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CONTROL WITH USE OF PREVIOUS EXPERIENCE

In the paper the problem of determining a minimax control policy is solved for a linear system with disturbances belonging to the exponential class. It is assumed that some previous information is available resulting in fixing the second moment of the a priori distribution of the parameter. It is proved that for the performance index (5) a minimax control policy always exists. The paper is a continuation of the paper [7] in which a minimax equalizer control policy was found which exists only in special cases.

1. Problem statement. Let us define a discrete linear system with complete observations and random horizon

$$(1) \quad x_{n+1} = \alpha_n x_n + u_n + \gamma_n v_n, \quad x_0 = e, \quad n = 0, 1, \dots, N,$$

where x_n is the state variable, u_n is the control, $v_0, v_1, \dots$ are independent random variables with the same distribution, N is a random variable independent of $v_0, v_1, \dots$ with given distribution

$$P(N = i) = p_i, \quad i = 1, \dots, M, \quad p_M > 0, \quad \sum_{i=1}^M p_i = 1,$$

α_n, γ_n, e are given constants, $\gamma_n \neq 0$.

It is assumed that the observations available at time n are

$$X_n = (x_0, x_1, \dots, x_n) \quad \text{and} \quad U_{n-1} = (u_0, u_1, \dots, u_{n-1}).$$

Since equation (1) holds and $\gamma_i \neq 0$, it follows from the above that at time n the values of the random variables $v_0, v_1, \dots, v_{n-1}$ are known.

It is supposed that the control $u_n, n = 0, 1, \dots, M$, is a Borel function of X_n and U_{n-1} .

It is assumed that the random variables $v_0, v_1, \dots$ have the same distribution P_λ belonging to an exponential family, i.e. their density with respect to a σ -finite measure μ on $R = (-\infty, \infty)$ is

$$(2) \quad p(v, \lambda) = S(v, q) \exp [qA(\lambda) + vB(\lambda)],$$

where $\lambda \in \Lambda$ is a parameter. We suppose that a natural parametrization is chosen for which the conditions of the paper [5] hold. Then we have

$$(3) \quad E_\lambda(v_n) = q\lambda \quad \text{for some } q > 0$$

and

$$(4) \quad E_\lambda(v_n^2) = q_1 \lambda^2 + q_2 \lambda + q_3 \quad \text{for some constants } q_1, q_2, q_3.$$

We assume that $p(v, \lambda)$ is known with the only exception of the parameter λ .

Random variables and stochastic processes belonging to the exponential class are considered in [1], [3], [4], [5].

Let $u_0, u_1, \dots, u_M$ be controls. The vector $U = (u_0, u_1, \dots, u_M)$ is called a *control policy*. It is assumed that the performance index connected with policy U is

$$(5) \quad J = \sum_{i=0}^N (\xi_i x_i^2 + 2\eta_i x_i \lambda + \zeta_i \lambda^2 + k_i u_i^2),$$

where

$$\xi_i \zeta_i - \eta_i^2 \geq 0, \quad \xi_i > 0, \quad k_i > 0.$$

Let P_λ be a distribution of the random variables $v_0, v_1, \dots$ and let P be a distribution of N . The risk function for the control policy U is

$$R(\lambda, U) = E_P[E_\lambda(J)],$$

where $E_P(\cdot)$ denotes the expectation with respect to the distribution P of the random variable N .

Let π be the a priori distribution of the parameter λ . We assume that we have a priori information about λ resulting in the fact that $\pi \in \Gamma$. In this paper we suppose that Γ is the set of all distributions π for which

$$(6) \quad E_\pi(\lambda^2) = m_2,$$

where $m_2 \in (\Lambda)^2 - \{0\}$ is given.

The Bayes risk connected with distribution π is

$$r(\pi, U) = E_\pi(R(\lambda, U)).$$

We consider only control policies U for which the Bayes risk $r(\pi, U)$ exists and is finite for each $\pi \in \Gamma$. The set of all these policies is denoted by Δ .

