An extension of a result of C. J. Smyth to polynomials in several variables

by

A. BAZYLEWICZ (Warszawa)

H. L. Montgomery and A. Schinzel in their paper [2] have asked whether for every positive integer d there exists a number C(d) < 1 with the following property:

Every polynomial $F(z_1, ..., z_n)$ with integral coefficients of the total degree d such that F(0, ..., 0) = 1 has a zero $(\varrho_1, ..., \varrho_n)$ satisfying $\max_{1 \le i \le n} |\varrho_i| \le C(d)$.

We present here a solution of this problem.

By analogy with polynomials in one variable, a polynomial $F \in C[z_1, ..., z_n]$ is called *reciprocal* if

$$F(z_1, ..., z_n) = \text{const } z_1^{d_1} ... z_n^{d_n} F(z_1^{-1}, ..., z_n^{-1})$$

where d_i is the degree of F with respect to z_i .

Our principal result is the following:

THEOREM 1. Let $F \in \mathbb{Z}[z_1, ..., z_n]$ be a non-reciprocal polynomial such that F(0, ..., 0) = 1.

Then F has a zero $(\varrho_1, \ldots, \varrho_n)$ satisfying

$$\max_{1 \le i \le n} |\varrho_i| \le \theta_0^{-1/d}$$

where d is the total degree of F and θ_0 is the least Pisot-Vijayaraghavan number.

For n=1 this result has been obtained by C. J. Smyth [4] as a corollary to his theorem on the Mahler measure.

For the proof of the theorem we need another definition.

DEFINITION. A polynomial $F \in C[z_1, ..., z_n]$ of degree d_i with respect to z_i is called self-inversive if there exists a $c \in C$ such that

$$z_1^{d_1} \dots z_n^{d_n} \overline{F}(z_1^{-1}, \dots, z_n^{-1}) = cF$$

where the bar denotes the complex conjugation.

Clearly, for $F \in R[z_1, ..., z_n]$ the notions of reciprocal and self-inversive polynomials coincide.

The proof is based on three lemmata.

LEMMA 1. Let $G \in C[z_1, ..., z_n]$ be of degree d_i with respect to z_i , $1 \le i \le n$, $E \subset C$, $card E > max d_i$. If for all $(\varrho_1, ..., \varrho_n) \in E^n$ we have $G(\varrho_1, ..., \varrho_n) = 0$, then G = 0.

Proof. See Lemma 8 in [3].

LEMMA 2. Let E be the set of all primitive roots of unity of order not divisible by 3. If $F \in C[z_1, ..., z_n]$, F(0, ..., 0] = 1 and for all $\varepsilon \in E^n$ the polynomial $F_n(z) = F(\varepsilon_1 z, ..., \varepsilon_n z)$ is self-inversive then also $F(z_1, ..., z_n)$ is self-inversive.

Proof. Let F be of total degree d and of degree d_i with respect to z_i . We have

(1)
$$F_{\varepsilon}(z) = \sum_{i=0}^{d} z^{i} F_{i}(\varepsilon_{1}, \ldots, \varepsilon_{n})$$

where F_i is the sum of all terms of degree i in F.

The assumption that $F_s(z)$ is self-inversive means that

$$z^d \bar{F}_{\varepsilon}(z^{-1}) = c(\varepsilon) F_{\varepsilon}(z), \quad \text{where} \quad c(\varepsilon) \in C, \text{ if } F_d(\varepsilon_1, \ldots, \varepsilon_n) \neq 0.$$

This condition together with (1) implies that

for all $\varepsilon \in E^n$. The polynomials

$$F_d(z_1, \ldots, z_n)$$

$$\times \big(\big(\prod_{i=1}^{n} z_{i}^{d_{i}} \big) \bar{F}_{d-j}(z_{1}^{-1}, \ldots, z_{n}^{-1}) - \big(\prod_{i=1}^{n} z_{i}^{d_{i}} \big) \bar{F}_{d}(z_{1}^{-1}, \ldots, z_{n}^{-1}) F_{j}(z_{1}, \ldots, z_{n}) \big)$$

vanish for j = 0, 1, ..., d in all points $s \in E^n$.

