Thus the existence of B-ideals follows, for example, from the Continuum Hypothesis. Now, it is easy to see that each B-ideal is a P-ideal (but not necessarily maximal). Indeed, suppose that $A_n \in J$, where J is a B-ideal. We may assume that the sequence is increasing. If $B_n = A_n \setminus n$, then the B_n's are in J and $\min_n B_n \to \infty$, and hence, for some Z, $B = \bigcup\{B_n\colon n \in Z\}$ is in J. But $A_n \subseteq_* B_n \subseteq B$ for each $n \in \omega$, and hence J is a P-ideal.

Finally, we prove the following

PROPOSITION. *Assuming* CH, *there is a B-ideal, and hence a P-ideal, which cannot be extended to a P-point. In particular, there are nonmaximal B-ideals.*

Proof. The Balcar–Frankiewicz–Mills Theorem shows that the space $G(2^\omega)$ (the Gleason space of the Cantor set) can be embedded into ω^* as a closed P-set X. Hence the family

$$F = \{A \subseteq \omega\colon X \subseteq A^*\}$$

is a P-filter. If F were extendible to a P-point p then, since $\{p\} = \bigcap\{A^*\colon A \in p\}$ and $A \cap X \neq \emptyset$ for each $A \in p$, we would have $p \in X$, which is impossible, because X is separable and without isolated points. The dual $J = \{\omega \setminus A\colon A \in F\}$ is then a P-ideal not extendible to a P-point and, in fact, it is a B-ideal: suppose that $A_n \in J$ and $\min_n A_n \to \infty$. Let $A \in J$ almost contain each A_n and let $e_n = A_n \setminus A$. There is an infinite $Z \subseteq \omega$ such that $\{e_n\colon n \in Z\}$ is a disjoint family. It is possible to form 2^ω almost disjoint subunions $\bigcup\{e_n\colon n \in Z_\alpha\}$, for almost disjoint $Z_\alpha \subseteq Z$. One of them is in J, for otherwise we would have 2^ω nonempty open-closed disjoint subsets of X, which is impossible as $G(2^\omega)$ has countable cellularity.

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Nielsen reduction in free groups with operators

by

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Abstract. The Nielsen method is generalized to an equivariant situation, in which the variables of a free group are freely permuted by an operator group G. Critical elements $W = A\cdot x(A)^{-1}$, $x \in G$ occur, which are analysed in detail. An equivariant Grushko–Neumann Theorem is deduced and applications to low-dimensional CW-complexes are given.

§ I. Introduction. Let G be an arbitrary group, $F(a_1, \ldots, a_n)$ a free group of finite rank, and let $\bar{F}$ be the normal closure of F in $G * F$. $\bar{F}$ is freely generated by the xa_ix^{-1}, $x \in G$, with G operating on $\bar{F}$ by conjugation. Alternatively we may think of $\bar{F}$ as a free group with basis $x(a_i)$ ($\hat{=}\ xa_ix^{-1}$), $x \in G$, which is freely permuted by G. The *length* of an element W of $\bar{F}$ is understood to be the length with respect to the (in general infinite) basis $x(a_i)$ and is denoted by $|W|$.

If $W_1, \ldots, W_m$ are finitely many elements of $\bar{F}$, then we denote by $\mathrm{Gp}(W_1, \ldots, W_m)$ the subgroup of $\bar{F}$ generated by the W_i; by $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$ we denote the smallest G-invariant subgroup of $\bar{F}$ containing the W_i, i.e. the subgroup, which is generated by all $x(W_i)$, $x \in G$. $(W_1, \ldots, W_m)$ is called a *G-generating system* of $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$. *A G-generating system is called (G-) free* or a (G-) *basis* of $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$, if the $x(W_i)$, $x \in G$, $i = 1, \ldots, m$ are free in the ordinary sense. If a G-invariant subgroup of $\bar{F}$ has a G-basis, then this subgroup is said to be G-free.

$\mathrm{Gp}(W_1, \ldots, W_m)$ remains unchanged if the m-tuple $(W_1, \ldots, W_m)$ is subject to *Nielsen transformations* (NT), i.e. a finite sequence of the following elementary transformations:

(1)
(i) $W_i \to W_i^{-1}$ for some i (*inversion*),
(ii) $W_i \to W_iW_j$, $i \neq j$ (*multiplication*),
(iii) *deletion* of some W_i, where $W_i = 1$.

For $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$ we may enlarge this list by

(2) (iv) $W_i \to x(W_i)$ for some i, $x \in G$ ((G-) *conjugation*).

Transformations which are finite sequences of (i) ... (iv) are called *relative Nielsen transformations* (RNT). (The notion "relative" comes from the applications to pairs of 2-complexes, see § III, (c).) By (i), (ii) and (iv) it is possible

(3) to multiply W_i on the left or right with $x(W_j)^{\pm 1}$, $i \neq j$ (*generalized multiplication*).

A relative Nielsen transformation (not containing deletions) transforms a G-free m-tupel $(W_1, \ldots, W_m)$ into another G-free m-tupel.