A policy $U^{(0)}$ such that

$$\sup_{\pi \in \Gamma} r(\pi, U^{(0)}) = \inf_{U \in \Delta} \sup_{\pi \in \Gamma} r(\pi, U)$$

is called a *minimax policy*.

We now find the minimax control policies for all P_λ, P, J satisfying the above conditions.

2. Distributions belonging to the exponential class. Suppose that the random variables $v_0, v_1, \dots$ are distributed according to the density (2). Denote by $\pi_{\beta,r}$ the a priori distribution of the parameter λ with density

$$(7) \quad g(\lambda; \beta, r) = C(\beta, r) B'(\lambda) \exp[\beta A(\lambda) + r B(\lambda)].$$

If the distribution $\pi_{\beta,r}$ is assigned to λ then the density of the a posteriori distribution of λ , given $\sum_{i=0}^{n-1} v_i$, is $g(\lambda; \beta_n, r_n)$, where

$$\beta_n = \beta + nq, \quad r_n = r + \sum_{i=0}^{n-1} v_i.$$

These a priori distributions will play an important role in our further considerations.

In [5] it was proved that there are only six distributions (and linear transformations of them) belonging to the exponential class and satisfying the conditions (3) and (4). We list them below. The measure μ with respect to which the densities are given is the Lebesgue or the counting measure.

(a) Let the random variables v_n be distributed according to the Poisson law

$$p(v, \lambda) = \frac{\lambda^v}{v!} e^{-\lambda} \quad (\lambda > 0).$$

Then

$$E_\lambda(v_n) = \lambda, \quad E_\lambda(v_n^2) = \lambda^2 + \lambda.$$

Moreover, according to (7), the a priori distribution $\pi_{\beta,r}$ of the parameter λ is of the form

$$g(\lambda; \beta, r) = \begin{cases} \frac{\beta^r}{\Gamma(r)} \lambda^{r-1} e^{-\beta\lambda}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0, \end{cases}$$

which implies

$$E_\pi(\lambda) = r/\beta, \quad E_\pi(\lambda^2) = r(r+1)/\beta^2$$

for $(\beta, r) \in S = \{\beta > 0, r > 0\}$.

(b) Suppose that the random variables v_n have the gamma distribution

$$p(v, \lambda) = \begin{cases} \frac{1}{\Gamma(q) \lambda^q} v^{q-1} e^{-v/\lambda}, & \text{if } v > 0 \\ 0, & \text{if } v \leq 0 \quad (\lambda > 0). \end{cases}$$

Then

$$E_\lambda(v_n) = q\lambda, \quad E_\lambda(v_n^2) = q(q+1)\lambda^2$$

According to (7) the a priori distribution $\pi_{\beta,r}$ of the parameter λ is of the form

$$g(\lambda; \beta, r) = \begin{cases} \frac{r^{\beta+1}}{\Gamma(\beta+1)} \cdot \frac{1}{\lambda^{\beta+2}} e^{-r/\lambda}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0, \end{cases}$$

and

$$E_{\pi}(\lambda) = r/\beta, \quad E_{\pi}(\lambda^2) = r^2/\beta(\beta-1)$$

for $(\beta, r) \in S = \{\beta > 1, r > 0\}$.

(c) Suppose that the random variables v_n have the negative binomial distribution

$$p(v, \lambda) = \frac{\Gamma(q+v)}{\Gamma(q)v!} \cdot \frac{\lambda^v}{(1+\lambda)^{q+v}} \quad (\lambda > 0).$$

We obtain

$$E_{\lambda}(v_n) = q\lambda, \quad E_{\lambda}(v_n^2) = q(q+1)\lambda^2 + q\lambda.$$

Let q be a natural number. Let us put $p = \lambda/(1+\lambda)$. In this case the distribution $p(v, \lambda)$ takes the more familiar form

$$p(v, \lambda) = \binom{q+v-1}{v} p^v (1-p)^q.$$

According to (7) we have

$$g(\lambda; \beta, r) = \begin{cases} \frac{1}{B(\beta+1, r)} \cdot \frac{\lambda^{r-1}}{(1+\lambda)^{\beta+r+1}}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0, \end{cases}$$

and

$$E_{\pi}(\lambda) = r/\beta, \quad E_{\pi}(\lambda^2) = r(r+1)/\beta(\beta-1)$$

for $(\beta, r) \in S = \{\beta > 1, r > 0\}$.

