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Poincaré series and Kloosterman sums for SL(3, *Z*)

by

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CONTENTS

1. Introduction. For $z = x+iy$, $y > 0$, let

$$\Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi inz}$$

by the Ramanujan cusp form of weight 12. Ramanujan conjectured that

$$\tau(n) = O(n^{11/2+\varepsilon})$$

for any $\varepsilon > 0$. This conjecture was proved by Deligne [2] in 1974. Actually, Deligne proved the more general result (Petersson conjecture) that

$$a(n) = O(n^{(k-1)/2+\varepsilon}) \tag{1.1}$$

where $a(n)$ is the nth Fourier coefficient of a holomorphic cusp form of weight k associated to a congruence subgroup of SL(2, *Z*).

Ramanujan's conjecture can be generalized to non-holomorphic cusp forms (Maass wave forms) associated to arithmetic discrete subgroups of GL(r, *R*), $r \geqslant 2$, and in this form, the conjecture is still open even for $r = 2$. We now briefly describe the generalized Ramanujan conjecture.

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Let $G_r = \mathrm{GL}(r, \boldsymbol{R})$, $\Gamma_r = \mathrm{SL}(r, \boldsymbol{Z})$, $O(r) =$ the orthogonal group, and $Z =$ the center of G_r consisting of scalar matrices. We define the generalized upper half space

$$H^r = G_r/O(r)Z.$$

By the Iwasawa decomposition, every $\tau \in H^r$ has a unique coset representative of the form

$$\tau = \begin{bmatrix} 1 & x_{1,2} & \dots & x_{1,r} \\ & 1 & & \vdots \\ & & & x_{r-1,r} \\ & & & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 \dots y_{r-1} & & & & \\ & y_1 y_2 \dots y_{r-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{bmatrix} \tag{1.2}$$

where $y_1, \dots, y_{r-1} > 0$ and $x_{i,j} \in \boldsymbol{R}$ for $j > i$.

If $\nu_1, \nu_2, \dots, \nu_{r-1}$ are complex parameters, we define a function $I_{\nu_1,\dots,\nu_{r-1}}\colon H^r \to C$ by requiring that

$$I_{\nu_1,\dots,\nu_{r-1}}(\tau) = \prod_{i=1}^{r-1} \prod_{j=1}^{r-1} y_i^{c_{ij}\nu_j} \tag{1.3}$$

where

$$c_{ij} = \begin{cases} (r-i)j, & 1 \leqslant j \leqslant i, \\ (r-j)i, & i \leqslant j \leqslant r-1. \end{cases}$$

Let $\mathscr{D}$ denote the algebra of G_r-invariant differential operators on H^r. Then $I_{\nu_1,\dots,\nu_{r-1}}$ is an eigenfunction of $\mathscr{D}$, and hence determines a character $\lambda_{\nu_1,\dots,\nu_{r-1}}$ on $\mathscr{D}$ by the formula

$$D I_{\nu_1,\dots,\nu_{r-1}} = \lambda_{\nu_1,\dots,\nu_{r-1}}(D) I_{\nu_1,\dots,\nu_{r-1}} \qquad (D \in \mathscr{D}).$$

Let I_r denote the identity matrix on G_r. For a positive integer M, let

$$\Gamma_r(M) = \{\gamma \in \Gamma_r \mid \gamma \equiv I_r (\mathrm{mod}\ M)\}$$

be the principal congruence subgroup (mod M).

DEFINITION 1.1. A function φ on H^r is called an *automorphic form of type* $\nu_1, \dots, \nu_{r-1}$ *for* $\Gamma_r(M)$ if

(1.4) $\qquad \varphi(\gamma\tau) = \varphi(\tau) \quad$ for $\quad \gamma \in \Gamma_r(M), \quad \tau \in H^r$,

(1.5) $\qquad D\varphi = \lambda_{\nu_1,\dots,\nu_{r-1}}(D) \cdot \varphi \quad$ for $\quad D \in \mathscr{D}$,

(1.6) $\varphi(\varrho\tau)$ has polynomial growth in $y_1, \dots, y_{r-1}$ on the region $\{\tau \mid y_i \geqslant 1$ $(i = 1, 2, \dots, r-1)\}$, for every $\varrho \in \Gamma_r(M) \backslash \Gamma_r$.

If, furthermore, φ satisfies

$$\int_{\Gamma_r(M) \cap U \backslash U} \varphi(\varrho u \tau)\, du = 0 \qquad (\tau \in H^r) \tag{1.7}$$

for every $\varrho \in \Gamma_r(M) \backslash \Gamma_r$, and each group U of the form

$$U = \begin{bmatrix} I_{r_1} & & & * \\ & I_{r_2} & & \\ & & \ddots & \\ 0 & & & I_{r_s} \end{bmatrix} \subset G_r$$

then φ is called a *cusp form*. This implies that $\varphi \in \mathscr{L}^2(\Gamma_r(M) \backslash H^r)$.

Let N_r denote the group of upper triangular matrices with unit diagonal, and for integers $n_1, \dots, n_{r-1}$, define a character θ of N_r by

$$\theta(x) = \prod_{1 \leqslant j \leqslant r-1} e^{\frac{2\pi i}{M} n_j x_{j,j+1}} \tag{1.8}$$

where

$$x = \begin{bmatrix} 1 & x_{1,2} & \dots & x_{1,r} \\ & 1 & & \vdots \\ & & & x_{r-1,r} \\ & & & 1 \end{bmatrix} \in N_r.$$

For a cusp form φ for $\Gamma_r(M)$ and $\varrho \in \Gamma_r(M) \backslash \Gamma_r$, let

$$\varphi^{\varrho}_{n_1,\dots,n_{r-1}}(\tau) = \int_{\Gamma_r(M) \cap N_r \backslash \Gamma_r} \varphi(\varrho u \tau) \overline{\theta(u)}\, du \tag{1.9}$$

where du is the Haar measure for N_r.

Then φ has a Fourier expansion (see [16], [19], [1], [9], [7])

$$\varphi(\tau) = \sum_{n_1=1}^{\infty} \sum_{\substack{n_2=-\infty \\ n_2 \neq 0}}^{\infty} \dots \sum_{\substack{n_{r-1}=-\infty \\ n_{r-1} \neq 0}}^{\infty} \sum_{g \in R} \varphi^{\varrho_g}_{n_1,\dots,n_{r-1}}\left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \tau\right) \tag{1.10}$$

where R is a set of coset representatives for $\Gamma_{r-1}(M) \cap N_{r-1} \backslash \Gamma_{r-1}$, and for each $g \in R$, we choose ϱ_g such that

$$\begin{bmatrix} \varrho_g \cdot g & 0 \\ 0 & 1 \end{bmatrix} \in \Gamma_r(M).$$

Now, by the multiplicity one theorem of Shalika [19], it follows that there exists a constant $a^{\varrho}_{n_1,\dots,n_{r-1}}$ such that

$$\varphi^{\varrho}_{n_1,\dots,n_{r-1}}(\tau) = a^{\varrho}_{n_1,\dots,n_{r-1}} \prod_{i=1}^{r-1} |n_i|^{-\frac{1}{2}i(r-i)} \varphi^{\varrho}_{1,\dots,1}(d\tau) \tag{1.11}$$

where

$$d = \begin{bmatrix} n_1 n_2 \dots n_{r-1} & & & & \\ & n_2 \dots n_{r-1} & & & \\ & & \ddots & & \\ & & & n_{r-1} & \\ & & & & 1 \end{bmatrix}$$

The constants $a^{\varrho}_{n_1,\dots,n_{r-1}}$ are called the *Fourier coefficients* of φ. If φ is an eigenform for the Hecke algebra, then the Fourier coefficients satisfy multiplicative properties.

GENERALIZED RAMANUJAN CONJECTURE. *Let φ be an automorphic cusp form of type $(\nu_1, \dots, \nu_{r-1})$ for $\Gamma_r(M)$ with Fourier expansion given by* (1.9), (1.10), (1.11). *Then*

$$\mathrm{Re}(\nu_1) = \mathrm{Re}(\nu_2) = \dots = \mathrm{Re}(\nu_{r-1}) = 1/r$$

and

$$a^{\varrho}_{n,1,\dots,1,1} = O(n^{\varepsilon})$$

for every $\varepsilon > 0$, and every $\varrho \in \Gamma_r(M)\backslash\Gamma_r$.

If, in addition, φ is an eigenfunction for the Hecke algebra, then the above conjecture implies that

$$|a^{\varrho}_{n,1,\dots,1,1}| \leqslant r \cdot |a^{\varrho}_{1,\dots,1}|.$$

This conjecture was first explicitly stated by A. Selberg [18] for the case $r = 2$. The conjecture can be rephrased in the adelic language and is equivalent to the non-occurrence of complementary series representations. We indicate how this is done at the Archimedean place.

If φ is a Hecke eigenform of type $\nu_1, \dots, \nu_{r-1}$ for Γ_r, let ϱ_φ be the restriction of the right regular representation of G_r (defined on $\mathscr{L}^2(\Gamma_r\backslash G_r)$) to the subspace spanned by $\{\varphi(\tau)|\ \tau \in H^r\}$. As φ is square-integrable, ϱ_φ must be unitary, and this is a constraint on the possible values of $\nu_1, \dots, \nu_{r-1}$. Let B denote the Borel subgroup of upper triangular matrices of type

$$b = \begin{bmatrix} 1 & x_{1,2} & \dots & x_{1,r} \\ & 1 & & \vdots \\ & & & x_{r-1,r} \\ & & & 1 \end{bmatrix} \begin{bmatrix} y_0 y_1 \dots y_{r-1} & & & \\ & y_0 y_1 \dots y_{r-2} & & \\ & & \ddots & \\ & & & y_0 \end{bmatrix}$$

in G_r. Then ϱ_φ is the representation induced by the character $\delta^{-1/2} I_{\nu_1,\dots,\nu_{r-1}}$ of B where

$$I_{\nu_1,\dots,\nu_{r-1}}(b) = \prod_{i=1}^{r-1} \prod_{j=1}^{r-1} |y_i|^{c_{ij}\nu_j}$$

as in (1.3), and δ is the modular function of B (cf. [6])

$$\delta(b) = \prod_{i=1}^{r-1} |y_i|^{i(r-i)}.$$

Now $\delta^{-1/2} I_{\nu_1,\dots,\nu_{r-1}}$ is a unitary character of B if

$$\mathrm{Re}(\nu_1) = \mathrm{Re}(\nu_2) = \dots = \mathrm{Re}(\nu_{r-1}) = 1/r,$$

and in this case ϱ_φ or φ is called *unitary principal series*. If ϱ_φ is unitary, but $\delta^{-1/2} I_{\nu_1,\dots,\nu_{r-1}}$ is not, then φ is called *complementary series*. The generalized Ramanujan conjecture asserts that complementary series do not occur in $\mathscr{L}^2(\Gamma_r(M)\backslash H^r)$. For example, if $r = 3$, ϱ_φ will be complementary series if $\nu_1 = \sigma + it$, $\nu_2 = \sigma - it$, $1/6 < \sigma < 1/2$, $t \in \boldsymbol{R}$, $\sigma \neq 1/3$.

Langlands [11] has made some very general conjectures on liftings of representations that would settle not only the generalized Ramanujan conjecture, but also Artin's conjecture on the holomorphicity of the Artin L-functions associated to non-abelian Galois groups. At present, however, the so-called "Langlands program" is generally considered to be a long way from completion.

The purpose of this work is to provide an approach to the generalized Ramanujan conjecture for $\mathrm{GL}(2, \boldsymbol{R})$ and $\mathrm{GL}(3, \boldsymbol{R})$ *via* Kloosterman sums. Such connections first appeared in the work of Kloosterman [10], who essentially showed that

$$a(n) = O(n^{k/2-1/8+\varepsilon}) \tag{1.12}$$

where $a(n)$ is the nth Fourier coefficient of a holomorphic cusp form of weight k for a congruence subgroup $\Gamma \subset \mathrm{SL}(2, \boldsymbol{Z})$. In 1948, Weil [21] elucidated the connection between the Riemann hypothesis for curves over F_p and the classical Kloosterman sum

$$S(m, n; p) = \sum_{\substack{a=1 \\ a\bar{a} \equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} e^{2\pi i\left(\frac{am+\bar{a}n}{p}\right)}.$$

Using multiplicative properties of the Kloosterman sums, this led to the bound

$$|S(m, n; c)| \leqslant \tau(c)(m, n, c)^{1/2} c^{1/2}$$

where $\tau(c)$ denotes the number of divisors of c and (m, n, c) is the greatest common divisor of m, n, c. A consequence of this is the estimate

$$a(n) = O(n^{k/2-1/4+\varepsilon}),$$

a considerable improvement over (1.12).

Connections between Kloosterman sums and holomorphic Poincaré series were already given in 1932 by H. Petersson [15]. A. Selberg [18] went considerably further by investigating the non-holomorphic Poincaré series

$$P_n(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im}(\gamma z))^s e^{2\pi i n \gamma z} \qquad (\operatorname{Re}(s) > 1)$$

where $n > 0$, $\Gamma_\infty = \left\{ \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \middle| \ m \in \mathbf{Z} \right\} \cap \Gamma$, $z = x+iy \in \mathbf{C}$, $y > 0$ and γz denotes the usual action by linear fractional transformations. He, for the first time, brought into play the deep connections between Kloosterman sums, which occur in the Fourier expansion of $P_n(z, s)$, and the spectral theory of $\Gamma \backslash H^2$. By use of this theory, Selberg obtained the meromorphic continuation of the zeta function

$$\sum_{c=1}^{\infty} \frac{S(m, n; c)}{c^{2s}} \qquad (\operatorname{Re}(s) > 3/4). \tag{1.13}$$

As a consequence, he obtained the bounds

$$1/4 < \operatorname{Re}(\nu) < 3/4,$$
$$a_n = O(n^{1/4+\varepsilon})$$

where a_n denotes the nth Fourier coefficient of a cusp form of type ν for a congruence subgroup $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$. Recently, [14], [5], the Fourier coefficient bound has been improved to

$$a_n = O(n^{1/5+\varepsilon})$$

by use of certain liftings in accordance with the "Langland's philosophy".

In order to generalize the Kloosterman sum approach, we consider Poincaré series for G_r. For integers $n_1, \ldots, n_{r-1}$, let θ be the character of N_r given by (1.8). An E-function is a smooth function $E\colon H^r \to \mathbf{C}$ satisfying the conditions

$$E(x\tau) = \theta(x) E(\tau), \qquad x \in N_r,\ \tau \in H^r, \tag{1.14}$$

$$|E(\tau)| = O(1), \qquad \tau \in H^r. \tag{1.15}$$

For complex parameters $\nu_1, \ldots, \nu_{r-1}$, let $I_{\nu_1,\ldots,\nu_{r-1}}$ be the function given by (1.3). The Poincaré series $P_{n_1,\ldots,n_{r-1}}(\tau; \nu_1, \ldots, \nu_{r-1})$ is defined as

$$P_{n_1,\ldots,n_{r-1}}(\tau; \nu_1, \ldots, \nu_{r-1}) = \sum_{\gamma \in \Gamma_r \cap N_r \backslash \Gamma_r} I_{\nu_1,\ldots,\nu_{r-1}}(\gamma\tau) E(\gamma\tau) \tag{1.16}$$

where the series on the right-hand side of (1.16) converges uniformly and absolutely on compact subsets of H^r if $\operatorname{Re}(\nu_i) > 2/r$, $i = 1, 2, \ldots, r-1$. If, in addition, $E(\tau)$ has exponential decay in y_i ($y_i \to \infty$) for all $i = 1, 2, \ldots, r-1$, with τ given by (1.2), then it is easily seen that $P_{n_1,\ldots,n_{r-1}}$ is square-integrable.

In Theorem 5.1, we compute the Fourier expansion of $P_{n_1,n_2}(\tau; \nu_1, \nu_2)$ for the Poincaré series associated to G_3. It is shown that

$$\int_0^1\!\int_0^1\!\int_0^1 P_{n_1,n_2}(\tau; \nu_1, \nu_2) e^{-2\pi i(m_1x_1+m_2x_2)} dx_1\, dx_2\, dx_3 \tag{1.17}$$

$$= y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} \sum_{w \in W} \sum_{\varepsilon_1,\varepsilon_2 = \pm 1} \sum_{D_1,D_2=1}^{\infty} D_1^{-3\nu_1} D_2^{-3\nu_2} \times$$
$$S_w(\varepsilon_1 m_1, \varepsilon_2 m_2, n_1, n_2; D_1, D_2) J_w(y_1, y_2; \nu_1, \nu_2; \varepsilon_1 m_1, \varepsilon_2 m_2; n_1, n_2; D_1, D_2)$$

where W is the Weyl group of permutation matrices of $\mathrm{GL}(3, \mathbf{Z})$, S_w are generalized Kloosterman sums (see § 4), and J_w are certain integrals given in Table (5.4). Note that the triple integral on the left side of (1.17) is just (1.9) applied to P_{n_1,n_2}.

The six GL(3) Kloosterman sums S_w can be described as follows. Firstly,

$$S_w = \begin{cases} \delta_{D_1,1} \cdot \delta_{D_2,1}, & w = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \\ \delta_{D_1,1} S(m_2, n_2; D_2), & w = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \\ \delta_{D_2,1} S(m_1, n_1; D_1), & w = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \end{cases}$$

where

$$\delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

For the long element $w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$,

$$S_w = S(D_2 m_2, n_1; D_1) S(D_1 m_1, n_2; D_2)$$

if $(D_1, D_2) = 1$. Finally, the remaining two Kloosterman sums, corresponding to the non-trivial Weyl group elements of order 3 (in the case $D_1 = p$, $D_2 = p^2$ or $D_1 = p^2$, $D_2 = p$) are associated to a certain surface over F_p (see Appendix by M. Larsen). Using the Riemann hypothesis for varieties over finite fields, proved by Deligne (cf. [3]), the multiplicative properties of S_w lead to Larsen's bound (Theorem 1 of the Appendix)

$$|S_w| \leqslant \min\{\tau(D_1)^{\varkappa}(n_2, D_2/D_1)D_1^2, \tau(D_2)(m_1, n_1, D_1)D_2\}, \tag{1.18}$$

$$w = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \varkappa = \frac{\log 3}{\log 2}$$

and a similar estimate (with indices 1 and 2 interchanged) when $w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

It is an interesting problem to determine the Kloosterman sums, and their associated algebraic varieties, for the groups Γ_r, $r \geqslant 4$.

The meromorphic continuations of $P_{n_1,n_2}(\tau; \nu_1, \nu_2)$ and of the Fourier coefficients (1.17) are obtained in Sections 6 and 7. It is shown that if $\{\varphi^{(j)}\}_{j=1,\ldots,h}$ is an orthonormal basis for the cusp forms of fixed type (λ_1, λ_2) for Γ_3, then $P_{n_1,n_2}(\tau; \nu_1, \nu_2)$ has a meromorphic continuation in ν_1, ν_2 with polar divisors at the lines

$$2\nu_1 + \nu_2 - 1 = \begin{cases} \bar{\lambda}_1 + 2\bar{\lambda}_2 - 1 - 2N, \\ \bar{\lambda}_1 - \bar{\lambda}_2 - 2N, \\ 1 - 2\bar{\lambda}_1 - \bar{\lambda}_2 - 2N, \end{cases} \tag{1.19}$$

$$\nu_1 + 2\nu_2 - 1 = \begin{cases} 1 - \bar{\lambda}_1 - \bar{\lambda}_2 - 2N, \\ \bar{\lambda}_2 - \bar{\lambda}_1 - 2N, \\ 2\bar{\lambda}_1 + \bar{\lambda}_2 - 1 - 2N, \end{cases} \tag{1.20}$$

where $N \geqslant 0$ is an integer.

In Section 6 (see (6.16) for a precise version) we show that the inner product (for $\mathrm{Re}(w_1) > \mathrm{Re}(\nu_1)$, $\mathrm{Re}(w_2) > \mathrm{Re}(\nu_2)$)

$$\int_{\Gamma_3\backslash H^3} P_{n_1,n_2}(\tau; \nu_1, \nu_2)\overline{P_{m_1,m_2}(\tau; w_1, w_2)}\,\frac{dx_1\,dx_2\,dx_3\,dy_1\,dy_2}{(y_1 y_2)^3}$$

also has polar divisors at the above lines with residue (for the lines $N = 0$) proportional to

$$n_1^{2\nu_1+\nu_2}\, n_2^{\nu_1+2\nu_2}\, m_1^{2w_1+w_2-1}\, m_2^{w_1+2w_2-1} \sum_{j=1}^{h} a_{n_1,n_2}^{(j)}\, \overline{a_{m_1,m_2}^{(j)}} \tag{1.21}$$

where $a_{n_1,n_2}^{(j)}$ denotes the n_1th, n_2th Fourier coefficient of $\varphi^{(j)}$ as defined in (1.11), and the constant of proportionality is independent of n_1, n_2, m_1, m_2. It is interesting that the continuous spectrum does not contribute any non-trivial polar divisors.

There are additional intriguing spectral results; for example, double poles on the diagonal $\nu_1 = \nu_2$ correspond to the Gelbart–Jacquet lift [5] from G_2 to G_3.