In analogy to the "absolute" case, where G does not occur, the following *Relative Nielsen Theorem* holds:

THEOREM 1. *If* $\overline{\mathrm{Gp}(W_1, \ldots, W_m)} = \bar{F}$, *then there exists a finite sequence* Φ *of length-reducing or length-preserving elementary relative Nielsen transformations which transforms* $(W_1, \ldots, W_m)$ *into the standard basis* $(a_1, \ldots, a_n)$ *of* $\bar{F}$.

It follows that $m \geqslant n$. If moreover $m = n$, then Φ does not contain deletions, and thus $(W_1, \ldots, W_n)$ turns out to be a basis.

But in contrast to the absolute case, there arise difficulties in generalizing Theorem 1 to a relative Nielsen reduction method for arbitrary $(W_1, \ldots, W_m)$, i.e., if $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$ may be a *proper* subgroup of $\bar{F}$. We describe these difficulties together with a brief "history of Theorem 1".

In 1944 Renée Peiffer, in her study [14] of identities of relations, mentions the case "$m = n$, G a free group of finite rank". The proof of her Theorem 4 contains the claim that length-reductions can be performed as in the absolute case. Elvira Strasser Rapaport gave a complete proof of this case about 20 years later ([15], Theorem 1). Rapaport was interested in applications to the Andrews–Curtis conjecture, and her approach was refined subsequently by R. Craggs [4].

W. Browning and the second author have independently treated the more general case "$m = n$, G arbitrary" ([2], [11] Thm 7). Both point out that a relative Nielsen reduction method would have to consider elements of the form

$$W = A \cdot x(A)^{-1}, \quad A \neq 1,\ x \neq 1, \tag{4}$$

which cancel "essentially" against G-conjugates of themselves. But, fortunately, these elements do not occur in the case $m = n$, as a G-basis of $\bar{F}$ must project to a basis of F under the natural projection $\bar{F} \to F$ given by $x(a_i) \to a_i$, whereas $W = A \cdot x(A)^{-1}$ projects to 1.

We will call elements of type (4) *critical elements*. They are responsible for the additional difficulties in the relative case:

EXAMPLE 1. Let $W \neq 1$ be the element $W = a \cdot x(a)^{-1}$, where $F(a)$ is of rank 1 and x is a generator of $G = \mathbf{Z}_n$, $n > 1$. Then

$$W \cdot x(W) \cdot \ldots \cdot x^{n-1}(W) = 1. \tag{5}$$

Thus W is not a basis of $\overline{\mathrm{Gp}(W)}$. Further the result of every relative Nielsen transformation applied to W fulfills (5) or the inverse relation and hence cannot be a basis either.

This shows that, in general, there exists no relative Nielsen transformation converting $(W_1, \ldots, W_m)$ into a G-basis of $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$.

Moreover, $\overline{\mathrm{Gp}(W)}$ of Example 1 is not even G-free, for otherwise we could apply Theorem 1 to W, expressed as a G-word in a basis $(W_1, \ldots, W_m)$ of $\overline{\mathrm{Gp}(W)} = \overline{\mathrm{Gp}(W_1, \ldots, W_m)} = \bar{F}(W_1, \ldots, W_m)$. This would yield $m = 1$ and establish W as a basis of $\overline{\mathrm{Gp}(W)}$, which it is not.

Thus a relative Nielsen–Schreier subgroup theorem does not hold in general.

Denk [5] avoided the restriction $m = n$ in Browning's and Metzler's proof of Theorem 1 by applying the Grushko–Neumann Theorem. An analysis of critical elements in the reduction process had been bypassed once more.

Such an analysis is the goal of the present paper (Main Theorem and its proof in § II). It turns out that — as in (5) — relations of G cause the "misbehaviour" of critical elements (compare the considerations leading to (18) below): As a consequence of the main result we obtain a relative Nielsen–Schreier subgroup theorem for (locally) free operator groups G (Theorem 2).

Likewise we will deduce Theorem 1 from the main theorem, as well as a relative (= operator) version of the Grushko–Neumann Theorem (Theorem 3). In addition to these consequences, § III contains applications to presentations and low-dimensional CW-complexes which were the motivation for this paper: A direct application of Theorem 2 is a basis theorem for identities of presentations (Theorem 4). Theorem 5 shows that the crucial collapsing argument of P. Wright's paper [18] can be replaced by an algebraic one on relative Nielsen transformations of bases.

The paper contains a condensed version of the first author's thesis [6].

§ II. Critical elements and the reduction process. By G-conjugation, a critical element W can be *normalized* to $W' = A' \cdot x'(A')^{-1}$, $A' \neq 1$, such that A' — considered as a reduced word in $G * F$ — starts in F. A' is called the *essential part* of W, x' the *operator part* of W. Let us call critical elements W_1, W_2 *equivalent* if they determine the same essential part. Thus the sublist K of critical elements of an m-tupel $(W_1, \ldots, W_m)$ is partitioned into equivalence classes K_β. Let $\mathrm{Gp}(K_\beta)$ denote the subgroup of $\bar{F}$, which is generated by all $W_i \in K_\beta$, and let $\overline{\mathrm{Gp}(K_\beta)}$ be its G-closure. $U_\beta \subseteq G$ is defined to be the subgroup of G generated by all operator parts of the $W_i \in K_\beta$.