(d) Suppose that the random variables v_n have the binomial distribution

$$p(v, \lambda) = \binom{q}{v} \lambda^v (1-\lambda)^{q-v} \quad (0 < \lambda < 1).$$

In this case

$$E_{\lambda}(v_n) = q\lambda, \quad E_{\lambda}(v_n^2) = q(q-1)\lambda^2 + q\lambda.$$

The a priori distribution $\pi_{\beta,r}$ is now a beta distribution

$$g(\lambda; \beta, r) = \begin{cases} \frac{1}{B(r, \beta-r)} \lambda^{r-1} (1-\lambda)^{\beta-r-1}, & \text{if } 0 < \lambda < 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$E_{\pi}(\lambda) = r/\beta, \quad E_{\pi}(\lambda^2) = r(r+1)/\beta(\beta+1)$$

for $(\beta, r) \in S = \{\beta - r > 0, r > 0\}$.

(e) Let the random variables v_n be normally distributed with variance 1

$$p(v, \lambda) = \frac{1}{\sqrt{2\pi}} e^{-(v-\lambda)^2/2} \quad (-\infty < \lambda < \infty).$$

Then

$$E_{\lambda}(v_n) = \lambda, \quad E_{\lambda}(v_n^2) = \lambda^2 + 1.$$

Let, according to (7), the a priori distribution $\pi_{\beta,r}$ of the parameter λ be given by the density

$$g(\lambda; \beta, r) = \sqrt{\frac{\beta}{2\pi}} e^{-\beta(\lambda - r/\beta)^2/2},$$

which implies

$$E_{\pi}(\lambda) = \frac{r}{\beta}, \quad E_{\pi}(\lambda^2) = \frac{r^2}{\beta} + \frac{1}{\beta}$$

for $(\beta, r) \in S = \{\beta > 0, -\infty < r < \infty\}$.

(f) Finally, let the random variables v_n have the generalized exponential hyperbolic secant distribution (GEHS) [2], [5]. Its density with respect to the Lebesgue measure is

$$p(v, \lambda) = \frac{1}{(1+\lambda^2)^{q/2}} \exp(v \operatorname{arctg} \lambda) \cdot S(v, q),$$

where

$$S(v, q) = \frac{2^{q-2}}{\pi} B\left(\frac{q}{2} - i\frac{v}{2}, \frac{q}{2} + i\frac{v}{2}\right) = \frac{2^{q-2} \Gamma^2(q/2)}{\pi \Gamma(q)} \prod_{k=0}^{\infty} \left[1 + \frac{v^2}{(q+2k)^2}\right]^{-1} \\ (-\infty < v < \infty, -\infty < \lambda < \infty, q > 0).$$

Here

$$E_{\lambda}(v_n) = q\lambda, \quad E_{\lambda}(v_n^2) = q(q+1)\lambda^2 + q.$$

According to (7) we have

$$g(\lambda; \beta, r) = \frac{2^\beta (\beta + 1)}{\pi} B\left(\frac{\beta + 2}{2} - i\frac{r}{2}, \frac{\beta + 2}{2} + i\frac{r}{2}\right) \frac{1}{(1 + \lambda^2)^{(\beta + 2)/2}} \exp(r \operatorname{arctg} \lambda)$$

and

$$E_\pi(\lambda) = \frac{r}{\beta}, \quad E_\pi(\lambda^2) = \frac{r^2}{\beta(\beta - 1)} + \frac{1}{\beta - 1}$$

for $(\beta, r) \in S = \{\beta > 1, -\infty < r < \infty\}$.