On the basis of the above results, it can be shown that the Kloosterman zeta function

$$\sum_{w\in W}\sum_{D_1,D_2=1}^{\infty} S_w(m_1, m_2, n_1, n_2; D_1, D_2)\, D_1^{-3\nu_1} D_2^{-3\nu_2}$$

has a meromorphic continuation in ν_1, ν_2 with polar divisors at the lines (1.19), (1.20) with residue proportional to

$$n_1^{1-2\nu_1-\nu_2}\, n_2^{1-\nu_1-2\nu_2}\, m_1^{1-2\nu_1-\nu_2}\, m_2^{1-\nu_1-2\nu_2} \sum_{j=1}^{h} a_{n_1,n_2}^{(j)}\, \overline{a_{m_1,m_2}^{(j)}}. \tag{1.22}$$

It is reasonable to conjecture, that for any fixed $w \in W$, the zeta function

$$Z_w(\nu_1, \nu_2) = \sum_{D_1,D_2=1}^{\infty} S_w(m_1, m_2, n_1, n_2; D_1, D_2)\, D_1^{-3\nu_1} D_2^{-3\nu_2}$$

has a meromorphic continuation in ν_1, ν_2. In Section 8, heuristic evidence is developed, which suggests that for

$$w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$Z_w(\nu_1, \nu_2)$ has polar divisors at the lines (1.19), while for

$$w = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$Z_w(\nu_1, \nu_2)$ has polar divisors at the lines (1.20). We are, therefore, led to

CONJECTURE 1.2. *The partial Kloosterman zeta function*

$$Z(\nu_1, \nu_2) = \sum_{w^3=1} Z_w(\nu_1, \nu_2) \tag{1.23}$$

has a meromorphic continuation in ν_1, ν_2 with polar divisors at the lines (1.19), (1.20), *and with residue proportional to* (1.22) *at these lines, where the constant of proportionality is independent of m_1, m_2, n_1, n_2 at the lines $N = 0$.*

The conjecture has the following consequence.

THEOREM 1.3. *Suppose Conjecture 1.2 is true. Then the generalized Ramanujan conjecture holds for $r = 3$.*

Proof. First, the bound given in (1.18) implies that the partial Kloosterman zeta function (1.23) is holomorphic in the region $\mathrm{Re}(\nu_1) > \frac{1}{3}$, $\mathrm{Re}(\nu_2) > \frac{1}{3}$. Hence Conjecture 1.2 implies that every automorphic cusp form of type (λ_1, λ_2) satisfies $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{1}{3}$, or equivalently that complementary series representations do not occur in the decomposition of the right regular representations of $\mathrm{GL}(3, \boldsymbol{R})$ on $\mathscr{L}^2(\Gamma_3 \backslash H^3)$. As for the finite primes, set

$$\nu_1 = \bar{\lambda}_2 + \tfrac{1}{3}\varepsilon, \quad \nu_2 = \bar{\lambda}_1 + \tfrac{1}{3}\varepsilon,$$

and choose $m_1 = n_1 = 1$, $m_2 = n_2 = n$. Then Conjecture 1.2 implies that

$$|Z(\bar{\lambda}_2 + \tfrac{1}{3}\varepsilon, \bar{\lambda}_1 + \tfrac{1}{3}\varepsilon)| \sim \frac{c_1 |n|^{-2\varepsilon} \sum_{j=1}^{h} |a_{1,n}^{(j)}|^2}{\varepsilon} \tag{1.24}$$

as $\varepsilon \to 0+$, where c_1 is some constant independent of n. On the other hand, using the bound (1.18), and recalling that

$$\sum_{D_1=1}^{\infty} \frac{1}{D_1^{1+\varepsilon}} \sim \frac{1}{\varepsilon}$$

as $\varepsilon \to 0+$, one finds that

$$|Z(\bar{\lambda}_2 + \tfrac{1}{3}\varepsilon, \bar{\lambda}_1 + \tfrac{1}{3}\varepsilon)| < c_2/\varepsilon \tag{1.25}$$

for some constant c_2 independent of n. Combining (1.24) and (1.25), the theorem follows.

While we believe Conjecture 1.2 to be true, it should be pointed out that there is also reason to be cautious. The conjecture predicts that the zeta function $Z(\nu_1, \nu_2)$ has absolute convergence right up to the point where the first pole occurs (if one restricts one's attention to the line $\nu_1 = \nu_2$). This is in contradiction to the analogous situation with the Selberg–Kloosterman zeta function formed with $\mathrm{GL}(2)$ Kloosterman sums, where the Selberg–Linnik conjecture predicts that cancellation between the sums will cause the zeta function to have a pole-free region extending somewhat to the left of the line of absolute convergence.

Finally, let us compare this method with the results obtainable by representation theory. We have already pointed out that for a cusp form of type (ν_1, ν_2) to exist, the representation ϱ_∞ must be unitary, which implies that $\mathrm{Re}(\nu_1)$, $\mathrm{Re}(\nu_2) < \frac{1}{2}$. This is the same precise bound obtained by assuming that the poles of the Poincaré series correspond to the poles of the "big cell" Kloosterman zeta function

$$Z_w(\nu_1, \nu_2), \qquad w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

By contrast, the Kloosterman sums occurring as coefficients in (1.23) are smaller than "big cell" sums, so Conjecture 1.2 predicts a result stronger than that obtainable by representation theory.

2. Poincaré series. Let

$$\tau = \begin{bmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{bmatrix} \in H^3. \tag{2.1}$$

We now redefine the notion of an E-function on H^3. For every pair of integers n_1, n_2, an E-function is a function

$$E_{n_1,n_2}\colon H^3 \to \boldsymbol{C}$$

satisfying the condition

$$E_{n_1,n_2}\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{bmatrix} \tau\right) = e(n_1 \xi_1 + n_2 \xi_2) E_{n_1,n_2}(\tau) \tag{2.2}$$

for all ξ_1, ξ_2, $\xi_3 \in \boldsymbol{R}$. Clearly, such a function is independent of x_3. In practice, we shall usually work with the special E-function

$$E_{n_1,n_2}(\tau) = e(n_1 z_1 + n_2 z_2) \tag{2.3}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. An E-function which is also an eigenfunction of $\mathscr{D}$ (see [1], p. 21) is necessarily a Whittaker function for $\mathrm{GL}(3, \boldsymbol{R})$.

Now let us fix the notation

$$\Gamma = \mathrm{SL}(3, \boldsymbol{Z}),$$

$$\Gamma_\infty = \left\{\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} \in \mathrm{SL}(3, \boldsymbol{Z})\right\}$$

where Γ_∞ is the minimal parabolic subgroup of Γ.

For $\tau \in H^3$, let

$$I_{\nu_1,\nu_2}(\tau) = y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} \tag{2.4}$$

and for fixed integers n_1, n_2, let $E_{n_1,n_2}(\tau)$ be an E-function satisfying the growth condition

$$E_{n_1,n_2}(\tau) = O(1) \quad (\text{for } \tau \in H^3,\ y_1, y_2 = O(1)). \tag{2.5}$$

Let ν_1, ν_2 be two complex variables where $\mathrm{Re}(\nu_1)$, $\mathrm{Re}(\nu_2) > 2/3$, and $E_{n_1,n_2}(\tau)$ as above. The general Poincaré series for the minimal parabolic subgroup Γ_∞ is

$$P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} I_{\nu_1,\nu_2}(\gamma\tau)\,E_{n_1,n_2}(\gamma\tau). \tag{2.6}$$

By abuse of notation, we have not expressed the dependence on E_{n_1,n_2} in the symbol $P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2)$. In view of (2.5), it is clear that the series on the right-hand side of (2.6) converges absolutely and uniformaly on compact subsets of H^3 as long as $\mathrm{Re}(\nu_1)$, $\mathrm{Re}(\nu_2) > 2/3$.

Now, let Δ_1, Δ_2 be the two generators ([1], pp. 33–34) for the algebra of $GL(3,\,\boldsymbol{R})$ invariant differential operators acting on H^3. It is easily shown that P_{n_1,n_2} satisfies the following differential recursion relations, which we have tabulated for two different choices of E-functions. Here λ_1, λ_2 are defined by the eigenvalue relations

$$\Delta_1 I_{\nu_1,\nu_2}(\tau) = \lambda_1 I_{\nu_1,\nu_2}(\tau), \qquad \Delta_2 I_{\nu_1,\nu_2}(\tau) = \lambda_2 I_{\nu_1,\nu_2}(\tau).$$

For $E_{n_1,n_2}(\tau) = e(n_1x_1+n_2x_2)$, we have

$$\begin{aligned}\Delta_1 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2) &= \lambda_1 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2)\\ &\quad -4\pi^2\big[n_1^2 P_{n_1,n_2}(\tau;\,\nu_1+\tfrac{4}{3},\,\nu_2-\tfrac{2}{3})\\ &\quad +n_2^2 P_{n_1,n_2}(\tau;\,\nu_1-\tfrac{2}{3},\,\nu_2+\tfrac{4}{3})\big],\\ \Delta_2 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2) &= \lambda_2 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2)\\ &\quad -4\pi^2\big[n_1^2(1-\nu_1-2\nu_2)P_{n_1,n_2}(\tau;\,\nu_1+\tfrac{4}{3},\,\nu_2-\tfrac{2}{3})\\ &\quad +n_2^2(2\nu_1+\nu_2-1)P_{n_1,n_2}(\tau;\,\nu_1-\tfrac{2}{3},\,\nu_2+\tfrac{4}{3})\big].\end{aligned}\tag{2.7}$$

For the case $E_{n_1,n_2}(\tau) = e(n_1z_1+n_2z_2)$, we have

$$\begin{aligned}\Delta_1 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2) &= \lambda_1 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2)\\ &\quad -4\pi^2 n_1n_2 P_{n_1,n_2}(\tau;\,\nu_1+\tfrac{1}{3},\,\nu_2+\tfrac{1}{3})\\ &\quad -6\pi\big[n_1\nu_1 P_{n_1,n_2}(\tau;\,\nu_1+\tfrac{2}{3},\,\nu_2-\tfrac{1}{3})\\ &\quad +n_2\nu_2 P_{n_1,n_2}(\tau;\,\nu_1-\tfrac{1}{3},\,\nu_2+\tfrac{2}{3})\big],\\ \Delta_2 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2) &= \lambda_2 P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2)\\ &\quad +8\pi^2 n_1n_2(\nu_2-\nu_1)P_{n_1,n_2}(\tau;\,\nu_1+\tfrac{1}{3},\,\nu_2+\tfrac{1}{3})\\ &\quad +6\pi\big[n_1\nu_1(\nu_1+2\nu_2-1)P_{n_1,n_2}(\tau;\,\nu_1+\tfrac{2}{3},\,\nu_2-\tfrac{1}{3})\\ &\quad +n_2\nu_2(-2\nu_1-\nu_2+1)P_{n_1,n_2}(\tau;\,\nu_1-\tfrac{1}{3},\,\nu_2+\tfrac{2}{3})\big].\end{aligned}\tag{2.8}$$

The most useful property of a Poincaré series is that its inner product against an automorphic form is essentially the double Mellin transform of the n_1th, n_2th Fourier coefficient of that form. This is expressed in the following proposition.

PROPOSITION 2.1. *Let* $\varphi\in\mathscr{L}^2(\Gamma\backslash H^3)$. *Then*

$$\langle P_{n_1,n_2},\,\varphi\rangle = \int_0^\infty\!\!\int_0^\infty \varphi_{n_1,n_2}\left(\begin{bmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{bmatrix}\right) \times y_1^{2\nu_1+\nu_2}\,y_2^{\nu_1+2\nu_2}\,E_{n_1,n_2}\left(\begin{bmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{bmatrix}\right)\frac{dy_1\,dy_2}{(y_1y_2)^3},$$

where

$$\varphi_{n_1,n_2}(\tau) = \int_0^1\!\!\int_0^1\!\!\int_0^1 \bar{\varphi}\left(\begin{bmatrix} 1 & \xi_2 & \xi_3\\ & 1 & \xi_1\\ & & 1\end{bmatrix}\tau\right)e(n_1\xi_1+n_2\xi_2)\,d\xi_1\,d\xi_2\,d\xi_3.$$

Proof. By the Rankin–Selberg unfolding method

$$\begin{aligned}\langle P_{n_1,n_2},\,\varphi\rangle &= \int_{\Gamma\backslash H^3} P_{n_1,n_2}(\tau;\,\nu_1,\,\nu_2)\,\bar{\varphi}(\tau)\frac{dx_1\,dx_2\,dx_3\,dy_1\,dy_2}{(y_1y_2)^3}\\ &= \int_{\Gamma_\infty\backslash H^3} \bar{\varphi}(\tau)\,y_1^{2\nu_1+\nu_2}\,y_2^{\nu_1+2\nu_2}\,E_{n_1,n_2}(\tau)\frac{dx_1\,dx_2\,dx_3\,dy_1\,dy_2}{(y_1y_2)^3}.\end{aligned}$$

To complete the proof, we note that

$$\int_{\Gamma_\infty\backslash H^3} = \int_0^\infty\!\!\int_0^\infty\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1$$

and that

$$E_{n_1,n_2}(\tau) = e(n_1x_1+n_2x_2)\,E_{n_1,n_2}\left(\begin{bmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{bmatrix}\right),$$

the last assertion following from (2.1), (2.2) and the identity

$$\tau = \begin{bmatrix} 1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{bmatrix}\begin{bmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{bmatrix}.$$

Let us close this section by considering the maximal parabolic Poincaré series. There are precisely two non-conjugate maximal parabolic subgroups of Γ. Namely

$$P_{2,1} = \left\{\begin{bmatrix} * & * & *\\ * & * & *\\ 0 & 0 & 1\end{bmatrix}\in\Gamma\right\},\qquad P_{1,2} = \left\{\begin{bmatrix} 1 & * & *\\ 0 & * & *\\ 0 & * & *\end{bmatrix}\in\Gamma\right\}.$$

For $z = x+iy \in H^2$, $\Gamma^2 = \mathrm{SL}(2, \boldsymbol{Z})$, $\Gamma^2_\infty = \left\{\begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \in \Gamma^2\right\}$ and $n \geqslant 0$, let

$$(2.9)\qquad P_n(z, s) = \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \left(\mathrm{Im}(\gamma z)\right)^s e^{2\pi i n \gamma(z)}$$

be the Poincaré series for Γ^2. We can now define the $\mathrm{SL}(3, \boldsymbol{Z})$ maximal parabolic Poincaré series induced from (2.9).

With the convention

$$f(\tau)|[\gamma] = f(\gamma\tau) \qquad (\gamma \in \mathrm{GL}(3, \boldsymbol{R}))$$

for any $f\colon H^3 \to \boldsymbol{C}$, we define for $n \geqslant 0$

$$(2.10)\qquad \begin{aligned} P_n(\tau; s_1, s_2) &= \sum_{\gamma \in P_{2,1} \backslash \Gamma} (y_1^2 y_2)^{s_1} P_n(x_2+iy_2, s_2)|[\gamma], \\ \tilde{P}_n(\tau; s_1, s_2) &= \sum_{\gamma \in P_{1,2} \backslash \Gamma} (y_1 y_2^2)^{s_1} P_n(x_1+iy_1, s_2)|[\gamma]. \end{aligned}$$

Now, for $g \in \mathrm{GL}(3, \boldsymbol{R})$, let ${}^\iota g = w\,{}^t g^{-1} w$ with $w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$. Note that this preserves the Iwasawa decomposition, so ι gives rise to an involution on H^3 which we also write ι.

But, ${}^\iota P_{2,1} = P_{1,2}$ since

$$w \begin{bmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 1 \end{bmatrix}^{-1} w = w \begin{bmatrix} a' & d' & 0 \\ b' & e' & 0 \\ c' & f' & 1 \end{bmatrix} w = \begin{bmatrix} 1 & f' & c' \\ 0 & e' & b' \\ 0 & d' & a' \end{bmatrix}.$$

Hence, we see that

$$(2.11)\qquad P_n({}^\iota\tau; s_1, s_2) = \tilde{P}_n(\tau; s_1, s_2).$$

We also have

$$\begin{aligned} P_n(\tau; s_1, s_2) &= \sum_{\gamma \in P_{2,1} \backslash \Gamma} (y_1^2 y_2)^{s_1} P_n(x_2+iy_2, s_2)|[\gamma] \\ &= \sum_{\gamma \in P_{2,1} \backslash \Gamma} (y_1^2 y_2)^{s_1} \Big(\sum_{\gamma' \in \Gamma^2_\infty \backslash \Gamma^2} y_2^{s_2} e^{2\pi i n z_2}|[\gamma'] \Big)|[\gamma]. \end{aligned}$$

Since

$$\Gamma^2_\infty \backslash \Gamma^2 \cong \Gamma_\infty \backslash P_{2,1}$$

we obtain

$$(2.12)\quad P_n(\tau; s_1, s_2) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y_1^{2s_1} y_2^{s_1+s_2} e^{2\pi i n z_2}|[\gamma] = P_{0,n}(\tau; 2s_1, s_1+s_2).$$

3. Invariants of $\Gamma_\infty \backslash \Gamma$ and Bruhat decompositions. Let $g \in \Gamma = \mathrm{SL}(3, \boldsymbol{Z})$, and define the involution

$$(3.1)\qquad {}^\iota g = w\,{}^t g^{-1} w, \qquad w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

Thus if

$$g = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ A_1 & B_1 & C_1 \end{bmatrix}$$

then

$${}^\iota g = \begin{bmatrix} a_{11}a_{22}-a_{12}a_{21} & a_{13}a_{21}-a_{11}a_{23} & a_{12}a_{23}-a_{13}a_{22} \\ a_{12}A_1-a_{11}B_1 & a_{11}C_1-a_{13}A_1 & a_{13}B_1-a_{12}C_1 \\ a_{21}B_1-a_{22}A_1 & a_{23}A_1-a_{21}C_1 & a_{22}C_1-a_{23}B_1 \end{bmatrix}.$$

If (A_1, B_1, C_1) is the bottom row of g and (A_2, B_2, C_2) is the bottom row of ${}^\iota g$, then $A_1, B_1, C_1, A_2, B_2, C_2$ depend only on the orbit of g in $\Gamma_\infty \backslash \Gamma$. They are subject to the one relation

$$(3.2)\qquad A_1C_2+B_1B_2+C_1A_2 = 0.$$

In fact, [1] and [20] have proved

Proposition 3.1. *If A_1, B_1, C_1 are coprime integers and A_2, B_2, C_2 are coprime integers such that (3.2) is satisfied then there exists a unique orbit of $\Gamma_\infty \backslash \Gamma$ with the given coordinates. Furthermore, if*

$$g = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ A_1 & B_1 & C_1 \end{bmatrix}$$

then

$$\begin{aligned} A_2 &= a_{21}B_1 - a_{22}A_1, \\ B_2 &= a_{23}A_1 - a_{21}C_1, \\ C_2 &= a_{22}C_1 - a_{23}B_1. \end{aligned}$$

Let W denote the Weyl group of $\mathrm{GL}(3, \boldsymbol{R})$. We also define

$$G_\infty = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} \in \mathrm{GL}(3, \boldsymbol{R}), \qquad D = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix} \in \mathrm{GL}(3, \boldsymbol{R}).$$

For $w \in W$, let

$$(3.3)\qquad G_w = G_\infty w D G_\infty$$

so that

$$\mathrm{GL}(3, \mathbf{R}) = \bigcup_{w \in W} G_w \quad \text{(disjoint union)}. \tag{3.4}$$

We call (3.3) and (3.4) the *Bruhat decomposition.*

Let $g \in \Gamma_\infty \backslash \Gamma$ with

$$g = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ A_1 & B_1 & C_1 \end{bmatrix} \tag{3.5}$$

have coordinates (A_1, B_1, C_1), (A_2, B_2, C_2). We shall give the explicit Bruhat decomposition of such a coset representative. There are precisely six cases to consider, and they are contained in the following six propositions.

PROPOSITION 3.2. *Let $g \in \Gamma_\infty \backslash \Gamma$ have coordinates $A_1 = A_2 = B_1 = B_2 = 0$, C_1, $C_2 \neq 0$. Then*

$$g = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & C_1 \end{bmatrix} = \begin{bmatrix} a_{11} & & \\ & \frac{C_2}{C_1} & \\ & & C_1 \end{bmatrix} \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ & 1 & \frac{a_{23} C_1}{C_2} \\ & & 1 \end{bmatrix}.$$

PROPOSITION 3.3. *Let $g \in \Gamma_\infty \backslash \Gamma$ have coordinates $A_1 = A_2 = B_1 = 0$, C_1, $B_2 \neq 0$. Then*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-a_{11} C_1}{B_2} & \\ & 1 & \frac{a_{23}}{C_1} \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{-B_2}{C_1} & & \\ & \frac{1}{B_2} & \\ & & C_1 \end{bmatrix} \begin{bmatrix} 1 & \frac{-C_2}{B_2} & 0 \\ & 1 & a_{13} B_2 \\ & & 1 \end{bmatrix}.$$

PROPOSITION 3.4. *Let $g \in \Gamma_\infty \backslash \Gamma$ have coordinates $A_1 = A_2 = B_2 = 0$, B_1, $C_2 \neq 0$. Then*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & \frac{a_{22}}{B_1} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & & \\ & B_1 & \\ & & \frac{-1}{a_{11} B_1} \end{bmatrix} \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ & 1 & \frac{C_1}{B_1} \\ & & 1 \end{bmatrix}$$

PROPOSITION 3.5. *Let $g \in \Gamma_\infty \backslash \Gamma$ with invariants $A_1 = 0$, B_1, $A_2 \neq 0$. Then*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{a_{11} B_1}{A_2} & \frac{-b_{11}}{A_2} \\ & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{A_2}{B_1} & & \\ & B_1 & \\ & & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{a_{22} B_1}{A_2} & \frac{a_{23} B_1}{A_2} \\ & 1 & \frac{C_1}{B_1} \\ & & 1 \end{bmatrix}$$

where $b_{11} = a_{11} a_{22} - a_{12} a_{21}$.