In virtue of Lemma 1 we get

$$\left(\prod_{i=1}^n z_i^{d_i}\right) \bar{F}_{d-j}(z_1^{-1}, \ldots, z_n^{-1}) = \left(\prod_{i=1}^n z_i^{d_i}\right) \bar{F}_{d}(z_1^{-1}, \ldots, z_n^{-1}) F_{j}(z_1, \ldots, z_n),$$

because $F_d(z_1, \ldots, z_n) \neq 0$, hence, by (1),

(2)
$$(\prod_{i=1}^n z_i^{d_i}) \bar{F}(z_1^{-1}, \ldots, z_n^{-1}) = (\prod_{i=1}^n z_i^{d_i}) \bar{F}_d(z_1^{-1}, \ldots, z_n^{-1}) F(z_1, \ldots, z_n).$$

The polynomial on the left-hand side is with respect to z_i of degree at most d_i , i.e., the degree of F, hence

$$\left(\prod_{i=1}^{n} z_{i}^{d_{i}}\right) \overline{F}_{d}\left(z_{1}^{-1}, \ldots, z_{n}^{-1}\right) \in C$$

and (2) shows that F is self-inversive.

LEMMA 3. Let K be a complex multiplication field, $\sqrt{3} \notin K$ and let Σ be the set of all injections of K into C. Let P be a monic non self-inversive polynomial with coefficients being integers of K and $P(0) \neq 0$.

If for
$$\sigma \in \Sigma$$

$$P^{\sigma}(z) = \prod_{i=1}^{\deg P} (z - \alpha_{\sigma i})$$

then

$$\max_{\sigma \in \Sigma} \prod_{i=1}^{\deg P} \max(1, |\alpha_{\sigma i}|) \ge \theta_0$$

where θ_0 is the least Pisot-Vijayaraghavan number.

Proof. See Theorem 1 in [1].

Proof of Theorem 1. By Lemma 2 there exists an $\epsilon \in E^n$ such that $F_{\epsilon}(z)$ is not self-inversive.

By Lemma 3, with $P = z^d F_a(z^{-1})$, there exists a $z_0 \in C$ such that

$$F_{*}(z_{0}^{-1}) = 0$$
 and $|z_{0}| \ge \theta_{0}^{1/d}$.

Hence, by the definition of F_s ,

$$F(\varepsilon_1 z_0^{-1}, \ldots, \varepsilon_n z_0^{-1}) = 0$$
 and $\max_{1 \le i \le n} |\varepsilon_i z_0^{-1}| \le \theta_0^{-1/d}$.

Remark. The above proof clearly works for all non self-inversive polynomials $F \in K[z_1, ..., z_n]$ with integral coefficients and F(0, ..., 0) = 1.

THEOREM 2. If $F \in Z[z_1, ..., z_n]$ of the total degree d satisfies F(0, ..., 0) = 1, then there exist C(d) < 1 and $(\varrho_1, ..., \varrho_n) \in C^n$ such that $F(\varrho_1, ..., \varrho_n) = 0$ and $\max |\varrho_i| \leq C(d)$.

Proof. In virtue of Theorem 1 we can assume that F is reciprocal. By the definition of a reciprocal polynomial, F(0, ..., 0) = 1 implies that the term $z_1^{d_1} ... z_n^{d_n}$ occurs in F, so $d = d_1 + ... + d_n$. Hence F really depends only on d variables at most and Corollary 1 of [2] gives our result.

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We note that

$$C(d) = 1 - \frac{c_0 (\log d)^3}{d^{d+6} e^{3d^2 + 4d}}$$

where the constant c_0 is absolute.

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