Let us consider the condition (6). Let $m_2 \in h(A) = (A)^2 - \{0\}$. Applying the condition (6) to the distribution $\pi_{\beta, r}$ we obtain the equation

$$(8) \quad k(\beta, r) = m_2,$$

quadratic with respect to the variable r . It is easy to check that, if $m_2 \in h(A)$, $(\beta, r) \in S$ and (β, r) is a solution of the equation (8), then the ratio r/β takes its values in the intervals

$$(9) \quad \begin{array}{lll} \text{(a)} & (0, \sqrt{m_2}), & \text{(b)} & (0, \sqrt{m_2}), & \text{(c)} & (0, \sqrt{m_2}), \\ \text{(d)} & (m_2, \sqrt{m_2}), & \text{(e)} & (-\sqrt{m_2}, \sqrt{m_2}), & \text{(f)} & (-\sqrt{m_2}, \sqrt{m_2}) \end{array}$$

for the distributions listed in points (a)–(f), respectively.

3. Bayes policies and their limits. Let us suppose that the parameter λ is distributed according to the law $\pi \in \Gamma$. A control policy U_π such that

$$r(\pi, U_\pi) = \inf_{U \in \Delta} r(\pi, U)$$

is called a *Bayes policy*.

Let $U_{\beta, r}^* = (u_0^*, u_1^*, \dots, u_M^*)$ be the control policy for which

$$u_M^* = 0,$$

$$(10) \quad u_n^* = -P_n x_n - Q_n \frac{r_n}{\beta_n} \quad (n = 0, 1, \dots, M-1),$$

where

$$P_n = \frac{\pi_{n+1}}{\pi_n} \cdot \frac{\alpha_n A_{n+1}}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}}, \quad Q_n = \frac{\pi_{n+1}}{\pi_n} \cdot \frac{\gamma_n q A_{n+1} + B_{n+1}}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}},$$

$$\beta_n = \beta + nq, \quad r_n = r + \sum_{i=0}^{n-1} v_i, \quad \pi_n = \sum_{i=n}^M p_i,$$

the constants A_n, B_n satisfy the equations

$$A_n = \xi_n + \frac{\pi_{n+1}}{\pi_n} \cdot \frac{k_n \alpha_n^2 A_{n+1}}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}},$$

$$B_n = \eta_n + \frac{\pi_{n+1}}{\pi_n} \cdot \frac{\alpha_n k_n (\gamma_n q A_{n+1} + B_{n+1})}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}},$$

and the boundary conditions $A_M = \xi_M, B_M = \eta_M$.

Policy $U_{\beta, r}^*$ is defined if $\beta \neq -nq$ ($n = 0, 1, \dots, M-1$).

This control policy is a Bayes one with respect to the a priori distribution $\pi_{\beta, r}$ of the parameter λ if $(\beta, r) \in S$ (see [7]).

From [7] it follows also that

$$(11) \quad R(\lambda, U_{\beta, r}^*) = a_0 x_0^2 + 2b_0 x_0 \lambda + c_0 r^2 + e_0 \lambda^2 + 2f_0 r \lambda + i_0 \lambda + j_0,$$

where

$$(12) \quad \begin{aligned} a_0 &= A_0, \\ b_0 &= B_0, \\ c_0 &= \sum_{i=0}^{M-1} \frac{s_i}{\beta_i^2}, \\ e_0 &= t_0 - 2q \sum_{i=1}^{M-1} i \frac{s_i}{\beta_i} + \sum_{i=1}^{M-1} t_{i0} \frac{s_i}{\beta_i^2}, \\ f_0 &= - \sum_{i=0}^{M-1} \frac{s_i}{\beta_i} + q \sum_{i=1}^{M-1} i \frac{s_i}{\beta_i^2}, \\ i_0 &= q_2 \left(y_0 + \sum_{i=1}^{M-1} i \frac{s_i}{\beta_i^2} \right), \\ j_0 &= q_3 \left(y_0 + \sum_{i=1}^{M-1} i \frac{s_i}{\beta_i^2} \right); \end{aligned}$$