PROPOSITION 3.6. *Let $g \in \Gamma_\infty \backslash \Gamma$ have invariants $A_2 = 0$, A_1, $B_2 \neq 0$. Then*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ A_1 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{a_{11}}{A_1} \\ & 1 & \frac{A_{21}}{A_1} \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & & \\ & \frac{1}{B_2} & \\ & & \frac{B_2}{A_1} \end{bmatrix} \begin{bmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_1} \\ & 1 & \frac{-b_{22} B_2}{A_1} \\ & & 1 \end{bmatrix}$$

where $b_{22} = a_{11} C_1 - a_{13} A_1$.

PROPOSITION 3.7. *Let $g \in \Gamma_\infty \backslash \Gamma$ have invariants A_1, $A_2 \neq 0$. Then*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ A_1 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-b_{21}}{A_2} & \frac{a_{11}}{A_1} \\ & 1 & \frac{a_{21}}{A_1} \\ & & 1 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} A_1 & & \\ & -\frac{A_2}{A_1} & \\ & & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_1} \\ & 1 & \frac{-B_2}{A_2} \\ & & 1 \end{bmatrix}$$

where $b_{21} = a_{12} A_1 - a_{11} B_1$.

Propositions 3.2 to 3.7 are easily verified by direct calculations. Now, for $w \in W$, let us define

$$\Gamma_w = {}^t(w^{-1}\Gamma_\infty w) \cap \Gamma_\infty. \tag{3.6}$$

Clearly Γ_w is a group which we give explicitly as follows:

$$\Gamma_w = \begin{cases} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, & w = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \\[2ex] \begin{bmatrix} 1 & m & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad m \in \mathbf{Z}, & w = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\[2ex] \begin{bmatrix} 1 & & \\ & 1 & n \\ & & 1 \end{bmatrix}, \quad n \in \mathbf{Z}, & w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\[2ex] \begin{bmatrix} 1 & & l \\ & 1 & n \\ & & 1 \end{bmatrix}, \quad l, n \in \mathbf{Z}, & w = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\[2ex] \begin{bmatrix} 1 & m & l \\ & 1 & \\ & & 1 \end{bmatrix}, \quad m, l \in \mathbf{Z}, & w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \\[2ex] \begin{bmatrix} 1 & m & l \\ & 1 & n \\ & & 1 \end{bmatrix}, \quad m, l, n \in \mathbf{Z}, & w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}. \end{cases}$$

Let

$$U = \left\{ \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{bmatrix} \middle| \varepsilon_i = \pm 1, \varepsilon_1 \varepsilon_2 \varepsilon_3 = +1 \right\}.$$

Note that

$$\begin{bmatrix} * & * & * \\ a_{21} & a_{22} & a_{23} \\ A_1 & B_1 & C_1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{bmatrix} = \begin{bmatrix} * & * & * \\ \varepsilon_1 a_{21} & \varepsilon_2 a_{22} & \varepsilon_3 a_{23} \\ \varepsilon_1 A_1 & \varepsilon_2 B_1 & \varepsilon_3 C_1 \end{bmatrix}$$

sends $A_1 \to \varepsilon_1 A_1$, $B_1 \to \varepsilon_2 B_1$, $C_1 \to \varepsilon_3 C_1$, $A_2 \to \varepsilon_1 \varepsilon_2 A_2$, $B_2 \to \varepsilon_1 \varepsilon_3 B_2$, $C_2 \to \varepsilon_2 \varepsilon_3 C_2$. Thus one obtains representatives of $\Gamma_\infty \backslash G_w \cap \Gamma \pmod{U}$ by fixing two signs of nonzero invariants. It is easily verified that Γ_w acts properly (on the right) on $\Gamma_\infty \backslash \Gamma \cap G_w/U$ with G_w given by (3.3). That is to say, if $g \in \Gamma \cap G_w/U$, $\tau \in \Gamma_w$ and $\Gamma_\infty g \tau = \Gamma_\infty g$, then $\tau =$ identity.

We shall now exhibit, for every $w \in W$, a canonical set of coset representatives R_w for the quotient space

$$\Gamma_\infty \backslash G_w \cap \Gamma / U \Gamma_w.$$

PROPOSITION 3.8. *For* $w = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$, *we have*

$$R_w = \left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right\}.$$

PROPOSITION 3.9. *For* $w = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, *we have*

$$R_w = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ -B_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

where $(B_2, C_2) = 1$, $B_2 > 0$, $C_2 \pmod{B_2}$, *and for each pair* (B_2, C_2), a_{11} *and* a_{12} *are uniquely chosen so that* $a_{11} C_2 + a_{12} B_2 = 1$.

PROPOSITION 3.10. *For* $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *we have*

$$R_w = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & B_1 & C_1 \end{bmatrix} \right\}$$

where $(B_1, C_1) = 1$, $B_1 > 0$, $C_1 \pmod{B_1}$, *and for each pair* (B_1, C_1), a_{22}, a_{23} *are uniquely chosen so that* $a_{22} C_1 - B_1 a_{23} = 1$.

PROPOSITION 3.11. *For* $w = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, *we have*

$$R_w = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \dfrac{A_2}{B_1} & \alpha C_2 & \beta C_2 \\ 0 & B_1 & C_1 \end{bmatrix} \right\}$$

where $(B_1, C_1) = 1$, $B_1 > 0$, $C_1 \pmod{B_1}$, $(A_2/B_1, C_2) = 1$, $A_2 > 0$, $C_2 \pmod{A_2}$, $B_1 B_2 + C_1 A_2 = 0$, *and for each pair* (B_1, C_1), α, β *are uniquely* *chosen so that* $\alpha C_2-\beta B_1=1$. *For every quintuple* $(B_1, C_1, A_2, B_2, C_2)$, a_{11}, a_{12}, a_{13} *are uniquely chosen so that the matrix has determinant one.*

PROPOSITION 3.12. *For* $w=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, *we have*

$$R_w=\left\{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&\dfrac{a_{21}B_1}{A_1}&a_{23}\\A_1&B_1&C_1\end{bmatrix}\right\}$$

where $(B_2, C_2)=1$, $B_2>0$, $C_2 \pmod{B_2}$, $(A_1/B_2, C_1)=1$, $A_1>0$, $C_1 \pmod{A_1}$, $A_1C_2+B_1B_2=0$, *and for every quintuple* $(A_1, B_1, C_1, B_2, C_2)$, a_{11}, a_{12}, a_{13}, a_{21}, a_{23} *are uniquely chosen so that the matrix has determinant one.*

PROPOSITION 3.13. *For* $w=\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}$, *we have*

$$R_w=\left\{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\A_1&B_1&C_1\end{bmatrix}\right\}$$

where A_1, $A_2>0$, B_1, $C_1 \pmod{A_1}$, B_2, $C_2 \pmod{A_2}$, $(A_1, B_1, C_1)=1$, $(A_2, B_2, C_2)=1$, $A_1C_2+B_1B_2+A_2C_1=0$, *and for every sextuple* $(A_1, B_1, C_1, A_2, B_2, C_2)$, a_{21}, a_{22}, a_{23} *are uniquely chosen so that*

$$A_2=a_{21}B_1-a_{22}A_1,\quad B_2=a_{23}A_1-a_{21}C_1,\quad C_2=a_{22}C_1-a_{23}B_1$$

and a_{11}, a_{12}, a_{13} *are further chosen uniquely so that the matrix has determinant one.*

4. SL(3, *Z*) Kloosterman sums. In the Fourier expansion of the Poincaré series (2.6), certain exponential sums, reminiscent of classical Kloosterman sums, appear. Although the actual Fourier expansion computations are carried out in Section 5, we shall now introduce and study the new Kloosterman sums that occur there. It is interesting that there are two distinct such SL(3, *Z*) Kloosterman sums. They are given as follows. Let m_1, m_2, n_1, $n_2\in \boldsymbol{Z}$. Then for D_1, $D_2\in \boldsymbol{Z}^+$, the first type of SL(3, *Z*) Kloosterman sum is

(4.1) $$S(m_1, m_2, n_1, n_2; D_1, D_2)=\sum e\left(\frac{m_1B_1+n_1(Y_1D_2-Z_1B_2)}{D_1}+\frac{m_2B_2+n_2(Y_2D_1-Z_2B_1)}{D_2}\right).$$

Summation is over B_1, $C_1 \pmod{D_1}$, B_2, $C_2 \pmod{D_2}$ such that $(D_1, B_1, C_1)=(D_2, B_2, C_2)=1$ and $D_1C_2+B_1B_2+C_1D_2\equiv 0 \pmod{D_1D_2}$; for such B_1, C_1, B_2, C_2, we have chosen Y_1, Z_1, Y_2, Z_2 such that

$$Y_1B_1+Z_1C_1\equiv 1 \pmod{D_1},$$
$$Y_2B_2+Z_2C_2\equiv 1 \pmod{D_2}.$$

We show in Lemmas 4.1 and 4.2 that the sum (4.1) is well defined; that is to say, it is independent of the choice of Y_1, Z_1, Y_2, Z_2 and also independent of the choice of representatives B_1, C_1 and B_2, C_2 of the residue classes (mod D_1) and (mod D_2), respectively.

The Kloosterman sum (4.1) can also be written as

(4.2) $$S(m_1, m_2, n_1, n_2; D_1, D_2)=\sum e_{n_1,n_2}(b_1)\,e_{m_2,m_1}(b_2)$$

where the summation is over $\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty$, $\gamma=b_1wdb_2$, with b_1, $b_2\in G_\infty$, where

$$w=\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix},\quad d=\begin{bmatrix}D_1&&\\&-D_2/D_1&\\&&1/D_2\end{bmatrix},$$

and where, if

$$b=\begin{bmatrix}1&\beta_2&\beta_3\\&1&\beta_1\\&&1\end{bmatrix}\in G_\infty$$

and r_1, $r_2\in\boldsymbol{Z}$, then $e_{r_1,r_2}(b)=e(r_1\beta_1+r_2\beta_2)$.

The second type of SL(3, *Z*) Kloosterman sum arises only when $D_1|D_2$. It is given as

(4.3) $$S(m_1, n_1, n_2; D_1, D_2)=\sum e\left(\frac{m_1C_1+n_1\bar{C}_1C_2}{D_1}+\frac{n_2\bar{C}_2}{D_2/D_1}\right),$$

where the summation is over $C_1 \pmod{D_1}$, $C_2 \pmod{D_2}$, $(C_1, D_1)=(C_2, D_2/D_1)=1$, and where $\bar{C}_1$, $\bar{C}_2$ are chosen so that $C_1\bar{C}_1\equiv 1 \pmod{D_1}$ and $C_2\bar{C}_2\equiv 1 \pmod{D_2/D_1}$.

We now show that the Kloosterman sum (4.1) is well defined.

LEMMA 4.1. *If* $(D_1, B_1, C_1)=1$, $D_1\neq 0$,

$$D_1C_2+B_1B_2+C_1D_2\equiv 0 \pmod{D_1D_2},$$

and if $X_1D_1+Y_1B_1+Z_1C_1=X_1'D_1+Y_1'B_1+Z_1'C_1$, *then*

$$\frac{Y_1D_2-Z_1B_2}{D_1}\equiv\frac{Y_1'D_2-Z_1'B_2}{D_1} \pmod 1.$$

Proof. Changing the value of C_2 if necessary, we may assume that $D_1 C_2+B_1 B_2+C_1 D_2=0$. The vectors $\langle C_2, B_2, D_2\rangle$ and $\langle \xi_1, \eta_1, \zeta_1\rangle = \langle X_1-X_1', Y_1-Y_1', Z_1-Z_1'\rangle$ are both orthogonal to $\langle D_1, B_1, C_1\rangle$, so their vector cross product is parallel to $\langle D_1, B_1, C_1\rangle$. Hence, there exists a rational number λ such that

$$\langle \eta_1 D_2-\zeta_1 B_2, \zeta_1 C_2-\xi_1 D_2, \xi_1 B_2-\eta_1 C_2\rangle = \lambda \langle D_1, B_1, C_1\rangle.$$

Since $(D_1, B_1, C_1)=1$, λ must be an integer, and so

$$\frac{Y_1 D_2-Z_1 B_2}{D_1} = \lambda+\frac{Y_1' D_2-Z_1' B_2}{D_1}. \blacksquare$$

It is not *a priori* clear that the sum in (4.1) does not depend on the choice of representatives B_1, B_2 of the residue classes mod D_1, D_2, respectively; and indeed, there is a slight nuance regarding this point. The sum would more properly be written

$$\sum_{\substack{B_1 (\mathrm{mod}\, D_1)\\ B_2 (\mathrm{mod}\, D_2)}} \sum_{\substack{C_1 (\mathrm{mod}\, D_1)\\ C_2 (\mathrm{mod}\, D_2)}} e(\ ),$$

where the inner sum is subject to the conditions $(D_1, B_1, C_1)=(D_2, B_2, C_2)=1$, $D_1 C_2+B_1 B_2+C_1 D_2 \equiv 0 \pmod{D_1 D_2}$. We now have the following:

LEMMA 4.2. *The sum*

$$S_{B_1,B_2}(m_1, m_2, n_1, n_2; D_1, D_2)$$
$$=\sum e\left(\frac{m_1 B_1+n_1(Y_1 D_2-Z_1 B_2)}{D_1}+\frac{m_2 B_2+n_2(Y_2 D_1-Z_2 B_1)}{D_2}\right)$$

(The summation being subject to the conditions $(D_1, B_1, C_1)=(D_2, B_2, C_2)=1$, *and* $D_1 C_2+B_1 B_2+C_1 D_2 \equiv 0 \pmod{D_1 D_2}$.*) depends only on the residue classes of* $B_1 \pmod{D_1}$ *and* $B_2 \pmod{D_2}$.

Proof. If $B_1' = B_1+\lambda D_1$, we show that

$$S_{B_1,B_2}(m_1, m_2, n_1, n_2; D_1, D_2) = S_{B_1',B_2}(m_1, m_2, n_1, n_2; D_1, D_2).$$

For $C_1 \pmod{D_1}$, $C_2 \pmod{D_2}$ satisfying $(D_1, B_1, C_1)=(D_2, B_2, C_2)=1$, $D_1 C_2+B_1 B_2+C_1 D_2 \equiv 0 \pmod{D_1 D_2}$, we associate $C_1 \pmod{D_1}$ and $C_2' \pmod{D_2}$ where $C_2' = C_2-\lambda B_2$, so that $(D_1, B_1', C_1)=(D_2, B_2, C_2')=1$ and $D_1 C_2'+B_1' B_2+C_1 D_2 \equiv 0 \pmod{D_1 D_2}$. It follows that $Y_1 B_1'+Z_1 C_1 \equiv 1 \pmod{D_1}$, $Y_2' B_2+Z_2 C_2' \equiv 1 \pmod{D_2}$ where $Y_2' = Y_2+\lambda Z_2$. Summing over all C_1 and C_2, we obtain

$$S_{B_1,B_2}(m_1, m_2, n_1, n_2; D_1, D_2) = S_{B_1',B_2}(m_1, m_2, n_1, n_2; D_1, D_2),$$

the point being that $Y_2' D_1-Z_2 B_1' = Y_2 D_1-Z_2 B_1$. $\blacksquare$

Thus the $\mathrm{SL}(3, \mathbf{Z})$ Kloosterman sum $S(m_1, m_2, n_1, n_2; D_1, D_2)$ is well-defined. We now proceed to develop some of its properties.

PROPERTY 4.3. *If* $p_1 q_1 \equiv p_2 q_2 \equiv 1 \pmod{D_1 D_2}$, $p_1, q_1, p_2, q_2 \in \mathbf{Z}$, *then*

$$S(p_1 m_1, p_2 m_2, q_1 n_1, q_2 n_2; D_1, D_2) = S(m_1, m_2, n_1, n_2; D_1, D_2).$$

Proof. Given $(D_1, B_1, C_1)=(D_2, B_2, C_2)=1$, $D_1 C_2+B_1 B_2+C_1 D_2 \equiv 0 \pmod{D_1 D_2}$, let $Y_1 B_1+Z_1 C_1 \equiv 1 \pmod{D_1}$, $Y_2 B_2+Z_2 C_2 \equiv 1 \pmod{D_2}$. Also, let

$$\begin{aligned} B_1' &= p_1 B_1, & B_2' &= p_2 B_2,\\ C_1' &= p_1 p_2 C_1, & C_2' &= p_1 p_2 C_2,\\ Y_1' &= q_1 Y_1, & Y_2' &= q_2 Y_2,\\ Z_1' &= q_1 q_2 Z_1, & Z_2' &= q_1 q_2 Z_2. \end{aligned}$$

We have

$$(D_1, B_1', C_1')=(D_2, B_2', C_2')=1, \quad D_1 C_2'+B_1' B_2'+C_1' D_2 \equiv 0 \pmod{D_1 D_2},$$
$$Y_1' B_1'+Z_1' C_1' \equiv 1 \pmod{D_1}, \quad Y_2' B_2'+Z_2' C_2' \equiv 1 \pmod{D_2}.$$

Now

$$e\left(\frac{p_1 m_1 B_1+q_1 n_1(Y_1 D_2-Z_1 B_2)}{D_1}+\frac{p_2 m_2 B_2+q_2 n_2(Y_2 D_1-Z_2 B_1)}{D_2}\right)$$
$$= e\left(\frac{m_1 B_1'+n_1(Y_1' D_2-Z_1' B_2')}{D_1}+\frac{m_2 B_2'+n_2(Y_2' D_1-Z_2' B_1')}{D_2}\right).$$

Summing we obtain Property 4.3. $\blacksquare$

PROPERTY 4.4. $S(m_1, m_2, n_1, n_2; D_1, D_2) = S(n_1, n_2, m_1, m_2; D_1, D_2)$.

Proof. Given $(D_1, B_1, C_1)=(D_2, B_2, C_2)=1$ such that $D_1 C_2+B_1 B_2+C_1 D_2 \equiv 0 \pmod{D_1 D_2}$, let $X_1 D_1+Y_1 B_1+Z_1 C_1 = 1$ and $X_2 D_2+Y_2 B_2+Z_2 C_2 = 1$. Also, let

$$\begin{aligned} B_1' &= Y_1 D_2-Z_1 B_2, & B_2' &= Y_2 D_1-Z_2 B_1,\\ C_1' &= Z_2, & C_2' &= Z_1,\\ Y_1' &= X_2 B_1-Y_2 C_1, & Y_2' &= X_1 B_2-Y_1 C_2,\\ Z_1' &= C_2, & Z_2' &= C_1. \end{aligned}$$

We have

$$Y_1' B_1'+Z_1' C_1' \equiv Y_1 B_1+Z_1 C_1+D_1 C_2(X_1 Z_2+Y_1 Y_2+Z_1 X_2) \pmod{D_1 D_2},$$

so

$$Y_1' B_1'+Z_1' C_1' \equiv 1 \pmod{D_1}.$$

Similarly,

$$Y_2' B_2'+Z_2' C_2' \equiv 1 \pmod{D_2}.$$

Also, we have

$$D_1 C_2' + B_1' B_2' + C_1' D_2 \equiv D_1 D_2 (X_1 Z_2 + Y_1 Y_2 + Z_1 X_2) \equiv 0 \pmod{D_1 D_2}$$

and

$$Y_1' D_2 - Z_1' B_2' \equiv B_1 (X_2 D_2 + Y_2 B_2 + Z_2 C_2) \equiv B_1 \pmod{D_1 D_2}$$

and similarly

$$Y_2' D_1 - Z_2' B_1' \equiv B_2 \pmod{D_1 D_2}.$$

Thus

$$e\left(\frac{m_1 B_1 + n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1)}{D_2}\right)$$
$$= e\left(\frac{n_1 B_1' + m_1 (Y_1' D_2 - Z_1' B_2')}{D_1} + \frac{n_2 B_2' + m_2 (Y_2' D_1 - Z_2' B_1')}{D_2}\right).$$

Summing, we obtain Property 4.4.

Property 4.5. $S(m_1, m_2, n_1, n_2; D_1, D_2) = S(m_2, m_1, n_2, n_1; D_2, D_1)$.

Proof. Trivial.

Property 4.6. *If* $m_1 \equiv m_1'$, $n_1 \equiv n_1' \pmod{D_1}$, $m_2 \equiv m_2'$, *and* $n_2 \equiv n_2' \pmod{D_2}$, *then*

$$S(m_1, m_2, n_1, n_2; D_1, D_2) = S(m_1', m_2', n_1', n_2'; D_1, D_2).$$

Proof. Trivial.

