamma$) there exists an odd number, say $2k+1$, of products

$b'_{1s_1} b'_{2s_2} \dots b'_{\gamma s_\gamma}$ of value 1. Let us now consider the submatrix $B = [B_1, \dots, B_\gamma]$ of $B(\mathcal{C})$ over the real field formed by the columns of $B(\mathcal{C})$ corresponding to the columns of the matrix B' . We shall show that $\det B \neq 0$. Since for the corresponding entries of matrices B' and B we have $|b_{ij}| = b'_{ij}$ ($i, j = 1, 2, \dots, \gamma$), we also obtain

$$|b_{1s_1} b_{2s_2} \dots b_{\gamma s_\gamma}| = b'_{1s_1} b'_{2s_2} \dots b'_{\gamma s_\gamma}.$$

Hence, in the determinant

$$\det B = \sum (-1)^{I_s} b_{1s_1} b_{2s_2} \dots b_{\gamma s_\gamma}$$

exactly $2k+1$ terms $(-1)^{I_s} b_{1s_1} b_{2s_2} \dots b_{\gamma s_\gamma}$ have value either 1 or -1 . Therefore, we obtain $\det B \neq 0$ and both matrices B and $B(\mathcal{C})$ have the same rank γ .

3. Gradable digraphs. In this section, Theorems 1 and 2 are used to characterize gradable digraphs introduced by Harary et al. in [6], where some applications and structural properties of such digraphs are also discussed (see [8] and [9] for some special classes of gradable and non-gradable digraphs).

For a digraph $D = (V(D), A(D))$, let $1_A: A(D) \rightarrow R$ be the function such that $1_A(e) = 1$ for each arc e of D . It is easy to see that, for each semipath (semicycle, resp.) P , $1_A[P]$ is the number of arcs whose orientation coincides with the ordering (cyclic ordering, resp.) of vertices along the semipath (semicycle, resp.) P minus the number of arcs with the opposite orientation. In [6] the number $1_A[P]$ is called an *orientation index* of P .

A digraph D is said to be *gradable* if there exists a function f mapping the vertex set $V(D)$ into the set R of real numbers such that if (v, u) is an arc of D , then

$$f(v) - f(u) = 1 = 1_A((v, u)).$$

The following theorem provides a characterization of gradable digraphs and generalizes Theorem 10.5 from [6].

THEOREM 3. *If T is a spanning tree of a digraph D and $\mathcal{C}$ is a cycle basis of D , then the following statements are equivalent:*

- (i) D is gradable.
- (ii) $1_A[C] = 0$ for every semicycle C of D .
- (iii) $1_A[P] = 1_A[Q]$ for every two vertices $v, u \in V(D)$ and every two semipaths P and Q joining v to u in D .
- (iv) $1_A[C] = 0$ for every semicycle $C \in \mathcal{C}$.
- (v) $1_A[C] = 0$ for every fundamental semicycle $C \in \mathcal{C}(T)$.
- (vi) $1_A[T(v, u)] = 1$ for every arc $(v, u) \in A(D) - A(T)$, where $T(v, u)$ is the unique semipath joining v to u in the tree T .

Proof. The equivalence of statements (i)–(v) follows easily from Theorems 1 and 2. We now show that (iii) implies (vi), and (vi) implies (v). For an arc $(v, u) \in A(D) - A(T)$ and the semipath $(v, (v, u), u)$ we have

$$I_A[(v, (v, u), u)] = I_A((v, u)) = 1.$$

Thus, for the semipath $T(v, u)$, by (iii), we obtain

$$I_A[T(v, u)] = 1.$$

Now, for a fundamental cycle $C \in \mathcal{C}(T)$ there exists an arc $(v, u) \in A(D) - A(T)$ such that C is formed by (v, u) and $T(v, u)$. If the orientation of C is chosen to agree with that of the arc (v, u) , then according to (vi) we obtain

$$I_A[C] = I_A((v, u)) + I_A[T(u, v)] = I_A((v, u)) - I_A[T(v, u)] = 0.$$

4. The windy postman problem. For a finite, connected, undirected graph $G = (V, E)$ without loops and multiple edges, let $G_1 = (V, A_1)$ and $G_2 = (V, A_2)$ be any antisymmetric and symmetric orientations of G , respectively. To each edge $e = xy$ in G there correspond two arcs $a' = (x, y)$ and $a'' = (y, x)$ in G_2 , and only one of them in G_1 . Thus, to every cycle C in G there correspond two circuits C' and C'' in G_2 and only one semicycle C^0 in G_1 . Now, let $l: A(G_2) \rightarrow R$ be a non-negative function. The *windy postman problem* (see [10] and [4]) can be defined as follows: find a shortest (with respect to l) directed closed path P in G_2 such that for every edge e of G at least one of the corresponding arcs a' and a'' belongs to P . It is proved in [4] that the windy postman problem is NP-complete. For the class of pairs (G, l) , in which for every cycle C in G the corresponding circuits C' and C'' in G_2 have the same length, i.e., $l[C'] = l[C'']$, the windy postman problem can be solved by a polynomial-time algorithm [4]. The problem of checking the condition $l[C'] = l[C'']$ for all cycles C of G is related to the problem of functions which sum to zero on semicycles.

We define a function $F_l: A(G_1) \rightarrow R$ by

$$F_l((v, u)) = l((v, u)) - l((u, v))$$

for every arc (v, u) of G_1 .

THEOREM 4. For a cycle C of G we have $l[C'] = l[C'']$ if and only if $F_l[C^0] = 0$.

Proof. For a semicycle C^0 in G_1 , let $a'_1, \dots, a'_k$ be the arcs in C^0 whose orientation coincides with the cyclic ordering of vertices along C^0 , and let $a''_{k+1}, \dots, a''_n$ be the arcs in C^0 with the opposite orientation. Then $a'_1, \dots, a'_k$ and $a''_1, \dots, a''_n$ are the arcs of the circuits C' and C'' , respectively. Now, we

have

$$\begin{aligned}
 F_l[C^0] &= \sum_{i=1}^k F_l(a_i) - \sum_{i=k+1}^n F_l(a_i') \\
 &= \sum_{i=1}^k (l(a_i) - l(a_i')) - \sum_{i=k+1}^n (l(a_i') - l(a_i)) \\
 &= \sum_{i=1}^n l(a_i) - \sum_{i=1}^n l(a_i') = l[C'] - l[C''].
 \end{aligned}$$

This proves the theorem.

Let $\mathcal{C}$ be a fundamental cycle set of G . Guan proved in [4] that the following conditions are equivalent:

- (a) $l[C'] = l[C'']$ for every cycle C of G ;
- (b) $l[C'] = l[C'']$ for every cycle $C \in \mathcal{C}$.

He also stated without proof that in (b) the fundamental cycle set $\mathcal{C}$ may be replaced by any cycle basis of G . We now prove this fact.

THEOREM 5. *Let $\mathcal{C}$ be any cycle basis of a graph G . Then conditions (a) and (b) are equivalent.*

Proof. It is sufficient to show that (b) implies (a). If $\mathcal{C}$ is a cycle basis of G , then $\mathcal{C}^0 = \{C^0: C \in \mathcal{C}\}$ is a cycle basis of G_1 . According to (b) and Theorem 4, we have $F_l[C^0] = 0$ for every $C^0 \in \mathcal{C}^0$. Now, by Theorems 2 and 1, we have $F_l[C^0] = 0$ for every semicycle C^0 of G_1 . Therefore, by Theorem 4, we obtain $l[C'] = l[C'']$ for every cycle C of G .

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