here the coefficients independent of β are given by the equations

$$\begin{aligned} s_n &= Q_n^2 (\pi_{n+1} A_{n+1} + \pi_n k_n), \\ t_0 &= \zeta_0 + \sum_{i=0}^{M-1} \pi_{i+1} (\gamma_i^2 q_1 A_{i+1} + 2\gamma_i q B_{i+1} + \zeta_{i+1}), \\ t_{n0} &= q_1 n + q^2 n(n-1), \\ y_0 &= \sum_{i=0}^{M-1} \pi_{i+1} \gamma_i^2 A_{i+1}. \end{aligned}$$

Denote by $U_m^+ = (u_0^+, u_1^+, \dots, u_M^+)$ the control policy for which

$$(13) \quad \begin{aligned} u_M^+ &= 0, \\ u_n^+ &= -P_n x_n - Q_n m \quad (n = 0, 1, \dots, M-1). \end{aligned}$$

Let us notice that

$$U_m^+ = \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow m}} U_{\gamma,s}^*.$$

From (11) and (12) it follows that

$$(14) \quad \begin{aligned} R(\lambda, U_m^+) &= \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow m}} R(\lambda, U_{\gamma,s}^*) \\ &= a_0 x_0^2 + 2b_0 x_0 \lambda + m^2 \sum_{i=0}^{M-1} s_i + t_0 \lambda^2 - 2m \sum_{i=0}^{M-1} s_i \lambda + q_2 y_0 \lambda + q_3 y_0 \end{aligned}$$

if there are $(\gamma, s) \in S$ such that $s \rightarrow \infty, s/\gamma \rightarrow m$.

We define now another control policy $U_{\beta,m}^- = (u_0^-, u_1^-, \dots, u_m^-)$, where

$$(15) \quad \begin{aligned} u_M^- &= 0, \\ u_0^- &= -P_0 x_0 - Q_0 m, \\ u_n^- &= -P_n x_n - Q_n \frac{r_n^{(m)}}{\beta_n} \quad (n = 1, \dots, M-1), \\ r_n^{(m)} &= \sum_{i=0}^{n-1} v_i + m\beta. \end{aligned}$$

The policy $U_{\beta,m}^-$ is defined also for $\beta = 0$. We have

$$U_{\beta,m}^- = \lim_{\substack{\gamma \rightarrow \beta \\ s/\gamma \rightarrow m}} U_{\gamma,s}^*$$

and

$$(16) \quad R(\lambda, U_{\beta,m}^-) = \lim_{\substack{\gamma \rightarrow \beta \\ s/\gamma \rightarrow m}} R(\lambda, U_{\gamma,s}^*)$$

if there are $(\gamma, s) \in S$ such that $\gamma \rightarrow \beta, s/\gamma \rightarrow m$.

4. Minimax control policy. In this section we need the following lemma:

LEMMA. Suppose that there is a sequence $\{\pi_{\beta^{(k)}, r^{(k)}}\}$, $k = 1, 2, \dots$, of a priori distributions belonging to Γ and having the density (7), $(\beta^{(k)}, r^{(k)}) \in S$, for which the corresponding sequence $\{U_{\beta^{(k)}, r^{(k)}}\}$ of Bayes control policies satisfies the condition

$$(17) \quad \lim_{k \rightarrow \infty} r(\pi_{\beta^{(k)}, r^{(k)}}, U_{\beta^{(k)}, r^{(k)}}^*) = c.$$

If there is a control policy $U^{(0)} \in \Delta$ such that $r(\pi, U^{(0)}) \leq c$ for each $\pi \in \Gamma$, then $U^{(0)}$ is a minimax control policy.

The lemma results from the general theorems of decision theory. Since I could not find it in the literature, I present a formal proof of this lemma.