Property 4.7. *If* $(D_1 D_2, D_1' D_2') = 1$ *and if*

$$\bar{D}_1 D_1 \equiv \bar{D}_2 D_2 \equiv 1 \pmod{D_1' D_2'}, \quad \overline{D_1'} D_1' \equiv \overline{D_2'} D_2' \equiv 1 \pmod{D_1 D_2},$$

then

$$S(m_1, m_2, n_1, n_2; D_1 D_1', D_2 D_2') =$$
$$S(\overline{D_1'}^2 D_2' m_1, \overline{D_2'}^2 D_1' m_2, n_1, n_2; D_1, D_2)\, S(\bar{D}_1^2 D_2 m_1, \bar{D}_2^2 D_1 m_2, n_1, n_2; D_1', D_2').$$

Proof. Let p and p' be found so that $pD_1 D_2 + p' D_1' D_2' = 1$. Given $(D_1, B_1, C_1) = (D_2, B_2, C_2) = 1$, $D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 \pmod{D_1 D_2}$ and $(D_1', B_1', C_1') = (D_2', B_2', C_2') = 1$, $D_1' C_2' + B_1' B_2' + C_1' D_2' \equiv 0 \pmod{D_1' D_2'}$, let

$$d_1 = D_1 D_1', \quad d_2 = D_2 D_2',$$
$$b_1 = p' D_1' D_2' B_1 + pD_1 D_2 B_1', \quad b_2 = p' D_1' D_2' B_2 + pD_1 D_2 B_2',$$
$$c_1 = p'^2 D_1'^2 D_2' C_1 + p^2 D_1^2 D_2 C_1' \quad c_2 = p'^2 D_1' D_2'^2 C_2 + p^2 D_1 D_2^2 C_2'.$$

Then

$$(d_1, b_1, c_1) = (d_2, b_2, c_2) = 1,$$
$$d_1 c_2 + b_1 b_2 + c_1 d_2 \equiv 0 \pmod{d_1 d_2}.$$

Let

$$y_1 = p' D_1' D_2' Y_1 + pD_1 D_2 Y_1', \quad y_2 = p' D_1' D_2' Y_2 + pD_1 D_2 Y_2',$$
$$z_1 = p' D_1' D_2'^2 Z_1 + pD_1 D_2^2 Z_1', \quad z_2 = p' D_1'^2 D_2' Z_2 + pD_1^2 D_2 Z_2'.$$

Then

$$y_1 b_1 + z_1 c_1 \equiv 1 \pmod{d_1}, \quad y_2 b_2 + z_2 c_2 \equiv 1 \pmod{d_2},$$

$$\frac{m_1 b_1 + n_1 (y_1 d_2 - z_1 b_2)}{d_1} \equiv \frac{m_1 p' D_2' B_1 + n_1 p' D_2'^2 (Y_1 D_2 - Z_1 B_2)}{D_1}$$
$$+ \frac{m_1 pD_2 B_1' + n_1 pD_2^2 (Y_1' D_2' - Z_1' B_2')}{D_1'} \pmod{1},$$

with a similar identity for d_2.

Summing, we obtain

$$S(m_1, m_2, n_1, n_2; d_1, d_2)$$
$$= S(p' D_2' m_1, p' D_1' m_2, p' D_2'^2 n_1, p' D_1'^2 n_2; D_1, D_2)$$
$$\times S(pD_2 m_1, pD_1 m_2, pD_2^2 n_1, pD_1^2 n_2; D_1', D_2').$$

But

$$\bar{D}_1 \equiv pD_2, \quad \bar{D}_2 \equiv pD_1 \pmod{D_1' D_2'},$$
$$\overline{D_1'} \equiv p' D_2', \quad \overline{D_2'} \equiv p' D_1' \pmod{D_1 D_2}$$

and Property 4.7 follows from Property 4.3. ∎

Property 4.8.

$$S(m_1, m_2, n_1, n_2; D_1, 1) = S(m_1, n_1; D_1)$$

and

$$S(m_1, m_2, n_1, n_2; 1, D_2) = S(m_2, n_2; D_2)$$

where

$$S(m, n; D) = \sum_{\substack{a \,(\mathrm{mod}\, D) \\ a\bar{a} \equiv 1 \,(\mathrm{mod}\, D) \\ (a, D) = 1}} e\left(\frac{am + \bar{a}n}{D}\right)$$

is the classical Kloosterman sum.

Proof. Trivial.

Property 4.9. *If* $(D_1, D_2) = 1$, *then*

$$S(m_1, m_2, n_1, n_2; D_1, D_2) = S(D_2 m_1, n_1, D_1)\, S(D_1 m_2, n_2, D_2).$$

Proof. This follows by combining Properties 4.7 and 4.8.

Finally, we conclude our discussion of the big cell Kloosterman sum (4.1) with an explicit evaluation in the case $D_1 = p$ where p is a prime. To this end, set

$$\delta(p, m) = \begin{cases} p-1, & p \mid m, \\ -1, & p \nmid m. \end{cases}$$

Property 4.10. *Let p be a prime. Then*

(a) $S(m_1, m_2; n_1, n_2; p, p) = \delta(p, n_1)\delta(p, m_2) + \delta(p, m_1)\delta(p, n_2) + p - 1$.

(b) *For* $l > 1$,

$$S(m_1, m_2, n_1, n_2; p, p^l) = \delta(p, n_1) S(m_2, n_2 p, p^l) + \delta(p, m_1) S(n_2, m_2 p, p^l).$$

Proof. To evaluate $S(m_1, m_2, n_1, n_2; p, p^l)$, we must sum over $B_1, C_1 \pmod p$, $B_2, C_2 \pmod{p^l}$ such that $(p, B_1, C_1) = (p, B_2, C_2) = 1$ and

$$pC_2 + B_1 B_2 + p^l C_1 \equiv 0 \pmod{p^{l+1}}. \tag{4.4}$$

For $l \geqslant 1$, this implies $B_1 B_2 \equiv 0 \pmod p$. We split the sum into subsums over the three possibilities:

(1) $B_1 \equiv B_2 \equiv 0 \pmod p$,

(2) $B_1 \equiv 0 \pmod p$ and $B_2 \not\equiv 0 \pmod p$,

(3) $B_2 \equiv 0 \pmod p$ and $B_1 \not\equiv 0 \pmod p$.

In case (1) we can choose $B_1 = 0$ by Lemma 4.2 and the sum is empty unless $l = 1$ where one gets $p-1$.

In case (2) we may choose $Z_1 = \bar{C}_1$, $Y_2 = \bar{B}_2$; $Z_2 = 0$ so that C_2 is uniquely specified $\pmod{p^l}$ by (4.4) and the sum is reduced to $\delta(p, n_1) S(m_2, n_2 p, p^l)$.

Finally, in case (3) choose $Y_1 = \bar{B}_1$, $Z_1 = 0$, $Y_2 = 0$, $Z_2 = \bar{C}_2$. Write $B_2 = pB_2'$ where B_2' is determined $\pmod{p^{l-1}}$. Then (4.4) becomes

$$C_2 + B_1 B_2' + p^{l-1} C_1 \equiv 0 \pmod{p^l}.$$

When $l = 1$, $B_2' = 0$ and the sum reduces to $\delta(p, m_1)\delta(p, n_2)$. For $l > 1$, since $p \nmid C_2$, $p \nmid B_2'$, the sum reduces to

$$\sum e\left(\frac{m_1 B_1}{p} + \frac{m_2 pB_2' - n_2 \bar{C}_2 B_1}{p^l}\right)$$

where the summation is over $B_1, C_1 \pmod p$ and $B_2' \pmod{p^{l-1}}$ such that $p \nmid B_1 B_2'$, and where $C_2 \equiv -(B_1 B_2' + p^{l-1} C_1) \pmod{p^l}$. Summing over C_1 first, we get 0 unless $p \mid n_2$. When $p \mid n_2$ we then sum over B_1 to obtain $\delta(p, m_1) S(m_2 p, n_2, p^l)$. Since $S(m_2 p, n_2, p^l) = 0$ unless $p \mid n_2$, the proof is complete.

We now consider the second type of Kloosterman sum (4.3).

Property 4.11. *$S(m_1, n_1, n_2; D_1, D_2)$ depends only on $m_1 \pmod{D_1}$, $n_1 \pmod{D_1}$ and $n_2 \pmod{D_2/D_1}$.*

Proof. Trivial.

Property 4.12. *If $p_1 q_1 \equiv 1 \pmod{D_1}$, then*

$$S(p_1 m_1, q_1 n_1, n_2; D_1, D_2) = S(m_1, n_1, n_2; D_1, D_2).$$

Proof. Trivial.

Property 4.13. *If $p_2 q_2 \equiv 1 \pmod{D_2}$, then*

$$S(m_1, p_2 n_1, q_2 n_2; D_1, D_2) = S(m_1, n_1, n_2; D_1, D_2).$$

Proof. Trivial.

Property 4.14.

$$S(m_1, n_1, n_2; 1, D_2) = \sum_{\substack{C_2 \pmod{D_2} \\ (C_2, D_2) = 1}} e\left(\frac{n_2 C_2}{D_2}\right).$$

Proof. Trivial.

Property 4.15. *Let $(D_2, D_2') = 1$, $D_1 \mid D_2$, $D_1' \mid D_2'$. Then*

$$S(m_1, n_1, n_2; D_1 D_1', D_2 D_2')$$
$$= S(m_1 \overline{D_1'}, n_1 D_2', n_2 \overline{D_2'}^2; D_1, D_2) \cdot S(m_1 \bar{D}_1, n_1 D_2, n_2 \bar{D}_2^2; D_1', D_2')$$

where

$$D_1 \bar{D}_1 \equiv 1 \pmod{D_1'}, \quad D_2 \bar{D}_2 \equiv 1 \pmod{D_2'},$$
$$D_1' \overline{D_1'} \equiv 1 \pmod{D_1}, \quad D_2' \overline{D_2'} \equiv 1 \pmod{D_2}.$$

Proof. We compute the right-hand side. It is given by

$$\sum_{\substack{C_1 \pmod{D_1} \\ C_2 \pmod{D_2}}} \sum_{\substack{C_1' \pmod{D_1'} \\ C_2' \pmod{D_2'}}} e\left(n_2\left(\frac{\overline{D_2'}^2 \bar{C}_2}{D_2/D_1} + \frac{\bar{D}_2^2 \overline{C_2'}}{D_2'/D_1'}\right) + m_1\left(\frac{\overline{D_1'} C_1}{D_1} + \frac{\bar{D}_1 C_1'}{D_1'}\right)\right.$$
$$\left. + n_1\left(\frac{D_2' \bar{C}_1 C_2}{D_1} + \frac{D_2 \overline{C_1'} C_2'}{D_1'}\right)\right) \tag{4.5}$$

$$= \sum_{\substack{C_1 \pmod{D_1} \\ C_2 \pmod{D_2}}} \sum_{\substack{C_1' \pmod{D_1'} \\ C_2' \pmod{D_2'}}} e\left(n_2\left(\frac{D_2' \overline{D_2'}^2 \bar{C}_2/D_1' + D_2 \bar{D}_2^2 C_2'/D_1}{D_2 D_2'/D_1 D_1'}\right)\right.$$
$$\left. + m_1\left(\frac{D_1' \overline{D_1'} C_1 + D_1 \bar{D}_1 C_1'}{D_1 D_1'}\right) + n_1\left(\frac{D_1' D_2' \bar{C}_1 C_2 + D_1 D_2 \overline{C_1'} C_2'}{D_1 D_1'}\right)\right),$$

the summations on both sides of (4.5) being subject to the conditions

$$(C_1, D_1) = (C_1', D_1') = (C_2, D_2 D_1^{-1}) = (C_2', D_2' D_1'^{-1}) = 1.$$

Let us now choose p_1 and p_1' such that

(4.6) $$p_1 D_1 + p_1' D_1' = 1,$$

and set

$$\mathscr{C}_1 = p_1' C_1 D_1' + p_1 C_1' D_1, \qquad \mathscr{C}_2 = C_2 D_1' D_2' + C_2' D_1 D_2.$$

Note that

(4.7) $$(C_1, D_1) = (C_1', D_1') = 1 \Leftrightarrow (\mathscr{C}_1, D_1 D_1') = 1,$$

(4.8) $$\left(C_2, \frac{D_2}{D_1}\right) = \left(C_2', \frac{D_2'}{D_1'}\right) = 1 \Leftrightarrow \left(\mathscr{C}_2, \frac{D_2 D_2'}{D_1 D_1'}\right) = 1.$$

(4.9) As C_1 varies (mod D_1) and C_1' varies (mod D_1'), $\mathscr{C}_1$ varies (mod $D_1 D_1'$).

(4.10) As C_2 varies (mod D_2) and C_2' varies (mod D_2'), $\mathscr{C}_2$ varies (mod $D_2 D_2'$).

(4.11) $$\overline{\mathscr{C}_1} \equiv p_1' \bar{C}_1 D_1' + p_1 \overline{C_1'} D_1 \pmod{D_1 D_1'}.$$

The last assertion follows from

$$\mathscr{C}_1 (p_1' \bar{C}_1 D_1' + p_1 \overline{C_1'} D_1) \equiv (p_1' C_1 D_1' + p_1 C_1' D_1)(p_1' \bar{C}_1 D_1' + p_1 \overline{C_1'} D_1)$$
$$\equiv p_1'^2 D_1'^2 + p_1^2 D_1^2 \equiv 1 \pmod{D_1 D_1'},$$

where the last congruence follows by squaring (4.6).

Now, take p_2 and p_2' so that

(4.12) $$p_2 D_2 + p_2' D_2' = 1.$$

We claim that

(4.13) $$\overline{\mathscr{C}_2} = p_2'^2 \bar{C}_2 \frac{D_2'}{D_1'} + p_2^2 \overline{C_2'} \frac{D_2}{D_1} \left(\bmod \frac{D_2 D_2'}{D_1 D_1'}\right).$$

In fact, if we multiply the right-hand side of (4.13) by $\mathscr{C}_2$, then we obtain

$$(C_2 D_1' D_2' + C_2' D_1 D_2)\left(p_2'^2 \bar{C}_2 \frac{D_2'}{D_1'} + p_2^2 \overline{C_2'} \frac{D_2}{D_1}\right)$$
$$\equiv C_2 \bar{C}_2 p_2'^2 D_2'^2 + C_2' \overline{C_2'} p_2^2 D_2^2 \equiv p_2'^2 D_2'^2 + p_2^2 D_2^2 \equiv 1 \left(\bmod \frac{D_2 D_2'}{D_1 D_1'}\right),$$

where the last congruence follows by squaring (4.12).

Consequently, we may write in a well defined way

$$\overline{\mathscr{C}_2} \equiv \overline{D_2'}^2 \bar{C}_2 \frac{D_2'}{D_1'} + \bar{D}_2^2 \overline{C_2'} \frac{D_2}{D_1} \left(\bmod \frac{D_2 D_2'}{D_1 D_1'}\right).$$

Hence

$$\overline{\mathscr{C}_1} \mathscr{C}_2 \equiv (p_1' \bar{C}_1 D_1' + p_1 \overline{C_1'} D_1)(C_2 D_1' D_2' + C_2' D_1 D_2)$$
$$\equiv p_1' \bar{C}_1 C_2 D_1'^2 D_2' + p_1 \overline{C_1'} C_2' D_1^2 D_2$$
$$\equiv \bar{C}_1 C_2 D_1' D_2' + \overline{C_1'} C_2' D_1 D_2 \pmod{D_1 D_1'},$$

the last congruence following from (4.6). We thus obtain

(4.14) $$\overline{\mathscr{C}_1} \mathscr{C}_2 \equiv D_1' D_2' \bar{C}_1 C_2 + D_1 D_2 \overline{C_1'} C_2' \pmod{D_1 D_1'}.$$

If we now combine (4.7) to (4.14) we see that the expression on the right-hand side of (4.5) is equal to

$$\sum_{\substack{\mathscr{C}_1 (\bmod D_1 D_1') \\ \mathscr{C}_2 (\bmod D_2 D_2')}} e\left(\frac{n_2 \bar{\mathscr{C}}_2}{D_2 D_2'/D_1 D_1'} + \frac{m_1 \mathscr{C}_1}{D_1 D_1'} + \frac{n_1 \bar{\mathscr{C}}_1 \mathscr{C}_2}{D_1 D_1'}\right) = S(m_1, n_1, n_2; D_1 D_1', D_2 D_2'),$$

where the summation is subject to the conditions

$$(\mathscr{C}_1, D_1 D_1') = (\mathscr{C}_2, (D_2 D_1^{-1})(D_2' D_1'^{-1})) = 1. \ ∎$$

PROPERTY 4.16. *Let p be a prime number. Then for $b > a > 0$*

$$S(m_1, n_1, n_2; p^a, p^b) = 0$$

unless $b = 2a$, or $n_2 \equiv 0 \pmod{p^{b-2a}}$ and $b > 2a$, or $n_1 \equiv 0 \pmod{p^{2a-b}}$ and $b < 2a$.

Proof. We first consider the case when $b > 2a$. Let $b = 2a + l$, $l > 0$. Then

(4.15) $$S(m_1, n_1, n_2; p^a, p^b) = \sum_{\substack{C_1 (\bmod p^a) \\ C_2 (\bmod p^b)}} e\left(\frac{n_2 \bar{C}_2 + p^l (m_1 C_1 + n_1 \bar{C}_1 C_2)}{p^{a+l}}\right),$$

where the summation is subject to the conditions $(C_1, p) = (C_2, p) = 1$, and where

$$C_1 \bar{C}_1 \equiv 1 \pmod{p^a}, \qquad C_2 \bar{C}_2 \equiv 1 \pmod{p^{a+l}}.$$

Now, suppose $n_2 \equiv 0 \pmod{p^c}$, $c < l$ and $n_2 \not\equiv 0 \pmod{p^{c+1}}$. Then let

$$C_2 = u_0 + u_1 p + \ldots + u_{a+l-c-1} p^{a+l-c-1} + \ldots + u_{2a+l-1} p^{2a+l-1}$$

where

$$1 \leqslant u_0 < p \quad \text{and} \quad 0 \leqslant u_i < p \quad (\text{for } 0 < i \leqslant 2a + l - 1).$$

Similarly,

$$\bar{C}_2 = u_0' + \ldots + u_{a+l-c-1}' p^{a+l-c-1} + \ldots + u_{2a+l-1}' p^{2a+l-1}$$

where

$$1 \leqslant u_0' < p \quad \text{and} \quad 0 \leqslant u_i' < p \quad (\text{for } 0 < i \leqslant 2a + l - 1).$$

If we substitute C_2 and $\bar{C}_2$ as given above into (4.15), replace n_2 by $n_2' p^c$ ($n_2 = n_2' p^c$), and pull out the sum corresponding to $0 \leqslant u'_{a+l-c-1} < p$, we obtain a sum

$$\sum_{0 \leqslant u'_{a+l-c-1} < p} e\left(\frac{n_2' u'_{a+l-c-1}}{p}\right) = 0.$$

This establishes property (4.16) in this case.

If $b < 2a$, let $b = 2a - l$ with $a > l > 0$. Consequently

$$S(m_1, n_1, n_2; p^a, p^b) = \sum_{\substack{C_1 (\mathrm{mod}\, p^a) \\ C_2 (\mathrm{mod}\, p^b)}} e\left(\frac{n_2 \bar{C}_2 p^l + m_1 C_1 + n_1 \bar{C}_1 C_2}{p^a}\right), \tag{4.16}$$

where the summation is subject to the conditions $(C_1, p) = (C_2, p) = 1$, and where $C_1 \bar{C}_1 \equiv 1 (\mathrm{mod}\, p^a)$, $C_2 \bar{C}_2 \equiv 1 (\mathrm{mod}\, p^{a-e})$.

Now, suppose $n_1 \equiv 0 (\mathrm{mod}\, p^c)$, $c < l$ and $n_1 \not\equiv 0 (\mathrm{mod}\, p^{c+1})$ so that $n_1 = n_1' p^c$ with $p \nmid n_1'$. We let

$$C_2 = u_0 + \ldots + u_{a-c-1} p^{a-c-1} + \ldots + u_{2a-l-1} p^{2a-l-1}$$

where

$$1 < u_0 < p \quad \text{and} \quad 0 \leqslant u_i < p \quad (1 \leqslant i \leqslant 2a-l-1).$$

Similarly

$$\bar{C}_2 = u_0' + \ldots + u'_{a-c-1} p^{a-c-1} + \ldots + u'_{2a-l-1} p^{2a-l-1}$$

where

$$1 < u_0' < p \quad \text{and} \quad 0 \leqslant u_i' < p \quad (1 \leqslant i \leqslant 2a-l-1).$$

If we substitute C_2 and $\bar{C}_2$ into (4.16), replace n_1 by $n_1' p^c$, and pull out the sum corresponding to $0 \leqslant u_{a-c-1} < p$, we obtain a sum

$$\sum_{0 \leqslant u_{a-c-1} < p} e\left(\frac{n_1' \bar{C}_1 u_{a-c-1}}{p}\right) = 0$$

since $(\bar{C}_1, p) = 1$. This completes the proof. ∎

PROPERTY 4.17. *Let p be a prime number. Then for $a \geqslant 1$,*

$$S(m_1, n_1, n_2; p^a, p^a) = \begin{cases} p^{2a} - p^{2a-1} & \text{if } p^a | m_1 \text{ and } p^a | n_1, \\ -p^{2a-1} & \text{if } p^{a-1} \| m_1 \text{ and } p^a | n_1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Omitted.