Suppose that $U^{(0)}$ is not a minimax control policy. Then there is a control policy $U' \in \Delta$ such that

$$\sup_{\pi \in \Gamma} r(\pi, U^{(0)}) > \sup_{\pi \in \Gamma} r(\pi, U') = c - \varepsilon,$$

where $\varepsilon > 0$. But, on the other hand,

$$c - \varepsilon = \sup_{\pi \in \Gamma} r(\pi, U') \geq r(\pi_{\beta(k), r(k)}, U') \geq r(\pi_{\beta(k), r(k)}, U_{\beta(k), r(k)}^*),$$

contrary to (17).

COROLLARY. If $r(\pi, U^{(0)}) = c$ for each $\pi \in \Gamma$ and $U^{(0)}$ is Bayes with respect to some $\pi \in \Gamma$ then $U^{(0)}$ is minimax.

Suppose that we apply the control policy $U_{\beta, r}^*$. From (11) it follows that for $\pi \in \Gamma$ the Bayes risk is

$$r(\pi, U_{\beta, r}^*) = e_0 m_2 + (2b_0 x_0 + 2f_0 r + i_0) E_\pi(\lambda) + a_0 x_0^2 + c_0 r^2 + j_0.$$

Denote

$$(18) \quad T(\beta, r) = a_0 x_0^2 + c_0 r^2 + j_0, \quad Z(\beta, r) = 2b_0 x_0 + 2f_0 r + i_0.$$

Let $m_2 \in h(\Delta)$ and let S_k be the set of these (β, r) for which $k(\beta, r) = m_2$. There occurs one of the following three cases:

- (a) There is a point $(\beta, r) \in S$ such that $k(\beta, r) = m_2$, $Z(\beta, r) = 0$;
- (b) $Z(\beta, r) > 0$ for each $(\beta, r) \in S \cap S_k$;
- (c) $Z(\beta, r) < 0$ for each $(\beta, r) \in S \cap S_k$.

Suppose that there occurs the case (a). Then for this (β, r) the Bayes risk $r(\pi, U_{\beta, r}^*)$ does not depend on $\pi \in \Gamma$. Since policy $U_{\beta, r}^*$ is Bayes with respect to some $\pi \in \Gamma$ it is minimax.

Denote

$$\mu_1 = \inf_{(\beta, r) \in S \cap S_k} (r/\beta), \quad \mu_2 = \sup_{(\beta, r) \in S \cap S_k} (r/\beta).$$

From (9) it follows that $\mu_2 = \sqrt{m_2}$.

Suppose that there occurs the case (b). Then from (14) we have for $\pi \in \Gamma$

$$r(\pi, U_m^+) = \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow m}} r(\pi, U_{\gamma, s}^*) = \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow m}} (e_0(\gamma) E_\pi(\lambda^2) + Z(\gamma, s) E_\pi(\lambda) + T(\gamma, s)),$$

where $e_0(\beta) = e_0$ is given by the fourth equation of (12), and $m = m_2$.

Since $E_\pi(\lambda^2) = m_2$, $E_\pi^2(\lambda) \leq E_\pi(\lambda^2)$ and $Z(\gamma, s) > 0$ for $(\gamma, s) \in S \cap S_k$ we obtain for these (γ, s)

$$\begin{aligned}
(19) \quad r(\pi, U_{\sqrt{m_2}}^+) &\leq \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow \sqrt{m_2}}} (e_0(\gamma) m_2 + Z(\gamma, s) \sqrt{m_2} + T(\gamma, s)) \\
&= t_0 m_2 + (2b_0 x_0 - 2\sqrt{m_2} \sum_{i=0}^{M-1} s_i + q_2 y_0) \sqrt{m_2} + a x_0^2 + m_2 \sum_{i=0}^{M-1} s_i + q_3 y_0 \\
&= (t_0 - \sum_{i=0}^{M-1} s_i) m_2 + (2b_0 x_0 + q_2 y_0) \sqrt{m_2} + a_0 x_0^2 + q_3 y_0 \stackrel{\text{df}}{=} c.
\end{aligned}$$

On the other hand, for $(\beta, r) \in S \cap S_k$ is

$$r(\pi_{\beta, r}, U_{\beta, r}^*) = e_0(\beta) m_2 + Z(\beta, r)(r/\beta) + T(\beta, r) \rightarrow c$$

if $\beta \rightarrow \infty$, $r/\beta \rightarrow \sqrt{m_2}$ and there is a sequence of $(\beta, r) \in S \cap S_k$ which satisfies these suppositions. Then from the lemma it results that the policy $U_{\sqrt{m_2}}^+$ is minimax.