PROPERTY 4.18. *$S(m_1, n_1, n_2; D_1, D_2) = 0$ unless $n_1 D_2 / D_1^2 \in \mathbf{Z}$.*

Proof. This follows immediately from the multiplicativity relation given in Property 4.15 and the vanishing criteria given in Properties 4.16 and 4.17.

If $D_1 | D_2$, define

$$S_1(m_1, m_2, n_1, n_2; D_1, D_2) = \delta_{m_2, n_1 D_2/D_1^2} S(m_1, n_1, n_2; D_1, D_2)$$

(δ = Kronecker delta).

PROPERTY 4.19.

$$S_1(m_1, m_2, n_1, n_2, D_1, D_2) = \frac{D_1^2}{D_2} S_1\left(n_2, n_1, m_2, m_1; \frac{D_2}{D_1}, D_2\right).$$

Proof. Trivial.

5. Fourier expansion of Poincaré series. Recall that an arbitrary $\varphi \in \mathscr{L}^2(\Gamma/H^3)$, $\Gamma = \mathrm{SL}(3, \mathbf{Z})$, has a Fourier expansion (see [16], [19]) given by

$$\varphi(\tau) = \sum_{m_2=-\infty}^{\infty} \varphi_{0,m_2}(\tau) + \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \varphi_{m_1,m_2}(\gamma\tau) \tag{5.1}$$

where

$$\Gamma^2 = \left\{ \begin{bmatrix} A & B & \\ C & D & \\ & & 1 \end{bmatrix} \middle| \; A, B, C, D \in \mathbf{Z},\ AD - BC = \pm 1 \right\},$$

$$\Gamma^2_\infty = \left\{ \begin{bmatrix} 1 & B & \\ & 1 & \\ & & 1 \end{bmatrix} \middle| \; B \in \mathbf{Z} \right\}$$

and

$$\varphi_{m_1,m_2}(\tau) = \int_0^1 \int_0^1 \int_0^1 \varphi\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{bmatrix} \tau \right) e(-m_1 \xi_1 - m_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3.$$

The main purpose of this section will be to give the Fourier expansion of the Poincaré series

$$P_{n_1,n_2}(\tau; \nu_1, \nu_2) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} I_{\nu_1,\nu_2}(\gamma\tau) E_{n_1,n_2}(\gamma\tau) \tag{5.2}$$

where $\mathrm{Re}(\nu_1)$, $\mathrm{Re}(\nu_2) > 2/3$, and for

$$\tau = \begin{bmatrix} y_1 y_2 & x_2 y_1 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{bmatrix} \in H^3,$$

$$I_{\nu_1,\nu_2}(\tau) = y_1^{2\nu_1 + \nu_2} y_2^{\nu_1 + 2\nu_2}$$

while $E_{n_1,n_2}(\tau)$ is an E-function as defined in (2.2). For later applications, we

always choose
$$E_{n_1,n_2}(\tau) = e(n_1(x_1+iy_1/M)+n_2(x_2+iy_2/M))$$
for some integer M. Since an E-function does not depend on x_3, it will be convenient to set

(5.3) $$E_{n_1,n_2}(\tau) \equiv E_{n_1,n_2}(z_1, z_2)$$

where $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$.

THEOREM 5.1. *Let* $\mathrm{Re}(\nu_1)$, $\mathrm{Re}(\nu_2) > 2/3$. *Then*

$$\int_0^1\int_0^1\int_0^1 P_{n_1,n_2}\left(\begin{bmatrix}1 & \xi_2 & \xi_3\\ & 1 & \xi_1\\ & & 1\end{bmatrix}\tau\right)e(-m_1\xi_1-m_2\xi_2)\,d\xi_1\,d\xi_2\,d\xi_3$$

$$= e(m_1x_1+m_2x_2)I_{\nu_1,\nu_2}(\tau)\sum_{w\in W}\sum_{\varepsilon_1,\varepsilon_2=\pm 1}\sum_{D_1,D_2=1}^{\infty} S_w(\varepsilon_1 m_1, \varepsilon_2 m_2, n_1, n_2; D_1, D_2)$$

$$\times D_1^{-3\nu_1}D_2^{-3\nu_2}J_w(y_1, y_2; \nu_1, \nu_2; \varepsilon_1 m_1, \varepsilon_2 m_2, n_1, n_2; D_1, D_2)$$

where S_w *are the Kloosterman sums and* J_w *are the integrals given in the following table:*

(5.4) **Table of Fourier expansion data**

If $w = \begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}$,

$S_w(m_1, m_2, n_1, n_2; D_1, D_2) = \delta_{D_1,1}\cdot\delta_{D_2,1}$ (Kronecker delta),
$J_w(y_1, y_2; \nu_1, \nu_2; m_1, m_2, n_1, n_2; D_1, D_2) = \delta_{m_1,n_1}\delta_{m_2,n_2}E_{n_1,n_2}(y_1, y_2)$.

If $w = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$,

$S_w(m_1, m_2, n_1, n_2; D_1, D_2) = \delta_{D_1,1}\sum e((am_2+\bar{a}n_2)D_2^{-1})$
(summation over $a = 1, \ldots, D_2$, $(a, D_2) = 1$, $a\bar{a} \equiv 1 \pmod{D_2}$),
$J_w(y_1, y_2; \nu_1, \nu_2; m_1, m_2, n_1, n_2; D_1, D_2)$
$$= \delta_{n_1,0}\delta_{m_1,0}\int_{-\infty}^{\infty}(\xi_2^2+y_2^2)^{-3\nu_2/2}E_{n_1,n_2}(0, -(\xi_2+iy_2)^{-1}D_2^{-2})e(-m_2\xi_2)\,d\xi_2.$$

If $w = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

$S_w(m_1, m_2, n_1, n_2; D_1, D_2) = \delta_{D_2,1}\sum e((am_1+\bar{a}n_1)D_1^{-1})$
(summation over $a = 1, \ldots, D_1$, $(a, D_1) = 1$, $a\bar{a} \equiv 1 \pmod{D_1}$),

$J_w(y_1, y_2; \nu_1, \nu_2; m_1, m_2, n_1, n_2; D_1, D_2)$
$$= \delta_{n_2,0}\delta_{m_2,0}\int_{-\infty}^{\infty}(\xi_1^2+y_1^2)^{-3\nu_1/2}E_{n_1,n_2}(-(\xi_1+iy_1)^{-1}D_1^{-2}, 0)e(-m_1\xi_1)\,d\xi_1.$$

If $w = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$,

$S_w(m_1, m_2, n_1, n_2; D_1, D_2) = \sum e(n_2\bar{C}_2/(D_2D_1^{-1})+m_1C_1D_1^{-1}+n_1\bar{C}_1C_2D_1^{-1})$,
(This term only occurs when $D_1|D_2$. Summation is over $C_1 \pmod{D_1}$, $C_2 \pmod{D_2}$, $(C_1, D_1) = 1$, $(C_2, D_2D_1^{-1}) = 1$, $C_2\bar{C}_2 \equiv 1 \pmod{D_2D_1^{-1}}$, $C_1\bar{C}_1 \equiv 1 \pmod{D_1}$.)

$$J_w(y_1, y_2; \nu_1, \nu_2; m_1, m_2, n_1, n_2; D_1, D_2) = \delta_{m_2D_1^2,n_1D_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\xi_1^2+y_1^2)^{-3\nu_1/2}Z_4^{-3\nu_2/2}$$
$$\times E\big((\xi_1\xi_4+iy_1Z_4^{1/2})(\xi_1^2+y_1^2)^{-1}(D_2D_1^{-2}), (\xi_4+iy_2(\xi_1^2+y_1^2)^{1/2})Z_4^{-1}(D_1D_2^{-2})\big)e(-m_1\xi_1)\,d\xi_1\,d\xi_4$$
where $Z_4 = \xi_4^2+\xi_1^2y_2^2+y_1^2y_2^2$.

If $w = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$,

$S_w(m_1, m_2, n_1, n_2; D_1, D_2) = \sum e(n_1\bar{C}_1/(D_1D_2^{-1})+m_1C_2D_2^{-1}+n_2\bar{C}_2C_1D_2^{-1})$.
(This term only occurs if $D_2|D_1$. Summation is over $C_1 \pmod{D_1}$, $C_2 \pmod{D_2}$, $(C_2, D_2) = 1$, $(C_1, D_1D_2^{-1}) = 1$, $C_1\bar{C}_1 \equiv 1 \pmod{D_1D_2^{-1}}$, $C_2\bar{C}_2 \equiv 1 \pmod{D_2}$.)

$$J_w(y_1, y_2; \nu_1, \nu_2; m_1, m_2, n_1, n_2; D_1, D_2) = \delta_{m_1D_2^2,n_2D_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Z_3^{-3\nu_1/2}(\xi_2^2+y_2^2)^{-3\nu_2/2}$$
$$\times E_{n_1,n_2}\big((\xi_3+iy_1(\xi_2^2+y_2^2)^{1/2})Z_3^{-1}(D_2D_1^{-2}), (\xi_2\xi_3+iy_2Z_3^{1/2})(\xi_2^2+y_2^2)^{-1}(D_1D_2^{-2})\big)e(-m_2\xi_2)\,d\xi_2$$
where $Z_3 = \xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2$.

If $w = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$,

$S_w(m_1, m_2, n_1, n_2; D_1, D_2)$
$$= \sum e\big((m_2B_1+n_1(Y_1D_2-Z_1B_2))D_1^{-1}, (m_1B_2+n_2(Y_2D_1-Z_2B_1))D_2^{-1}\big).$$
(Summation over $B_1, C_1 \pmod{D_1}$, $B_2, C_2 \pmod{D_2}$, $(D_1, B_1, C_1) = (D_2, B_2, C_2) = 1$ such that $D_1C_2+B_1B_2+D_2C_1 \equiv 0 \pmod{D_1D_2}$; here Y_1, Z_1, Y_2, Z_2 are chosen so $Y_1B_1+Z_1C_1 \equiv 1 \pmod{D_1}$, $Y_2B_2+Z_2C_2 \equiv 1 \pmod{D_2}$.)

$$J_w(y_1, y_2; \nu_1, \nu_2; m_1, m_2, n_1, n_2; D_1, D_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Z_3^{-3\nu_1/2}Z_4^{-3\nu_2/2}e(-m_1\xi_1-m_2\xi_2)$$
$$\times E_{n_1,n_2}((-\xi_1\xi_3-\xi_2y_1^2+iy_1Z_4^{1/2})Z_3^{-1}(D_2D_1^{-2}), (-\xi_2\xi_4-\xi_1y_2^2+iy_1Z_3^{1/2})Z_4^{-1}(D_1D_2^{-2}))\,d\xi_1\,d\xi_2\,d\xi_3$$
where $Z_3 = \xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2$, $Z_4 = \xi_4^2+\xi_1^2y_2^2+y_1^2y_2^2$, $\xi_4 = \xi_1\xi_2-\xi_3$.

Remark. Note that the sums arising from the $w = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

terms are classical Kloosterman sums. For $w = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$S_w(m_1, m_2, n_1, n_2; D_1, D_2) = S(m_1, n_1, n_2; D_1, D_2)$$

is the sum introduced in (4.3), while for $w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$,

$$S_w(m_1, m_2, n_1, n_2; D_1, D_2) = S(m_2, n_2, n_1; D_2, D_1).$$

Finally, when $w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$, S_w is the sum introduced in (4.1) with m_1 and m_2 exchanged, $S_w(m_1, m_2, n_1, n_2; D_1, D_2) = S(m_2, m_1, n_1, n_2; D_1, D_2)$.

Proof of Theorem 5.1. Let

$$U = \left\{ \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{bmatrix} \;\middle|\; \varepsilon_i = \pm 1,\ \varepsilon_1 \varepsilon_2 \varepsilon_3 = +1 \right\}$$

so that U has exactly four elements. We can then write

$$P_{n_1,n_2}(\tau; \nu_1, \nu_2) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / U} \sum_{u \in U} I_{\nu_1,\nu_2}(\gamma\tau) E_{n_1,n_2}(\gamma u \tau).$$

The Fourier expansion is, therefore, given by

$$(5.5)\quad \int_0^1\!\!\int_0^1\!\!\int_0^1 P_{n_1,n_2}\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{bmatrix} \tau\right) e(-m_1 \xi_1 - m_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3$$

$$= e(m_1 x_1 + m_2 x_2) \int_0^1\!\!\int_0^1\!\!\int_0^1 P_{n_1,n_2}\left(\begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ & y_1 & \xi_1 \\ & & 1 \end{bmatrix}\right) \times e(-m_1 \xi_1 - m_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3$$

$$= e(m_1 x_1 + m_2 x_2) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / U} \sum_{u \in U} \int_0^1\!\!\int_0^1\!\!\int_0^1 I_{\nu_1,\nu_2} E_{n_1,n_2}\left(\gamma u \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ & y_1 & \xi_1 \\ & & 1 \end{bmatrix}\right) \times e(-m_1 \xi_1 - m_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3.$$

But each u sends some ξ_i to $-\xi_i$, and this can be undone by a simple

variable change. Hence, we see that the above integral is equal to

$$(5.6)\quad e(m_1 x_1 + m_2 x_2) \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / U} \int_0^1\!\!\int_0^1\!\!\int_0^1 I_{\nu_1,\nu_2} \times E_{n_1,n_2}\left(\gamma \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ & y_1 & \xi_1 \\ & & 1 \end{bmatrix}\right) e(-m_1 \varepsilon_1 \xi_1 - m_2 \varepsilon_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3.$$

Now, Γ_w, given by (3.6), acts properly on the right on each $\Gamma_\infty \backslash G_w \cap \Gamma / U$.

Hence, using the Bruhat decompositions given in (3.3), (3.4) and Propositions 3.2 to 3.7, we can compute

$$(5.7)\quad I = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / U} \int_0^1\!\!\int_0^1\!\!\int_0^1 I_{\nu_1,\nu_2} E_{n_1,n_2}\left(\gamma \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ & y_1 & \xi_1 \\ & & 1 \end{bmatrix}\right) \times e(-m_1 \xi_1 - m_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3$$

$$= \sum_{w \in W} \sum_{\substack{\gamma \in R_w \\ \gamma = b_1 w d b_2 \\ b_1, b_2 \in G_\infty, d \in D}} e_{n_1,n_2}(b_1) \sum_{l \in \Gamma_w} \int_0^1\!\!\int_0^1\!\!\int_0^1 I_{\nu_1,\nu_2} \times E_{n_1,n_2}\left(w d b_2 l \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ & y_1 & \xi_1 \\ & & 1 \end{bmatrix}\right) e(-m_1 \xi_1 - m_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3$$

here

$$e_{n_1,n_2}\left(\begin{bmatrix} 1 & x_2 & * \\ & 1 & x_1 \\ & & 1 \end{bmatrix}\right) = e(n_1 x_1 + n_2 x_2)$$

and

$$R_w = \Gamma_\infty \backslash G_w \cap \Gamma / U \Gamma_w$$

is the set of representatives given in Section 3.

Note that we have used the basic Property 2.2 of an E-function to bring out the factor $e_{n_1,n_2}(b_1)$.

Now, by (3.6), we see that Γ_w is one of the following six types of subgroups of Γ_∞:

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}.$$

Since l lies in one of the above groups, a simple change of variables in ξ_1, ξ_2, ξ_3 on the right-hand side of (5.7) yields

$$(5.8)\qquad I=\sum_{w\in W}\sum_{\substack{\gamma\in\Gamma_\infty\backslash(G_w\cap\Gamma)/U\Gamma_w\\ \gamma=b_1wdb_2}} e_{n_1,n_2}(b_1)\int_{\mathcal{D}_w} I_{\nu_1,\nu_2}$$
$$\times E_{n_1,n_2}\left(wdb_2\begin{bmatrix} y_1y_2 & y_1\xi_2 & \xi_3\\ & y_1 & \xi_1\\ & & 1\end{bmatrix}\right)e(-m_1\xi_1-m_2\xi_2)\,d\xi_1\,d\xi_2\,d\xi_3$$

where in the $\xi_3\times\xi_2\times\xi_1$ space

$$(5.9)\qquad \mathcal{D}_w=\begin{cases}[0,1]\times[0,1]\times[0,1], & w=\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix},\\[2mm] [0,1]\times[-\infty,\infty]\times[0,1], & w=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\\[2mm] [0,1]\times[0,1]\times[-\infty,\infty], & w=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\\[2mm] [-\infty,\infty]\times[0,1]\times[-\infty,\infty], & w=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},\\[2mm] [-\infty,\infty]\times[-\infty,\infty]\times[0,1], & w=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\\[2mm] [-\infty,\infty]\times[-\infty,\infty]\times[-\infty,\infty], & w=\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}.\end{cases}$$

Actually when $w=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ further justification of the above is required; (5.8) holds since the ξ_3-integral is independent of the interval of integration provided it is connected and has length one. This follows from Table (5.12).

Let

$$b_2=\begin{bmatrix}1&\beta_2&\beta_3\\&1&\beta_1\\&&1\end{bmatrix}.$$

Consequently

$$b_2\begin{bmatrix}y_1y_2 & y_1\xi_2 & \xi_3\\ & y_1 & \xi_1\\ & & 1\end{bmatrix}=\begin{bmatrix}y_1y_2 & y_1(\xi_2+\beta_2) & \xi_3+\beta_2\xi_1+\beta_3\\ & y_1 & \xi_1+\beta_1\\ & & 1\end{bmatrix}.$$

If we now make the change of variables

$$\xi_1\to\xi_1-\beta_1,\qquad \xi_2\to\xi_2-\beta_2,\qquad \xi_3\to\xi_3-\beta_2\xi_1-\beta_3$$

then (5.8) becomes

(5.10)

$$I=I_{\nu_1,\nu_2}(\tau)E_{n_1,n_2}(y_1,y_2)\int_0^1\int_0^1\int_0^1 e\big((n_1-m_1)\xi_1+(n_2-m_2)\xi_2\big)\,d\xi_1\,d\xi_2\,d\xi_3$$
$$+\sum_{\substack{w\in W\\ w\neq\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}}}\sum_{\substack{\gamma\in\Gamma_\infty\backslash(G_w\cap\Gamma)/U\Gamma_w\\ \gamma=b_1wdb_2}} e_{n_1,n_2}(b_1)\,e_{m_1,m_2}(b_2)$$
$$\times\int_{\mathcal{D}_{w,b_2}} I_{\nu_1,\nu_2}E_{n_1,n_2}\left(wd\begin{bmatrix}y_1y_2 & y_1\xi_2 & \xi_3\\ & y_1 & \xi_1\\ & & 1\end{bmatrix}\right)$$
$$\times e(-m_1\xi_1-m_2\xi_2)\,d\xi_1\,d\xi_2\,d\xi_3$$

for a certain domain of integration $\mathcal{D}_{w,b_2}$ depending only on w and b_2.

We now tabulate w, b_2, d and $\mathcal{D}_{w,b_2}$ explicitly by using the canonical representatives and Bruhat decompositions as given in Propositions 3.3 to 3.13. It is remarkable that ultimately the integral $\int_{\mathcal{D}_{w,b_2}}$ depends only on d, except when $w=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$, $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ in which case an extra exponential factor arises which is absorbed in the Kloosterman sum.

w	d	b_2	$\mathscr{D}_{w,b_2}$
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -B_2 & & \\ & \frac{1}{B_2} & \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{-C_2}{B_2} & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$	$[0, 1]\times[-\infty, \infty]\times[0, 1]$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ & B_1 & \\ & & -\frac{1}{B_1} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ & 1 & \frac{C_1}{B_1} \\ & & 1 \end{bmatrix}$	$[0, 1]\times[0, 1]\times[-\infty, \infty]$
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{A_2}{B_1} & & \\ & B_1 & \\ & & \frac{1}{A_2} \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{\alpha C_2 B_1}{A_2} & \frac{\beta C_2 B_1}{A_2} \\ & 1 & \frac{C_1}{B_1} \\ & & 1 \end{bmatrix}$	$[-\infty, \infty]\times\left[\frac{\alpha C_2 B_1}{A_2}, 1+\frac{\alpha C_2 B_1}{A_2}\right]\times[-\infty, \infty]$
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} A_1 & & \\ & \frac{1}{B_2} & \\ & & \frac{B_2}{A_1} \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_1} \\ & 1 & \frac{-b_{22} B_2}{A_1} \\ & & 1 \end{bmatrix}$	$[-\infty, \infty]\times[-\infty, \infty]\times\left[\frac{-b_{22} B_2}{A_1}, 1-\frac{b_{22} B_2}{A_1}\right]$
$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$	$\begin{bmatrix} A_1 & & \\ & \frac{-A_2}{A_1} & \\ & & \frac{1}{A_2} \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_1} \\ & 1 & \frac{-B_2}{A_2} \\ & & 1 \end{bmatrix}$	$[-\infty, \infty]\times[-\infty, \infty]\times[-\infty, \infty]$

Now

$$\text{(5.11)} \qquad S_w(m_1, m_2, n_1, n_2; d) = \sum_{\substack{\gamma\in\Gamma_\infty\backslash(G_w\cap\Gamma)/U\Gamma_w \\ \gamma=b_1 w d b_2 \\ d \text{ fixed}}} e_{n_1,n_2}(b_1)\, e_{m_1,m_2}(b_2)$$

is the Kloosterman sum appearing in Theorem 5.1, except when

$w = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. It remains to be shown that the integral $\int_{\mathscr{D}_{w,b_2}}$ can be transformed into a suitable multiple of J_w.