Suppose now that there takes place the case (c). Let the v_n have the normal or GEHS distribution. We prove that the policy $U_{\sqrt{m_2}}^+$ is minimax. Similarly as in the previous case, we have for $\pi \in \Gamma$ and $(\gamma, s) \in S$

$$\begin{aligned}
r(\pi, U_{\sqrt{m_2}}^+) &= \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow \sqrt{m_2}}} r(\pi, U_{\gamma, s}^*) \\
&= \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow \sqrt{m_2}}} (e_0(\gamma) E_\pi(\lambda^2) + Z(\gamma, s) E_\pi(\lambda) + T(\gamma, s)).
\end{aligned}$$

But $E_\pi(\lambda^2) = m_2$, $E_\pi(\lambda) \geq -\sqrt{E_\pi(\lambda^2)} = -\sqrt{m_2}$ and $Z(\gamma, s) < 0$ for $(\gamma, s) \in S \cap S_k$, which gives for these (γ, s)

$$\begin{aligned}
(20) \quad r(\pi, U_{\sqrt{m_2}}^+) &\leq \lim_{\substack{s \rightarrow \infty \\ s/\gamma \rightarrow \sqrt{m_2}}} (e_0(\gamma) m_2 - Z(\gamma, s) \sqrt{m_2} + T(\gamma, s)) \\
&= (t_0 - \sum_{i=0}^{M-1} s_i) m_2 - 2b_0 x_0 \sqrt{m_2} + a_0 x_0^2 + q_3 y_0 \stackrel{\text{df}}{=} c,
\end{aligned}$$

since for normal and GEHS distributions is $q_2 = 0$.

On the other hand, for $(\beta, r) \in S \cap S_k$ is

$$r(\pi_{\beta, r}, U_{\beta, r}^*) = e_0(\beta) m_2 + Z(\beta, r)(r/\beta) + T(\beta, r) \rightarrow c$$

if $\beta \rightarrow \infty$, $r/\beta \rightarrow -\sqrt{m_2}$ and there are $(\beta, r) \in S \cap S_k$ satisfying these suppositions. Then the policy $U_{\sqrt{m_2}}^+$ is minimax.

Let the v_n have the Poisson distribution and let the case (c) occur. We prove that the policy $U_{0,0}^-$ is minimax.

From (11), (12) and (16) we have for $(\gamma, s) \in S$

$$r(\pi, U_{0,0}^-) = \lim_{\substack{\gamma \rightarrow 0 \\ s/\gamma \rightarrow 0}} (e_0(\gamma) E_\pi(\lambda^2) + Z(\gamma, s) E_\pi(\lambda) + T(\gamma, s)).$$

Now $E_\pi(\lambda^2) = m_2$, $E_\pi(\lambda) \geq 0$, $Z(\gamma, s) < 0$ for $(\gamma, s) \in S \cap S_k$ and for these (γ, s) we have

$$(21) \quad r(\pi, U_{0,0}^-) \leq \lim_{\substack{\gamma \rightarrow 0 \\ s/\gamma \rightarrow 0}} (e_0(\gamma) m_2 + T(\gamma, s)) \\ = \left(t_0 - 2 \sum_{i=1}^{M-1} s_i + \sum_{i=1}^{M-1} t_{i0} \frac{s_i}{i^2} \right) m_2 + a_0 x_0^2 \stackrel{\text{df}}{=} c.$$

In the last equality we take into account that for the Poisson distribution we have $q = 1$, $q_3 = 0$.

Here also $r(\pi_{\beta,r}, U_{\beta,r}) \rightarrow c$ if $(\beta, r) \in S \cap S_k$, $\beta \rightarrow 0$, $r/\beta \rightarrow 0$ and there are $(\beta, r) \in S \cap S_k$ which satisfy these suppositions. The thesis is proved.

In the same way one can prove that if the case (c) occurs and the v_n have the gamma or negative binomial distribution then the policy $U_{1,0}^-$ is minimax and

$$(22) \quad \sup_{\pi \in \Gamma} r(\pi, U_{1,0}^-) = \left(t_0 - 2q \sum_{i=1}^{M-1} \frac{i}{1+qi} s_i + \sum_{i=1}^{M-1} t_{i0} \frac{s_i}{(1+qi)^2} \right) m_2 + a_0 x_0^2.$$

Finally, let the v_n be distributed according to the binomial law. We prove that in the case (c) the policy U_{0,m_2}^- is minimax. We have for $\pi \in \Gamma$ and $(\gamma, s) \in S$

$$r(\pi, U_{0,m_2}^-) = \lim_{\substack{\gamma \rightarrow 0 \\ s/\gamma \rightarrow m_2}} (e_0(\gamma) m_2 + Z(\gamma, s) E_\pi(\lambda) + T(\gamma, s)) \\ \leq \lim_{\substack{\gamma \rightarrow 0 \\ s/\gamma \rightarrow m_2}} (e_0(\gamma) m_2 + Z(\gamma, s) m_2 + T(\gamma, s))$$

since now $0 < \lambda < 1$, which implies $E_\pi(\lambda) \geq E_\pi(\lambda^2) = m_2$.

Then we obtain

$$(23) \quad \sup_{\pi \in \Gamma} r(\pi, U_{0,m_2}^-) = -s_0 m_2^2 + \left(t_0 - \sum_{i=1}^{M-1} \frac{1+qi}{qi} s_i + 2b_0 x_0 + qy_0 \right) m_2 + a_0 x_0^2 \\ = c = \lim_{\substack{\beta \rightarrow 0 \\ r/\beta \rightarrow m_2}} r(\pi_{\beta,r}, U_{\beta,r}^*)$$

for $(\beta, r) \in S \cap S_k$ and we have the thesis after applying the lemma.

Let $U^{(0)}$ be a minimax policy. The value of the game is defined as follows:

$$c = \sup_{\pi \in \Gamma} r(\pi, U^{(0)}).$$

Thus we have proved the following theorem:

THEOREM. *If there is a point $(\beta, r) \in S$ such that*

$$k(\beta, r) = m_2, \quad Z(\beta, r) = 0,$$

where $k(\beta, r)$ and $Z(\beta, r)$ are defined by (8) and (18), respectively, then a minimax control policy is $U_{\beta, r}^*$ defined in (10).

If $Z(\beta, r) > 0$ for each $(\beta, r) \in S$ such that $k(\beta, r) = m_2$ then a minimax control policy is $U_{\sqrt{m_2}}^+$ defined by (13).

If $Z(\beta, r) < 0$ for each $(\beta, r) \in S$ such that $k(\beta, r) = m_2$ then a minimax control policy is

- (i) $U_{\sqrt{m_2}}^+$ for the normal and the GEHS distribution,
- (ii) $U_{0,0}^-$ for the Poisson distribution,
- (iii) $U_{1,0}^-$ for the gamma and the negative binomial distribution,
- (iv) U_{0,m_2}^- for the binomial distribution.

Policy $U_{\beta, m}^-$ is defined in (15).

The value c of the game is

(a) in the first case when policy $U_{\beta, r}^*$ is minimax,

$$c = e_0 m_2 + a_0 x_0^2 + c_0 r^2 + j_0;$$

(b) in the second case when policy $U_{\sqrt{m_2}}^+$ is minimax, c is given by the equation (19);

(c) in the third case when policies $U_{\sqrt{m_2}}^+$, $U_{0,0}^-$, $U_{1,0}^-$, U_{0,m_2}^- are minimax, c is given by the equations (20)–(23), respectively.

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