Let

$$wd\begin{bmatrix} y_1 y_2 & y_2\xi_2 & \xi_3 \\ & y_1 & \xi_1 \\ & & 1 \end{bmatrix} \equiv \begin{bmatrix} y_1' y_2' & y_1' x_2' & x_3' \\ & y_1' & x_1' \\ & & 1 \end{bmatrix} \bmod (O(3)\cdot \boldsymbol{Z})$$

where $\boldsymbol{Z}$ is the center of $\mathrm{GL}(3, \boldsymbol{R})$ consisting of scalar matrices. For convenience, led d have the same signs as those specified in Propositions 3.9 to 3.13 (i.e., $B_2 > 0$ in the first row below, etc.). We give x_1', x_2', x_3', y_1', y_2' in the following table.

Table (5.12)

w	x_1'	x_2'	x_3'	y_1'	y_2'
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$-B_2\xi_3$	$\frac{-\xi_2}{B_2^2(\xi_2^2+y_2^2)}$	$\frac{\xi_1}{B_2}$	$B_2 y_1(\xi_2^2+y_2^2)^{1/2}$	$\frac{y_2}{B_2^2(\xi_2^2+y_2^2)}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\frac{-\xi_1}{B_1^2(\xi_1^2+y_1^2)}$	$B_1(\xi_1\xi_2-\xi_3)$	$\frac{\xi_1\xi_3+\xi_2 y_1^2}{B_1(\xi_1^2+y_1^2)}$	$\frac{y_1}{B_1^2(\xi_1^2+y_1^2)}$	$B_1 y_2(\xi_1^2+y_1^2)^{1/2}$
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\frac{A_2}{B_1^2}\frac{(\xi_1\xi_3+\xi_2 y_1^2)}{\xi_1^2+y_1^2}$	$-B_1\xi_4/A_2^2 Z_4$, $\xi_4=\xi_1\xi_2-\xi_3$, $Z_4=\xi_4^2+y_2^2\xi_1^2+y_1^2 y_2^2$	$\frac{\xi_1}{B_1 A_2(\xi_1^2+y_1^2)}$	$\frac{A_2 y_1 Z_4^{1/2}}{B_1^2(\xi_1^2+y_1^2)}$	$\frac{B_1 y_2(\xi_1^2+y_1^2)^{1/2}}{A_2^2 Z_4}$
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\frac{B_2\xi_3}{A_1^2 Z_3}$, $Z_3=\xi_3^2+\xi_2^2 y_1^2+y_1^2 y_2^2$	$\frac{A_1}{B_2^2}\left(\xi_1-\frac{\xi_2\xi_3}{\xi_2^2+y_2^2}\right)$	$\frac{\xi_1\xi_3+\xi_2 y_1^2}{A_1 B_2 Z_3}$	$\frac{B_2 y_1(\xi_2^2+y_2^2)^{1/2}}{A_1^2 Z_3}$	$\frac{A_1 y_2 Z_3^{1/2}}{B_2^2(\xi_2^2+y_2^2)}$
$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$	$\frac{-A_2}{A_1^2}\frac{(\xi_1\xi_3+\xi_2 y_1^2)}{Z_3}$ $Z_3=\xi_3^2+\xi_2^2 y_1^2+y_1^2 y_2^2$	$\frac{-A_1}{A_2^2}\frac{\xi_2\xi_4+\xi_1 y_2^2}{Z_4}$, $\xi_4=\xi_1\xi_2-\xi_3$, $Z_4=\xi_4^2+\xi_1^2 y_2^2+y_1^2 y_2^2$	$\frac{\xi_3}{A_1 A_2 Z_3}$	$\frac{A_2 y_1 Z_4^{1/2}}{A_1^2 Z_3}$	$\frac{A_1 y_2 Z_3^{1/2}}{A_2^2 Z_4}$

It immediately follows from Table (5.12), (5.10) and (5.11) that we can verify Table (5.4) for $w = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{bmatrix}$. We treat the remainder of Table (5.4) on a case by case basis.

For $w = \begin{bmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix}$, we have

$$\int_{\mathscr{D}_{w,b_2}} = A_2^{-3\nu_2} B_1^{-3\nu_1} \int_{\frac{\alpha C_2 B_1}{A_2}}^{1+\frac{\alpha C_2 B_1}{A_2}} e\left(\left(\frac{n_1 A_2}{B_1^2} - m_2\right)\xi_2\right)$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z_4^{-3\nu_2/2} (\xi_1^2 + y_1^2)^{-3\nu_1/2}$$

$$\times E_{n_1,n_2}\left(\frac{A_2}{B_1^2}\left(\frac{-\xi_1 \xi_4 + iy_1 Z_4^{1/2}}{\xi_1^2 + y_1^2}\right), \frac{B_1}{A_2^2}\left(\frac{-\xi_4 + iy_2(\xi_1^2 + y_1^2)^{1/2}}{Z_4}\right)\right)$$

$$\times e(-m_1 \xi_1)\, d\xi_1\, d\xi_2\, d\xi_3.$$

Making the change of variables sending ξ_3 to $\xi_4 = \xi_1 \xi_2 - \xi_3$, and computing for $n_1 A_2/B_1^2 \neq m_2$

$$\text{(5.13)} \qquad \int_{\alpha C_2 B_1 A_2^{-1}}^{1+\alpha C_2 B_1 A_2^{-1}} e((n_1 A_2 B_1^{-2} - m_2)\xi_2)\, d\xi_2$$

$$= \frac{1}{2\pi i} e\left(\frac{n_1 \alpha C_2}{B_1} - \frac{m_2 \alpha C_2 B_1}{A_2}\right)\left[e\left(\frac{n_1 A_2}{B_1^2}\right) - 1\right]\left(\frac{n_1 A_2}{B_1^2} - m_2\right)^{-1}$$

gives

$$\int_{\mathscr{D}_{w,b_2}} = A_2^{-3\nu_2} B_1^{-3\nu_1} e\left(\frac{n_1 \alpha C_2}{B_1} - \frac{m_2 \alpha C_2 B_1}{A_2}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\xi_1^2 + y_1^2)^{-3\nu_1/2} Z_4^{-3\nu_2/2}$$

$$\times E_{n_1,n_2}\left(\frac{A_2}{B_1^2} \frac{-\xi_1 \xi_4 + iy_1 Z_4^{1/2}}{Z_3}, \frac{B_1}{A_2^2} \frac{-\xi_4 + iy_2(\xi_1^2 + y_1^2)^{1/2}}{Z_4}\right)$$

$$\times e(-m_1 \xi_1)\, d\xi_1\, d\xi_4 \cdot \delta$$

where

$$\delta = \begin{cases} 1 & \text{if } m_2 = n_1 A_2/B_1^2, \\ \left(e\left(\frac{n_1 A_2}{B_1^2}\right) - 1\right) \Big/ 2\pi i \left(\frac{n_1 A_2}{B_1^2} - m_2\right) & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{\gamma} e_{n_1,n_2}(b_1)\, e_{m_1,m_2}(b_2)\, e\left(\frac{n_1 \alpha C_2}{B_1} - \frac{m_2 \alpha C_2 B_1}{A_2}\right)$$

$$= \sum_{\substack{C_1 (\mathrm{mod}\, D_1) \\ C_2 (\mathrm{mod}\, D_2)}} e\left(\frac{n_2 \bar{C}_2}{D_2/D_1} + \frac{m_1 C_1}{D_1} + \frac{n_1 \bar{C}_1 C_2}{D_1}\right)$$

(the summation being subject to the conditions $(C_1, D_1) = (C_2, D_2 D_1^{-1}) = 1$, with $C_1 \bar{C}_1 \equiv 1 (\mathrm{mod}\, D_1)$, $C_2 \bar{C}_2 \equiv 1 (\mathrm{mod}\, D_2 D_1^{-1})$), where $D_1 = B_1$, $D_2 = A_2$. However, by Property 4.18 this sum vanishes unless $n_1 D_2/D_1^2 \in \mathbf{Z}$. But in this case, the integral (5.13) vanishes for $m_2 \neq n_1 A_2/B_1^2$. Hence, we obtain the result of Table (5.4).

When $w = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{bmatrix}$, we obtain in a similar manner

$$\int_{\mathscr{D}_{w,b_2}} = A_1^{-3\nu_1} B_2^{-3\nu_2} \int_{\frac{-b_{22} B_2}{A_1}}^{1-\frac{b_{22} B_2}{A_1}} e\left(\left(\frac{n_2 A_1}{B_2^2} - m_1\right)\xi_1\right) d\xi_1$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z_3^{-3\nu_1/2} (\xi_2^2 + y_2^2)^{-3\nu_2/2} E_{n_1,n_2}\left(\frac{B_2}{A_1^2}\left(\frac{\xi_3 + iy_1(\xi_2^2 + y_2^2)^{1/2}}{Z_3}\right),\right.$$

$$\left.\frac{A_1}{B_2^2}\left(\frac{-\xi_2 \xi_3 + iy_2 Z_3^{1/2}}{\xi_2^2 + y_2^2}\right)\right) e(-m_2 \xi_2)\, d\xi_2\, d\xi_3$$

and the computation for this case follows as before.

Finally, when $w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$, from Proposition 3.7, Table (5.12), and the above considerations, we obtain

$$\sum_{R_w} e\left(\frac{n_1 a_{21} + m_2 B_1}{A_1} - \frac{n_2 b_{21} + m_1 B_2}{A_2}\right) A_1^{-3\nu_1} A_2^{-3\nu_2} J_w$$

for the contribution of the right-hand side of (5.10) coming from

$$w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

When $Y_1 B_1 + Z_1 C_1 \equiv 1 (\mathrm{mod}\, A_1)$, then

$$Y_1 A_2 - Z_1 B_2 \equiv Y_1(a_{21} B_1 - a_{22} A_1) - Z_1(a_{23} A_1 - a_{21} C_1)$$

$$\equiv a_{21}(Y_1 B_1 + Z_1 C_1) \equiv a_{21} (\mathrm{mod}\, A_1).$$

Similarly, when $Y_2 B_2 + Z_2 C_2 \equiv 1 \pmod{A_2}$, we have

$$Y_2 A_1 - Z_2 B_1 \equiv b_{21} \pmod{A_2}.$$

After renaming B_2, C_2, C_1, Y_2, Z_2, Z_1 by their negatives, using the representatives for R_w given in Proposition 3.13, and setting $A_1 = D_1$, $A_2 = D_2$, we obtain the Kloosterman sum given in Table (5.4). This completes the proof of Theorem 5.1. ∎

6. Spectral decomposition of Poincaré series, cuspidal contribution. Let f, g be two square-integrable automorphic forms for $\Gamma = SL(3, \mathbf{Z})$. We define the inner product

$$\langle f, g\rangle = \int_{\Gamma\backslash H^3} f(\tau)\overline{g(\tau)}\,\frac{dx_1\,dx_2\,dx_3\,dy_1\,dy_2}{(y_1 y_2)^3}. \tag{6.1}$$

With respect to this inner product, the Hilbert space $\mathcal{L}^2(\Gamma\backslash H^3)$ decomposes into the direct sum

$$\mathcal{L}^2(\Gamma\backslash H^3) = \mathbf{C} \oplus \mathcal{L}^2_{\text{cusp}}(\Gamma\backslash H^3) \oplus \mathcal{L}^2_{\text{Eis}}(\Gamma\backslash H^3) \oplus \mathcal{L}^2_{\text{Res}}(\Gamma\backslash H^3)$$

where $\mathcal{L}^2_{\text{cusp}}$ is the space spanned by cuspidal automorphic wave forms, $\mathcal{L}^2_{\text{Eis}}$ is spanned by integrals of Eisenstein series, and $\mathcal{L}^2_{\text{Res}}$ is spanned by integrals of the non-trivial residues of Eisenstein series. We now describe these spaces.

For $\tau \in H^3$, let $\varphi_j(\tau)$ $(j = 1, 2, \ldots)$ be a complete orthonormal basis for $\mathcal{L}^2_{\text{cusp}}$. Also let

$$\varphi_0(\tau) = \left(\operatorname{Vol}(\Gamma\backslash H^3)\right)^{-1/2}$$

be the suitably normalized constant function.

To describe the continuous spectrum, we introduce three types of Eisenstein series. These are associated to the three nonconjugate parabolic subgroups of Γ. Namely

$$\Gamma_\infty = \left\{\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1\end{bmatrix} \in SL(3, \mathbf{Z})\right\},$$

$$P_{2,1} = \left\{\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1\end{bmatrix} \in \Gamma\right\}, \tag{6.2}$$

$$P_{1,2} = \left\{\begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & *\end{bmatrix} \in \Gamma\right\}.$$

Firstly, the minimal parabolic Eisenstein series associated to Γ_∞ is

defined as

$$E(\tau; s_1, s_2) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y_1^{2s_1+s_2} y_2^{s_1+2s_2}|[\gamma] \tag{6.3}$$

where $f(\tau)|[\gamma] = f(\gamma\tau)$ for any $f\colon H^3 \to \mathbf{C}$. It is easily seen that the series on the right-hand side of (6.3) converges absolutely and uniformly on compact subsets of H^3 for $\operatorname{Re}(s_1), \operatorname{Re}(s_2) > 2/3$. It is also well known that $E(\tau; s_1, s_2)$ has a meromorphic continuation and functional equation (cf. [17], [13], [12], [20], [8], [1]).

Now, let $u_j(z)$ $(j = 1, 2, \ldots, z = x+iy, y > 0)$ be an orthonormal basis of Maass waveforms for $SL(2, \mathbf{Z})$. Then $u_j(z)$ are characterized by the properties

$$u_j(\gamma z) = u_j(z) \quad \text{for all } \gamma \in SL(2, \mathbf{Z}),$$

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u_j(z) = \left(\tfrac14 + r_j^2\right)u_j(z) \quad (r_j^2 \in \mathbf{R}), \tag{6.4}$$

$$|u_j(z)| = O(1).$$

Let us also fix $u_0 = \left(\operatorname{Vol}(SL(2, \mathbf{Z})\backslash H^2)\right)^{-1/2}$, to have length one.

We can also define the two maximal parabolic Eisenstein series associated to u_j. They are

$$\begin{aligned} E_j(\tau; s) &= \sum_{\gamma\in P_{2,1}\backslash\Gamma} (y_1^2 y_2)^s u_j(x_2 + iy_2)|[\gamma], \\ \tilde{E}_j(\tau; s) &= \sum_{\gamma\in P_{1,2}\backslash\Gamma} (y_1 y_2^2)^s u_j(x_1 + iy_1)|[\gamma]. \end{aligned} \tag{6.5}$$

The series on the right-hand side of (6.5) converge absolutely and uniformly on compact subsets of H^3 provided $\operatorname{Re}(s) > 1$.

For $\tau \in H^3$, let us recall the involution

$$'\tau = w({}^t\tau)^{-1} w \pmod{KZ}$$

where $K = O(3)$, $Z =$ center of $GL(3, \mathbf{R})$ consisting of scalar matrices and $w = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$. In a manner similar to (2.11), we see that

$$\tilde{E}_j(\tau; s) = E_j('\tau; s). \tag{6.6}$$

It is well known that $E_j(\tau; s)$ and $\tilde{E}_j(\tau; s)$ can be meromorphically continued and satisfy a functional equation (see [12]). The space $\mathcal{L}^2_{\text{Eis}}$ is spanned by integrals of $E(\tau; s_1, s_2)$ and $E_j(\tau; s)$ $(j = 1, 2, \ldots)$

Finally, we consider $\mathcal{L}^2_{\text{Res}}$. Using [1] or [12] or Theorem 5.1, it can be shown that the minimal parabolic Eisenstein series $E(\tau; s_1, s_2)$ has polar divisors at the six lines

$$s_1 = 0 \quad \text{or} \quad 2/3,$$
$$s_2 = 0 \quad \text{or} \quad 2/3, \tag{6.7}$$
$$1-s_1-s_2 = 0 \quad \text{or} \quad 2/3.$$

Then $\mathscr{L}^2_{\text{Res}}$ is one-dimensional with basis, say, a suitable integral of

$$\operatorname*{Res}_{s_1=2/3} E(\tau; s_1, s_2) = c' E_0(\tau; s_2+\tfrac{1}{3}) \tag{6.8}$$

for some constant c'. This is obtained by comparing the Fourier expansion given in Theorem 5.1 (in the case $n_1 = n_2 = 0$) with the Fourier expansion of the maximal parabolic Eisenstein series $E_0(\tau; s)$ given in [4]. Here $E_0(\tau; s)$ is given by (6.5) with $u_0 = \big(\mathrm{Vol}(\mathrm{SL}(2, \mathbf{Z})\backslash H^2)\big)^{-1/2}$.

We now state Selberg's spectral expansion for an arbitrary $\Phi \in \mathscr{L}^2(\Gamma\backslash H^3)$. We have

$$\Phi(\tau) = \sum_{j=0}^{\infty}\Big(\langle\Phi, \varphi_j\rangle\varphi_j(\tau)+\frac{c_j}{4\pi i}\int_{\mathrm{Re}(s)=1/2}\langle\Phi, E_j(*, s)\rangle E_j(\tau; s)\,ds\Big) \tag{6.9}$$
$$+\frac{c}{(2\pi i)^2}\int_{\mathrm{Re}(s_1)=1/3}\int_{\mathrm{Re}(s_2)=1/3}\langle\Phi, E(*; s_1, s_2)\rangle E(\tau; s_1, s_2)\,ds_1\,ds_2$$

where c, c_j ($j = 0, 1, 2, \ldots$) are certain constants.

Now, for $\mathrm{Re}(v_1), \mathrm{Re}(v_2) > 2/3$ and $M \geqslant 1$, the Poincaré series

$$P_{n_1,n_2}(\tau; v_1, v_2; M) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y_1^{2v_1+v_2}\, y_2^{v_1+2v_2}\, e(n_1 z_1+n_2 z_2)|[\gamma] \tag{6.10}$$

where $n_1, n_2 > 0$ and

$$z_1 = x_1+(iy_1/M), \qquad z_2 = x_2+(iy_2/M)$$

lies in $\mathscr{L}^2(\Gamma\backslash H^3)$. The spectral decomposition (6.9) applied to the Poincaré series (6.10) will give the meromorphic continuation in v_1, v_2 of the Poincaré series. In the remainder of this section, we compute the cuspidal contribution of the spectral decomposition.

Accordingly, let φ be a cuspidal automorphic wave form for Γ. If $\mathscr{D}$ denotes the algebra of differential operators acting on H^3 and commuting with $\mathrm{GL}(3, \mathbf{R})$, we say φ is *of type* (λ_1, λ_2) if it has the same eigenvalues as $y_1^{2\lambda_1+\lambda_2} y_2^{\lambda_1+2\lambda_2}$ for any partial differential operator in $\mathscr{D}$. By considering the action of the six elements of the Weyl group

$$\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}, \begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}, \begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix}, \begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix}, \begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix}, \begin{bmatrix}&&1\\1&&\\&1&\end{bmatrix}$$

on $y_1^{2\lambda_1+\lambda_2} y_2^{\lambda_1+2\lambda_2}$, it can be shown that if φ is of type (λ_1, λ_2), it must also be of type

$$(\tfrac{2}{3}-\lambda_2, \tfrac{2}{3}-\lambda_1), \quad (\lambda_1+\lambda_2-\tfrac{1}{3}, \tfrac{2}{3}-\lambda_2), \quad (\tfrac{2}{3}-\lambda_1, \lambda_1+\lambda_2-\tfrac{1}{3}),$$
$$(1-\lambda_1-\lambda_2, \lambda_1), \quad (\lambda_2, 1-\lambda_1-\lambda_2).$$

Let us now consider the inner product $\langle P_{n_1,n_2}, \varphi\rangle$ which occurs in the spectral decomposition (6.9). By Proposition 2.1, we have

$$\langle P_{n_1,n_2}, \varphi\rangle = \int_0^\infty\int_0^\infty \varphi_{n_1,n_2}\left(\begin{bmatrix}y_1y_2&&\\&y_1&\\&&1\end{bmatrix}\right) \times y_1^{2v_1+v_2}\, y_2^{v_1+2v_2}\, e^{-2\pi(n_1y_1+n_2y_2)/M}\,\frac{dy_1\,dy_2}{(y_1y_2)^3} \tag{6.11}$$

where

$$\varphi_{n_1,n_2}(\tau) = \int_0^1\int_0^1\int_0^1 \bar{\varphi}\left(\begin{bmatrix}1&\xi_2&\xi_3\\&1&\xi_1\\&&1\end{bmatrix}\tau\right) e(n_1\xi_1+n_2\xi_2)\,d\xi_1\,d\xi_2\,d\xi_3.$$

Clearly, φ_{n_1,n_2} is an E-function and also an eigenfunction of type $(\bar{\lambda}_1, \bar{\lambda}_2)$ for $\mathscr{D}$. Since φ_{n_1,n_2} has polynomial growth in y_1, y_2 as $y_1, y_2 \to \infty$, it follows from Shalika's multiplicity one theorem [19] that φ_{n_1,n_2} must be a constant multiple of the Whittaker function

$$W_{\bar{\lambda}_1,\bar{\lambda}_2}\left(\begin{bmatrix}n_1n_2&&\\&n_1&\\&&1\end{bmatrix}\tau\right)$$

where (see [1], p. 161)

$$W_{\lambda_1,\lambda_2}\left(\begin{bmatrix}y_1y_2&&\\&y_1&\\&&1\end{bmatrix}\right) \tag{6.12}$$
$$= \frac{1}{4}\frac{1}{(2\pi i)^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} G(s_1, s_2; \lambda_1, \lambda_2)(\pi y_1)^{1-s_1}(\pi y_2)^{1-s_2}\,ds_1\,ds_2$$

(σ sufficiently large)

and

$$(6.13)\quad G(s_1, s_2; \lambda_1, \lambda_2) = \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right)\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}$$

$$(\alpha = 1-\lambda_1-2\lambda_2,\ \beta = \lambda_2-\lambda_1,\ \gamma = 2\lambda_1+\lambda_2-1).$$

Here $\alpha+\beta+\gamma = 0$ and α, β, γ are permuted by the six elements of the Weyl group.

Let us set

$$(6.14)\qquad \varphi_{n_1,n_2}(\tau) = \frac{\overline{A(n_1, n_2)}}{n_1 n_2} W_{\bar\lambda_1,\bar\lambda_2}\left(\begin{bmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{bmatrix}\tau\right).$$

We shall show that $\dfrac{A(n_1, n_2)}{n_1 n_2}$ must be the Fourier coefficient in the Fourier–Whittaker expansion of φ. That is to say (following [1], p. 65–66)

$$(6.15)\quad \varphi(\tau) = \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{A(n_1, n_2)}{n_1 n_2} W_{\lambda_1,\lambda_2}\left(\begin{bmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{bmatrix} g\tau\right)$$

where

$$\Gamma^2 = \left\{\begin{bmatrix} A & B & \\ C & D & \\ & & 1 \end{bmatrix} \middle|\ A, B, C, D\in Z,\ AD-BC = \pm 1\right\},$$

$$\Gamma^2_\infty = \left\{\begin{bmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{bmatrix}\in \mathrm{SL}(3, Z)\right\}.$$

To deduce (6.14), we simply note that for $g\in\Gamma^2_\infty\backslash\Gamma^2$,

$$\int_0^1\int_0^1\int_0^1 W_{\lambda_1,\lambda_2}\left(\begin{bmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{bmatrix} g \begin{bmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{bmatrix}\tau\right) e(n_1\xi_1+n_2\xi_2)\, d\xi_1\, d\xi_2\, d\xi_3$$

is zero unless g is the identity.

It now follows from (6.11) that

$$\langle P_{n_1,n_2}, \varphi\rangle = \frac{\overline{A(n_1, n_2)}}{n_1 n_2}\int_0^\infty\int_0^\infty W_{\bar\lambda_1,\bar\lambda_2}\left(\begin{bmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{bmatrix}\right) \times y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} e^{-2\pi(n_1y_1+n_2y_2)/M}\frac{dy_1\, dy_2}{(y_1 y_2)^3}.$$

Consequently

$$(6.16)\quad \langle P_{n_1,n_2}, \varphi\rangle = \frac{\overline{A(n_1, n_2)}}{n_1^{2\nu_1+\nu_2-1} n_2^{\nu_1+2\nu_2-1}} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(-2\pi n_1)^{k_1}}{k_1!}\frac{(-2\pi n_2)^{k_2}}{k_2!} M^{-k_1-k_2}$$
$$\times \int_0^\infty\int_0^\infty W_{\bar\lambda_1,\bar\lambda_2}\left(\begin{bmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{bmatrix}\right) y_1^{2\nu_1+\nu_2+k_1-2} y_2^{\nu_1+2\nu_2+k_2-2}\frac{dy_1\, dy_2}{y_1 y_2}$$
$$= \frac{\overline{A(n_1, n_2)}\, \pi^{2-3\nu_1-3\nu_2}}{4n_1^{2\nu_1+\nu_2-1} n_2^{\nu_1+2\nu_2-1}} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(-2n_1)^{k_1}}{k_1!}\frac{(-2n_2)^{k_2}}{k_2!\, M^{k_1+k_2}}$$
$$\times G(2\nu_1+\nu_2+k_1-1,\ \nu_1+2\nu_2+k_2-1;\ \bar\lambda_1, \bar\lambda_2).$$

Here, we have used the double Mellin transform [1, p. 161]

$$(6.17)\quad \int_0^\infty\int_0^\infty W_{\lambda_1,\lambda_2}\left(\begin{bmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{bmatrix}\right) y_1^{s_1-1} y_2^{s_2-1}\frac{dy_1\, dy_2}{y_1 y_2} = \frac{1}{4}\pi^{-s_1-s_2} G(s_1, s_2; \lambda_1, \lambda_2).$$

Since the gamma function has simple poles at the non-positive integers, we obtain from (6.13) and (6.17) the following proposition:

PROPOSITION 6.1. *Let φ be a cuspidal automorphic wave form for Γ of type (λ_1, λ_2). Let $P_{n_1,n_2}(\tau; \nu_1, \nu_2; M)$ be the Poincaré series (6.10). Then the inner product $\langle P_{n_1,n_2}, \varphi\rangle$ can be meromorphically continued in ν_1, ν_2 and has polar divisors at the lines*

$$2\nu_1+\nu_2-1 = \begin{cases} \bar\lambda_1+2\bar\lambda_2-1-N, \\ \bar\lambda_1-\bar\lambda_2-N, \\ 1-2\bar\lambda_1-\bar\lambda_2-N, \end{cases}$$

$$\nu_1+2\nu_2-1=\begin{cases}1-\bar\lambda_1-2\bar\lambda_2-N,\\ \bar\lambda_2-\bar\lambda_1-N,\\ 2\bar\lambda_1+\bar\lambda_2-1-N,\end{cases}$$

where N is any nonnegative even integer.

We now diagonalize the Poincaré series by setting $\nu_1=\nu_2$, and for simplicity we define

$$P_{n_1,n_2}(\nu;\tau)=P_{n_1,n_2}(\nu,\nu;\tau). \tag{6.18}$$

Then $P_{n_1,n_2}(\nu;\tau)$ will have poles when

$$3\nu-1=\begin{cases}\pm\bar\alpha-N,\\ \pm\bar\beta-N,\\ \pm\bar\gamma-N,\end{cases} \tag{6.19}$$

where $\alpha=1-\lambda_1-2\lambda_2$, $\beta=\lambda_2-\lambda_1$, $\gamma=2\lambda_1+\lambda_2-1$, and N is any nonnegative integer. There will be double poles when any two of the expressions on the right-hand side of (6.19) are equal. Let us consider when this can happen. Since α, β, γ are permuted by the six reflections of the Weyl group, the condition

$$\alpha=\pm\beta \quad\text{or}\quad \alpha=\pm\gamma \quad\text{or}\quad \beta=\pm\gamma$$

can only occur (modulo reflections of the Weyl group) if

$$\lambda_1=\lambda_2 \quad\text{or}\quad \lambda_1=\bar\lambda_2 \quad\text{and}\quad \operatorname{Re}(\lambda_1)=\operatorname{Re}(\lambda_2)=\tfrac13.$$

In the special case $\lambda_1=\lambda_2=\frac13$, there will be a fifth order pole. We can now assert

Proposition 6.2. *If $\varphi(\tau)$ is an automorphic cuspidal wave form of type $(\frac13,\frac13)$ with Fourier expansion* (6.15), *then $\langle P_{n_1,n_2},\varphi\rangle$ with P_{n_1,n_2} given by* (6.18) *has a fifth order pole at $\nu=\frac13$. Moreover,*

$$\lim_{\nu\to 1/3}(\nu-\tfrac13)^5\langle P_{n_1,n_2},\varphi\rangle=\tfrac{16}{243}\overline{A(n_1,n_2)}.$$

Proposition 6.3. *If $\varphi(\tau)$ is an automorphic cuspidal wave form of type (λ,λ) with $\lambda\neq\frac13$ then $\langle P_{n_1,n_2},\varphi\rangle$ with P_{n_1,n_2} given by* (6.18) *has double poles at $\nu=\bar\lambda$, $\frac23-\bar\lambda$ and a simple pole at $\nu=\frac13$. Moreover*

$$\lim_{\nu\to1/3}(\nu-\tfrac13)\langle P_{n_1,n_2},\varphi\rangle=\tfrac13\overline{A(n_1,n_2)}\,\Gamma\left(\frac{1-3\bar\lambda}{2}\right)^2\Gamma\left(\frac{3\bar\lambda-1}{2}\right)^2,$$

$$\lim_{\nu\to\bar\lambda}(\nu-\bar\lambda)^2\langle P_{n_1,n_2},\varphi\rangle=\frac{\pi^{2-6\bar\lambda}}{9(n_1n_2)^{3\bar\lambda-1}}\overline{A(n_1,n_2)}\,\Gamma(3\bar\lambda-1)\,\Gamma\left(\frac{3\bar\lambda-1}{2}\right)^2,$$

$$\lim_{\nu\to2/3-\bar\lambda}\left(\nu-(\tfrac23-\bar\lambda)\right)^2\langle P_{n_1,n_2},\varphi\rangle=\frac{\pi^{6\bar\lambda-2}}{9(n_1n_2)^{1-3\bar\lambda}}\overline{A(n_1,n_2)}\,\Gamma(1-3\bar\lambda)\,\Gamma\left(\frac{1-3\bar\lambda}{2}\right)^2.$$

Proposition 6.4. *If $\varphi(\tau)$ is an automorphic cuspidal wave form of type $(\lambda,\bar\lambda)$ with $\lambda\neq\frac13$ and $\operatorname{Re}(\lambda)=\frac13$, then $\langle P_{n_1,n_2},\varphi\rangle$ with P_{n_1,n_2} given by* (6.18) *has double poles at $\nu=\frac13(2\bar\lambda+\lambda)$, $\frac13(2-\lambda-2\bar\lambda)$ and simple poles at $\nu=\frac13\left(1\pm(\lambda-\bar\lambda)\right)$. Moreover,*

$$\lim_{\nu\to(1\pm(\lambda-\bar\lambda))/3}\left(\nu-\tfrac13\left(1\pm(\lambda-\bar\lambda)\right)\right)\langle P_{n_1,n_2},\varphi\rangle$$
$$=\frac{\overline{A(n_1,n_2)}\,\pi^{\pm(\bar\lambda-\lambda)}}{6(n_1n_2)^{\pm(\lambda-\bar\lambda)}}\Gamma\left(\frac{\pm(\lambda-\bar\lambda)+(1-\lambda-2\bar\lambda)}{2}\right)^2\Gamma\left(\frac{\pm(\lambda-\bar\lambda)-(1-\lambda-2\bar\lambda)}{2}\right),$$

$$\lim_{\nu\to(\lambda+2\bar\lambda)/3}\left(\nu-\tfrac13(\lambda+2\bar\lambda)\right)^2\langle P_{n_1,n_2},\varphi\rangle$$
$$=\frac{\overline{A(n_1,n_2)}\,\pi^{2-2\lambda-4\bar\lambda}}{9(n_1n_2)^{\lambda+2\bar\lambda-1}}\Gamma\left(\frac{2\lambda+\bar\lambda-1}{2}\right)\Gamma\left(\frac{3\bar\lambda-1}{2}\right)\Gamma(2\bar\lambda+\lambda-1),$$

$$\lim_{\nu\to(2-\lambda-2\bar\lambda)/3}\left(\nu-\tfrac13(2-\lambda-2\bar\lambda)\right)^2\langle P_{n_1,n_2},\varphi\rangle$$
$$=\frac{\overline{A(n_1,n_2)}\,\pi^{-2+2\lambda+4\bar\lambda}}{9(n_1n_2)^{1-\lambda-2\bar\lambda}}\Gamma\left(\frac{1-\bar\lambda-2\lambda}{2}\right)\Gamma\left(\frac{1-3\bar\lambda}{2}\right)\Gamma(1-2\lambda-\bar\lambda).$$

7. Continuous spectrum. Let $E_j(\tau;s)$ be the maximal parabolic Eisenstein series (6.5) associated to the Maass wave form $u_j(z)$. If $u_j(z)$ has eigenvalue $\frac14+r_j^2$ for the $\mathrm{GL}(2,\mathbf{R})$ Laplacian $-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$, then $E_j(\tau;s)$ will be of the type $(s-\frac16-\frac13ir_j,\ \frac13+\frac23ir_j)$. Clearly, $\bar E_j(\tau;s)$ will then be of the type $(\frac13+\frac23ir_j,\ s-\frac16-\frac13ir_j)$. Consequently, it will be sufficient to consider only $E_j(\tau;s)$ since a simple reflection will yield the corresponding results for $\bar E_j(\tau;s)$.

We now evaluate the inner product $\langle P_{n_1,n_2},E_j\rangle$ occurring on the right-hand side of the spectral decomposition of P_{n_1,n_2} as in (6.9). For $n_1>0$, $n_2>0$, let

$$\int_0^1\int_0^1\int_0^1\bar E_j\left(\begin{bmatrix}1&\xi_2&\xi_3\\ &1&\xi_1\\ &&1\end{bmatrix}\tau;s\right)e(n_1\xi_1+n_2\xi_2)\,d\xi_1\,d\xi_2\,d\xi_3 \tag{7.1}$$
$$=e_j(n_1,n_2;s)\,W_{\bar\mu_1,\bar\mu_2}\left(\begin{bmatrix}n_1n_2&&\\ &n_1&\\ &&1\end{bmatrix}\tau\right)$$

where $\mu_1=s-\frac16-\frac13ir_j$, $\mu_2=\frac13+\frac23ir_j$. Here $e_j(n_1,n_2;s)$ is the n_1th, n_2th Fourier coefficient of $E_j(\tau;s)$. The proof that a representation of type (7.1) holds is almost identical to the proof of (6.14). It now follows from

Proposition 2.1, (6.17) and (7.1) that

$$(7.2)\qquad \langle P_{n_1,n_2}, E_j\rangle = e_j(n_1, n_2; s)\int_0^\infty\int_0^\infty W_{\bar\mu_1,\bar\mu_2}\left(\begin{bmatrix} n_1 n_2 y_1 y_2 & & \\ & n_1 y_1 & \\ & & 1\end{bmatrix}\right)$$

$$\times y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} e^{-2\pi\frac{(n_1y_1+n_2y_2)}{M}}\frac{dy_1\,dy_2}{(y_1y_2)^3}$$

$$= \frac{e_j(n_1, n_2; s)\,\pi^{2-3\nu_1-3\nu_2}}{4n_1^{2\nu_1+\nu_2-2} n_2^{\nu_1+2\nu_2-2}}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty \frac{(-2n_1)^{k_1}}{k_1!}\,\frac{(-2n_2)^{k_2}}{k_2!\,M^{k_1+k_2}}$$

$$\times G(2\nu_1+\nu_2+k_1-1,\ \nu_1+2\nu_2+k_2-1;\ \bar\mu_1, \bar\mu_2).$$

Consequently,

$$(7.3)\qquad \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=1/2}\langle P_{n_1,n_2}, E_j(*; s)\rangle E_j(\tau; s)\,ds$$

$$= \frac{\pi^{2-3\nu_1-3\nu_2}}{4n_1^{2\nu_1+\nu_2-2} n_2^{\nu_1+2\nu_2-2}}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty \frac{(-2n_1)^{k_1}}{k_1!}\,\frac{(-2n_2)^{k_2}}{k_2!}$$

$$\times\frac{M^{-k_1-k_2}}{4\pi i}\int_{\mathrm{Re}(s)=1/2} e_j(n_1, n_2; s)$$

$$\times G(2\nu_1+\nu_2+k_1-1,\ \nu_1+2\nu_2+k_2-1;\ \bar\mu_1, \bar\mu_2)\,E_j(\tau; s)\,ds.$$

Equation (7.3) gives the meromorphic continuation in ν_1, ν_2 of the left-hand side of (7.3). Let us now examine where the possible poles can occur. Since $e_j(n_1, n_2; s)$ and $E_j(\tau; s)$ are holomorphic on the line $\mathrm{Re}(s) = 1/2$, the integral on the right-hand side of (7.3) converges absolutely and uniformly to a holomorphic function of ν_1, ν_2 as long as $G = G(2\nu_1+\nu_2+k_1-1,\ \nu_1+2\nu_2+k_2-1;\ \bar\mu_1, \bar\mu_2)$ does not have a singularity on that line. We recall that

$$(7.4)\qquad G = \frac{\Gamma\left(\frac{w_1+\frac12-\bar s+ir_j}{2}\right)\Gamma\left(\frac{w_1+\frac12-\bar s-ir_j}{2}\right)\Gamma\left(\frac{w_1+2\bar s-1}{2}\right)}{\Gamma\left(\frac{w_1+w_2}{2}\right)}\times$$

$$\times\Gamma\left(\frac{w_2-\frac12+\bar s-ir_j}{2}\right)\Gamma\left(\frac{w_2-\frac12+\bar s+ir_j}{2}\right)\Gamma\left(\frac{w_2-2\bar s+1}{2}\right)$$

where

$$(7.5)\qquad w_1 = 2\nu_1+\nu_2+k_1-1,\qquad w_2 = \nu_1+2\nu_2+k_2-1.$$

Hence, singularities can only occur if $\mathrm{Re}(w_1) = 0$ or $\mathrm{Re}(w_2) = 0$. In fact, (7.3) shows that the left-hand side of (7.3) is holomorphic in ν_1, ν_2 if $\mathrm{Re}(w_1) > 0$, $\mathrm{Re}(w_2) > 0$. To meromorphically continue the left-hand side of (7.3), we first assume $0 < \mathrm{Re}(w_1) < \varepsilon$, $0 < \mathrm{Re}(w_2) < \varepsilon$ for some $\varepsilon > 0$ sufficiently small. By shifting the s-integral to the right, we pick up residues when

$$\bar s = \tfrac12+w_1\pm ir_j,\quad \tfrac12-w_2\pm ir_j,\quad \tfrac12-\tfrac12 w_1,\quad \tfrac12+\tfrac12 w_2.$$

Here $\bar s = 1-s$. Let us adopt the convention

$$R^\pm = R^+ + R^-.$$

If we define

$$(7.6)\qquad I_j(\tau; \nu_1, \nu_2; n_1, n_2)$$

$$= \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=1/2}\langle P_{n_1,n_2}(*; \nu_1, \nu_2, M), E_j(*; s)\rangle E_j(\tau; s)\,ds$$

then it follows from (7.3), (7.4) and the above remarks that for $r_j \neq 0$ and $0 < \mathrm{Re}(w_1) < \varepsilon$, $0 < \mathrm{Re}(w_2) < \varepsilon$

$$(7.7)\qquad I_j(\tau; \nu_1, \nu_2; n_1, n_2) = \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=1/2+2\varepsilon}\langle P_{n_1,n_2}, E_j\rangle E_j(\tau; s)\,ds$$

$$-\frac{\pi^{2-3\nu_1-3\nu_2}}{4n_1^{2\nu_1+\nu_2-2} n_2^{\nu_1+2\nu_2-2}}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty \frac{(-2n_1)^{k_1}}{k_1!}\,\frac{(-2n_2)^{k_2}}{k_2!\,M^{k_1+k_2}}(R_1^\pm+R_2^\pm+R_3+R_4)$$

where

(7.8)

$$R_1^\pm = -2\Gamma(\mp ir_j)\,\Gamma\left(\frac{3w_1\pm 2ir_j}{2}\right)\Gamma\left(\frac{w_1+w_2\pm 2ir_j}{2}\right)\Gamma\left(\frac{w_2-2w_1\mp 2ir_j}{2}\right)$$

$$\times e_j(n_1, n_2; \tfrac12+\bar w_1\mp ir_j)\,E_j(\tau; \tfrac12+\bar w_1\mp ir_j),$$

$$R_2^\pm = 2\Gamma(\pm ir_j)\,\Gamma\left(\frac{3w_2\mp 2ir_j}{2}\right)\Gamma\left(\frac{w_1+w_2\mp 2ir_j}{2}\right)\Gamma\left(\frac{w_1-2w_2\pm 2ir_j}{2}\right)$$

$$\times e_j(n_1, n_2, \tfrac12-\bar w_2\mp ir_j)\,E_j(\tau; \tfrac12-\bar w_2\mp ir_j),$$

$$R_3 = \Gamma\left(\frac{\frac32 w_1+ir_j}{2}\right)\Gamma\left(\frac{\frac32 w_1-ir_j}{2}\right)\Gamma\left(\frac{w_2-\frac12 w_1-ir_j}{2}\right)\Gamma\left(\frac{w_2-\frac12 w_1+ir_j}{2}\right)$$

$$\times e_j(n_1, n_2; \tfrac12(1-\bar w_1))\,E_j(\tau; \tfrac12(1-\bar w_1)),$$

$$R_4 = -\Gamma\left(\frac{\frac32 w_2+ir_j}{2}\right)\Gamma\left(\frac{\frac32 w_2-ir_j}{2}\right)\Gamma\left(\frac{w_1-\frac12 w_2-ir_j}{2}\right)\Gamma\left(\frac{w_1-\frac12 w_2+ir_j}{2}\right)$$

$$\times e_j(n_1, n_2; \tfrac12(1+\bar w_2))\,E_j(\tau; \tfrac12(1+\bar w_2)),$$

$$w_1 = 2\nu_1+\nu_2+k_1-1,\qquad w_2 = \nu_1+2\nu_2+k_2-1.$$

Consequently (7.7) gives the meromorphic continuation of $I_j(\tau; \nu_1, \nu_2; n_1, n_2)$ to $-\frac23 < \mathrm{Re}(w_1)$, $-\frac23 < \mathrm{Re}(w_2)$. It is easily seen from (7.8) that all residues cancel in this region unless ir_j is real. Using the bound $\frac14+r_j^2 > \frac{3}{16}$ (see Selberg [18]) we obtain

PROPOSITION 7.1. *The integral given by* (7.6) *is holomorphic in the region* $\mathrm{Re}(2\nu_1+\nu_2) > \frac{1}{2}$, $\mathrm{Re}(\nu_1+2\nu_2) > \frac{1}{2}$.

Remark. This clearly holds for $j = 0$.

Finally, we consider the contribution of the spectral decomposition of $P_{n_1,n_2}(\tau; \nu_1, \nu_2)$ coming from the minimal parabolic Eisenstein series $E(\tau; s_1, s_2)$. We define

(7.9) $\quad I(\tau; \nu_1, \nu_2; n_1, n_2)$

$$= \frac{1}{(2\pi i)^2} \int\limits_{\mathrm{Re}(s_1)=1/3} \int\limits_{\mathrm{Re}(s_2)=1/3} \langle P_{n_1,n_2}(*; \nu_1, \nu_2), E(*; s_1, s_2)\rangle E(\tau; s_1, s_2)\, ds_1\, ds_2.$$

For $n_1, n_2 \neq 0$, let

$$\int_0^1\int_0^1\int_0^1 \overline{E\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{bmatrix}\tau; s_1, s_2\right)} e(n_1\xi_1 + n_2\xi_2)\, d\xi_1\, d\xi_2\, d\xi_3$$

$$= e_{n_1,n_2}(s_1, s_2)\, W_{\bar{s}_1,\bar{s}_2}\left(\begin{bmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{bmatrix}\tau\right).$$

It then follows as before that

$\langle P_{n_1,n_2}, E\rangle$

$$= \frac{e_{n_1,n_2}(s_1, s_2)\,\pi^{2-3\nu_1-3\nu_2}}{4n_1^{2\nu_1+\nu_2-2}\, n_2^{\nu_1+2\nu_2-2}} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(-2n_1)^{k_1}}{k_1!}\frac{(-2n_2)^{k_2}}{k_2!} G(w_1, w_2; \bar{s}_1, \bar{s}_2),$$

$$w_1 = 2\nu_1+\nu_2+k_1-1, \quad w_2 = \nu_1+2\nu_2+k_2-1.$$

Consequently,

(7.10) $\quad I(\tau; \nu_1, \nu_2; n_1, n_2)$

$$= \frac{\pi^{2-3\nu_1-3\nu_2}}{4n_1^{2\nu_1+\nu_2-2}\, n_2^{\nu_1+2\nu_2-2}} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(-2n_1)^{k_1}}{k_1!}\frac{(-2n_2)^{k_2}}{k_2!}\frac{1}{(2\pi i)^2}$$

$$\times \int\limits_{\mathrm{Re}(s_1)=1/3}\int\limits_{\mathrm{Re}(s_2)=1/3} e_{n_1,n_2}(s_1, s_2)\, G(w_1, w_2; \bar{s}_1, \bar{s}_2)\, E(\tau; s_1, s_2)\, ds_1\, ds_2.$$

Recall that

$G(w_1, w_2; \bar{s}_1, \bar{s}_2)$

$$\Gamma\left(\frac{w_1+\alpha}{2}\right)\Gamma\left(\frac{w_1+\beta}{2}\right)\Gamma\left(\frac{w_1+\gamma}{2}\right)\Gamma\left(\frac{w_2-\alpha}{2}\right)\Gamma\left(\frac{w_2-\beta}{2}\right)\Gamma\left(\frac{w_2-\gamma}{2}\right)$$

$$\Gamma\left(\frac{w_1+w_2}{2}\right)$$

$$\alpha = 1-\bar{s}_1-2\bar{s}_2, \quad \beta = \bar{s}_2-\bar{s}_1, \quad \gamma = 2\bar{s}_1+\bar{s}_2-1.$$

Since $\mathrm{Re}(s_1) = \frac{1}{3}$, $\mathrm{Re}(s_2) = \frac{1}{3}$, we see that $G(w_1, w_2; \bar{s}_1, \bar{s}_2)$ can have poles only if $\mathrm{Re}(w_1) \leqslant 0$ or $\mathrm{Re}(w_2) \leqslant 0$. By shifting the contours on the right-hand side of equation (7.10) we see that in the region $-2/3 < \mathrm{Re}(w_1) \leqslant 0$, $-2/3 < \mathrm{Re}(w_2) \leqslant 0$ the function $I(\tau; \nu_1, \nu_2; n_1, n_2)$ can have polar divisors only when $w_1 = 0$ or $w_2 = 0$. Consequently, we have

PROPOSITION 7.2. *The integral given by* (7.9) *is holomorphic in the region* $\mathrm{Re}(2\nu_1+\nu_2) > \frac{1}{3}$, $\mathrm{Re}(\nu_1+2\nu_2) > \frac{1}{3}$ *except for possible polar divisors at the lines*

$$2\nu_1+\nu_2 = 1, \quad \nu_1+2\nu_2 = 1.$$

8. The Poincaré series associated with a Bruhat cell. In this section, we denote

$$w_0 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix},$$

$$w_1 = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \quad w_4 = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix},$$

$$w_2 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \quad w_5 = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

We have conjectured that the zeta functions formed with the Kloosterman sums arising from the w_4 and w_5 Bruhat cells have poles corresponding to the cusp forms occurring in the spectral decomposition of $\mathscr{L}^2(\Gamma\backslash H^3)$. As we have seen, this conjecture has important implications for automorphic forms. The difficulty in supplying a proof is that the contributions of all Bruhat cells occur together in the Fourier expansions, seemingly inextricable.

One may investigate the Bruhat decomposition from the view-point of Minkowski reduction theory. Based on these ideas, we shall exhibit a modified Poincaré series which in a weak sense isolates the contribution of the w_4 cell. We shall show that if any cusp form of type (λ_1, λ_2) occurs in the spectral decomposition then this modified Poincaré series has poles among the three lines

$$2\nu_1+\nu_2-1 = \begin{cases} \bar{\lambda}_1+2\bar{\lambda}_2-1-N, \\ \bar{\lambda}_1-\bar{\lambda}_2-N, \\ 1-2\bar{\lambda}_1-\bar{\lambda}_2-N \end{cases}$$

for any nonnegative even integer N; in other words, the first of the two sets of lines in Proposition 6.1. We hold this in evidence that the Kloosterman zeta function

$$\sum_{D_1, D_2 = 1}^{\infty} S_{w_4}(m_1, m_2, n_1, n_2; D_1, D_2) D_1^{-3\nu_1} D_2^{-3\nu_2}$$

should have poles along the same lines.

Let Ω be a region in $\mathbf{R}^+ \times \mathbf{R}^+$. Form the E-function

$$E_{n_1,n_2}^{\Omega}(\tau) = \begin{cases} e(n_1 z_1 + n_2 z_2) & \text{if} \quad (y_1, y_2) \in \Omega, \\ 0 & \text{if} \quad (y_1, y_2) \notin \Omega. \end{cases}$$

Let $P_{n_1,n_2}^{\Omega}(\tau; \nu_1, \nu_2)$ be the Poincaré series formed with this E-function (cf. (2.6)). This series is convergent for $\mathrm{Re}(\nu_1)$, $\mathrm{Re}(\nu_2) > 2/3$. This function is square-integrable but discontinuous.

Let A_w $(w \in W)$ be the regions given in the following table:

w	A_w
$w_0 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	$\{y_1, y_2 \geqslant 1\}$
$w_1 = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$	$\{y_1, y_2 \leqslant 1\}$
$w_2 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$	$\{y_1 y_2 \geqslant 1,\ y_2 \leqslant 1\}$
$w_3 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$	$\{y_1 y_2 \geqslant 1,\ y_1 \leqslant 1\}$
$w_4 = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$	$\{y_1 y_2 \leqslant 1,\ y_2 \geqslant 1\}$
$w_5 = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$	$\{y_1 y_2 \leqslant 1,\ y_1 \geqslant 1\}$

Our philosophy is that if $\Omega = A_{w_i}$, then P_{n_1,n_2}^{Ω} will in some sense isolate the contribution of the w_i cell in the Bruhat decomposition. To make this precise, we consider the Fourier expansion. As in (5.8) the contribution of the w_j cell is

$$\sum_{\substack{\gamma \in \Gamma_\infty \backslash G_w \cap \Gamma / U \Gamma_w \\ \gamma = b_1 w d b_2}} e_{n_1,n_2}(b_1) \int_{\mathcal{D}'_{w_j}} I_{\nu_1,\nu_2} E_{n_1,n_2}\left(w d b_2 \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ & y_1 & \xi_1 \\ & & 1 \end{bmatrix} \right) \times e(-m_1 \xi_1 - m_2 \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3$$

where now (with D_1, D_2 fixed)

$$\mathcal{D}'_{w_j} = \xi(\xi_3, \xi_2, \xi_1) \in \{\mathcal{D}_{w_j} |\ (y'_1, y'_2) \in \Omega\},$$

y'_1 and y'_2 being given in Table (5.12). Now, we assert that as $y_1, y_2 \to \infty$, $\mathcal{D}'_{w_j}$ will expand to become all of $\mathcal{D}_{w_j}$ if $j = i$; otherwise, it will shrink away. Indeed we have

PROPOSITION 8.1. *Let $\Omega = A_{w_i}$, and, for fixed w_j, fixed D_1, D_2, let $\mathcal{D}'_{w_j}$ be as above. Let K be compact. If y_1, y_2 are sufficiently large, then*

$$K \subseteq \mathcal{D}'_{w_j} \quad \text{if} \quad j = i;$$
$$K \cap \mathcal{D}'_{w_j} = \emptyset \quad \text{if} \quad j \neq i.$$

Thus the contribution of the single Bruhat cell will predominate.

Actually, since we are mainly concerned with separating the w_4 contributions from those of w_1 and w_5, we may just as easily take $\Omega = \{y_2 \geqslant 1\} = A_{w_4} \cup A_{w_0} \cup A_{w_3}$. This region is easier to deal with since it is only defined by one inequality. We shall show that P_{n_1,n_2}^{Ω} has poles corresponding to cusp forms in $\mathcal{L}^2(\Gamma \backslash H^3)$.

THEOREM 8.2. *Let φ be a cuspidal automorphic wave form for Γ of type (ν_1, ν_2). Let $\Omega = \{y_2 \geqslant 1\}$. The inner product $\langle P_{n_1,n_2}^{\Omega}, \varphi \rangle$ can be meromorphically continued in (ν_1, ν_2) and has polar divisors at the lines*

$$2\nu_1 + \nu_2 - 1 = \begin{cases} \bar{\lambda}_1 + 2\bar{\lambda}_2 - 1 - 2N, \\ \bar{\lambda}_1 - \bar{\lambda}_2 - 2N, \\ 1 - 2\bar{\lambda}_1 - \bar{\lambda}_2 - 2N \end{cases}$$

where N is any nonnegative even integer.

Proof. The same as Proposition 6.1, except that instead of the integral (6.17) we have the Mellin transform restricted to the region Ω. Indeed, if $\alpha = 1 - \lambda_1 - 2\lambda_2$, $\beta = \lambda_2 - \lambda_1$, $\gamma = 2\lambda_1 + \lambda_2 - 1$ then

$$\int_1^\infty \int_0^\infty W_{\lambda_1,\lambda_2}\left(\begin{bmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{bmatrix} \right) y_1^{\mu_1 - 1} y_2^{\mu_2 - 1} \frac{dy_1}{y_1} \frac{dy_2}{y_2}$$

$$= \frac{1}{4} \Gamma\left(\frac{\mu_1+\alpha}{2}\right) \Gamma\left(\frac{\mu_1+\beta}{2}\right) \Gamma\left(\frac{\mu_1+\gamma}{2}\right)$$
$$\times \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s-\beta}{2}\right) \Gamma\left(\frac{s-\gamma}{2}\right) \Gamma\left(\frac{\mu_1+s}{2}\right)^{-1} (s-\mu_2)^{-1} \pi^{2-\mu_1-s} ds.$$

This may be deduced from (6.12) by Mellin inversion. The poles may be read off by taking $\mu_1 = 2\nu_1+\nu_2-1$, $\mu_2 = 1-\nu_1-2\nu_2$. Note that the Mellin-Barnes integral on the right is entire.

Appendix

Estimation of SL(3, Z) Kloosterman sums by Michael Larsen

Let $S(m_1, n_1, n_2; D_1, D_2)$ be the $SL(3, Z)$ Kloosterman sum (4.3). We have the following estimate.

THEOREM 1.

$$|S(m_1, n_1, n_2; D_1, D_2)| \leqslant \min\left(\tau(D_1)^{\varkappa} (n_2, D_2/D_1) D_1^2,\ \tau(D_2)(m_1, n_1, D_1) D_2\right)$$

where

$$\varkappa = \frac{\log 3}{\log 2} \quad \textit{and} \quad \tau(n) = \sum_{\substack{d|n \\ d \geqslant 1}} 1.$$

Proof. By Property 4.15, both sides of the inequality are multiplicative, so it suffices to consider the case $D_1 = p^a$, $D_2 = p^b$, $a \leqslant b$ and p prime. Since

$$S(m_1, n_1, n_2; D_1, D_2) = p^{2k+l} S(m_1 p^{-k}, n_1 p^{-k}, n_2 p^{-l}; D_1 p^{-k}, D_2 p^{-k-l})$$

whenever

$$p^k | (m_1, n_1, D_1) \quad \text{and} \quad p^l | (n_2, D_2/D_1)$$

we may assume $D_1 = 1$, $D_1 = D_2$ or $p \nmid (m_1, n_1) n_2$.

The first possibility is trivial; the second is covered by Property 4.17. For the third, we introduce the notation

$$\mathscr{S}(k) = \{0 \leqslant n \leqslant p^k,\ n \in \mathbf{Z} |\ (n, p) = 1\}.$$

If $p|n_1$ and $a > 1$, we have

$$S(m_1, n_1, n_2; p^a, p^b) = \sum_{\substack{C_1 \in \mathscr{S}(a-1) \\ C_2 \in \mathscr{S}(b)}} \sum_{k=0}^{p-1} e\left(\frac{m_1(C_1+kp^{a-1})+n_1 \bar{C}_1 C_2}{p^a} + \frac{n_2 \bar{C}_2}{p^{b-a}}\right)$$

where $C_1 \bar{C}_1 \equiv 1 \pmod{p^{a-1}}$, $C_2 \bar{C}_2 \equiv 1 \pmod{p^b}$. This equals

$$p \sum_{k=0}^{p-1} e\left(\frac{m_1 k}{p}\right) S(m_1, n_1/p, n_2; p^{a-1}, p^{b-1}) = 0,$$

since $p \nmid m_1$. When $a = 1$, one sees similarly that the sum is either 0 or p.

Likewise, if $p|m_1$, the sum $S(m_1, n_1, n_2; p^a, p^b)$ is equal to 0 or p.

We can therefore assume, without loss of generality, that $p \nmid m_1 n_1 n_2$, and thus by Property 4.16 that $D_1 = p^a$, $D_2 = p^{2a}$.

When $a = 1$, since $e((m_1 C_1 + n_1 \bar{C}_1 C_2 + n_2 \bar{C}_2)/p)$ depends only on the image of C_2 under the map $\mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$, it follows that

$$S(m_1, n_1, n_2; p, p^2) = p \sum_{C_1, C_2 \in \mathscr{S}(1)} e\left(\frac{m_1 C_1 + n_1 \bar{C}_1 C_2 + n_2 \bar{C}_2}{p}\right),$$

where $C_1 \bar{C}_1 \equiv C_2 \bar{C}_2 \equiv 1 \pmod{p}$.

Now, define $x = m_1 C_1$, $y = n_1 \bar{C}_1 C_2$, $z = n_2 \bar{C}_2$ and more generally

$$z(x, y) = m_1 n_1 n_2 x^{-1} y^{-1} \pmod{p^a}.$$

In this notation, the above sum is

$$p \sum_{\substack{x,y,z \in F_p \\ xyz = m_1 n_1 n_2}} e\left(\frac{x+y+z}{p}\right)$$

which by a theorem of Deligne [3] is of magnitude no greater than $3p^2$. Since

$$3p^2 = \tau(p)^{\varkappa} D_1^2 = \tau(p^2) D_2$$

we have the theorem in this case.

More generally,

$$S = S(m_1, n_1, n_2; p^a, p^{2a}) = p^a \sum_{x,y \in \mathscr{S}(a)} e\left(\frac{x+y+z(x, y)}{p^a}\right).$$

To compute this for $a > 1$, we use the congruence

$$(x+rp^k)^{-1} \equiv x^{-1} - rx^{-2} p^k + r^2 x^{-3} p^{2k} \pmod{p^{3k}}$$

and its consequence

$$\text{(A.1)} \qquad z(x+rp^k, y+sp^k)$$
$$\equiv z(x, y)\left(1-(rx^{-1}+sy^{-1}) p^k + (r^2 x^{-2} + rsx^{-1} y^{-1} + s^2 y^{-2}) p^{2k}\right) \pmod{p^{3k}}.$$

When $a = 2k$,

$$S = p^a \sum_{x,y \in \mathscr{S}(k)} \sum_{r=0}^{p^k-1} \sum_{s=0}^{p^k-1} e\left(\frac{x+rp^k+y+sp^k+z(x, y)-z(x, y)(rx^{-1}+sy^{-1}) p^k}{p^{2k}}\right)$$

$$= p^{2a} \sum_{\substack{x,y \in \mathscr{S}(k) \\ x \equiv y \equiv z(x,y) \pmod{p^k}}} e\left(\frac{x+y+z(x, y)}{p^{2k}}\right).$$

The last summation has only one term for each solution to

$$x^3 \equiv m_1 n_1 n_2 \pmod{p^k},$$

an equation which has no more than 3 roots for any pair (p, k). This fact follows from Hensel's lemma when $p \neq 3$ and the fact that $\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ is never a subgroup of $(\mathbf{Z}/3^k \mathbf{Z})^\times$. Consequently

$$|S| \leqslant 3p^{2a} < \min\left(\tau(D_1)^\varkappa D_1^2,\ \tau(D_2) D_2\right).$$

In the remaining case $a = 2k+1$,

$$S = p^a \sum_{\substack{x,y \in \mathcal{S}(k+1) \\ z = z(x,y)}} e\left(\frac{x+y+z}{p^a}\right) \sum_{r=0}^{p^k-1} e\left(\frac{r(1-zx^{-1})}{p^k}\right) \sum_{s=0}^{p^k-1} e\left(\frac{s(1-zy^{-1})}{p^k}\right)$$

$$= p^{2a-1} \sum_{\substack{x,y \in \mathcal{S}(k+1) \\ x \equiv y \equiv z \pmod{p^k}}} e\left(\frac{x+y+z}{p^a}\right)$$

$$= p^{2a-1} \sum_{\substack{x = y \in \mathcal{S}(k) \\ x \equiv z(x,x) \pmod{p^k}}} e\left(\frac{x+y+z}{p^a}\right) \times \sum_{r=0}^{p-1} \sum_{s=0}^{p-1} e\left(\frac{rp^k + sp^k + z(x+rp^k,\ y+sp^k) - z}{p^a}\right)$$

$$= p^{2a-1} \sum_{\substack{x = y \in \mathcal{S}(k) \\ x \equiv z(x,x) \pmod{p^k}}} e\left(\frac{x+y+z}{p^a}\right) \sum_{r=0}^{p-1} \sum_{s=0}^{p-1} e\left(\frac{(1-zx^{-1})r + (1-zy^{-1})s}{p^{k+1}}\right) e\left(\frac{x^{-1}(r^2+rs+s^2)}{p}\right)$$

the last equation coming from (A.1).

Since $x = y \equiv z \pmod{p^k}$ the inner sum above can be rewritten

$$\textstyle\sum_{\mathrm{I}} = \displaystyle\sum_{r=0}^{p-1} \sum_{s=0}^{p-1} e\left(\frac{x^{-1}(r^2+rs+s^2-Ar-As)}{p}\right)$$

for some A.

If $p \nmid 6$, setting $u = (r+s)/2 - A/3$, $v = (r-s)/2$, this becomes

$$\textstyle\sum_{\mathrm{I}} = \displaystyle e\left(\frac{-x^{-1} 3^{-1} A^2}{p}\right) \sum_{u=0}^{p-1} e\left(\frac{3x^{-1} u^2}{p}\right) \sum_{v=0}^{p-1} e\left(\frac{x^{-1} v^2}{p}\right).$$

This is a product of two Gauss sums and a root of unity, and is, therefore, of magnitude p.

For $p \mid 6$, we verify by cases that

$$\left|\textstyle\sum_{\mathrm{I}}\right| \leqslant \begin{cases} 2 & \text{if } p = 2, \\ 3\sqrt{3} & \text{if } p = 3. \end{cases}$$

Thus, in all cases,

$$|S| \leqslant 3p^{2a-1} \left|\textstyle\sum_{\mathrm{I}}\right| \leqslant 3\sqrt{3}\, p^{2a} < \min\left(\tau(D_1)^\varkappa D_1^2,\ \tau(D_2) D_2\right).$$

This completes the proof of the theorem.

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