The following list of (partly) elementary facts on critical elements will be of importance:

(6) If W is critical with essential part A and operator part x, then W^{-1} is critical with essential part A and operator part x^{-1}.

(7) If $W_1 = u_1(A \cdot x_1(A)^{-1})$ and $W_2 = u_2(A \cdot x_2(A)^{-1})$ are critical with essential part A and operator parts x_1 resp. x_2, then $W_1 \cdot u_1 x_1 u_2^{-1}(W_2)$ is critical with essential part A and operator part $x_1 \cdot x_2$.

Here and in the sequel the operator part 1 is assigned, if the result of a multiplication of critical elements is the "degenerate critical element" 1.

Let A_β be the essential part determined by K_β. Because of (6) and (7)

(8) every element of U_β can be realized as the operator part of a critical element $W \in \overline{\mathrm{Gp}(K_\beta)}$ with essential part A_β.

(9) All $W_i \in K_\beta$ have the same length $|W_i| = 2|A_\beta|$.

(10) The length of $W \in \overline{\mathrm{Gp}(K_\beta)}$ is a multiple of $2|A_\beta|$.

This follows, because $x(A_\beta)^{-1}$ and $y(A_\beta)$ in such a product cancel totally or not at all.

A converse of (8) is given by

(11) If a product $W \in \overline{\mathrm{Gp}(K_\beta)}$ has length $|W| \leqslant 2|A_\beta|$ then W is trivial or critical with essential part A_β. Its operator part is the product of the operator parts of the factors, i.e., if $W = \prod_\nu u_\nu(W_{i_\nu}^{\varepsilon_\nu})$ with $\varepsilon_\nu = \pm 1$, $W_{i_\nu} \in K_\beta$, and x_{i_ν} is the operator part of W_{i_ν}, then $\prod_\nu x_{i_\nu}^{\varepsilon_\nu}$ is the operator part of W.

(11) can be deduced from (10), (6) and (7) by first restricting to factorizations of W, in which no proper subproduct has the value 1. The general case then follows inductively.

(12) If the critical elements $W_i \in K_\beta$ have corresponding operator parts x_i, $i = 1, \ldots, r$, and if $(x_1, \ldots, x_r) \overset{\varphi}{\to} (x'_1, \ldots, x'_{r'})$, $r' \leqslant r$ is a Nielsen transformation of operators, then φ induces a relative Nielsen transformation $(W_1, \ldots, W_r) \overset{\Phi}{\to} (W'_1, \ldots, W'_{r'})$ of critical elements with essential part A_β.

In fact: an inversion gives rise to an inversion by (6), a multiplication to a generalized multiplication by (7), a deletion to a deletion. So Φ in particular consists of elementary steps which preserve or reduce the sum $\sum |W_i|$.

The x_i in (12) are called *freely independent*, if r' is necessarily equal to r, i.e., if there exists no φ containing a deletion.

Example 2. Let $F(a, b)$ be given and G be a free group of rank 1 generated by x. Choose $W_1 = a \cdot b \cdot x^{-1}(a^{-1})$, $W_2 = a \cdot x^{-1}(a^{-1})$, $W_3 = x^2(a) \cdot b^{-1}$. Then $W_1 \to W_1 \cdot W_3$ would be a length-reducing transformation, if the operators of the terminal letter of W_1 and the initial letter of W_3 were not different. In fact, there is no length-reducing generalized multiplication between W_1 and W_3. But the critical element W_2 gives rise to

$$k = W_2^{-1} \cdot x(W_2^{-1}) \cdot x^2(W_2^{-1}) = x^{-1}(a) \cdot x^2(a^{-1}), \tag{13}$$

which can be used to form the RNT $W_1 \to W_1 \cdot k \cdot W_3 = a$. Further transformations to the standard basis (a, b) of $F(a, b)$ are now obvious.

k itself is critical and acts as a *catalyst*: both halves of k react with one of the remaining factors of $W_1 \cdot k \cdot W_3$, and afterwards the desired length-reduction is possible.

Whereas ordinary Nielsen reduction can be performed by considering products of two and three factors, phenomena as in Example 2 suggest allowing the occurrence of catalysts between the factors in order to handle the operator case: Let $(W_1, \ldots, W_m)$ be given, then *the empty word* $k = 1$ *is called a catalyst.* $k \in \overline{\mathrm{Gp}(K_\beta)}$ is called a *catalyst for* $x(W_i^\varepsilon)$, $y(W_j^\eta)$, $\varepsilon, \eta \in \{-1, -1\}$, if $|k| = 2|A_\beta|$, $|x(W_i^\varepsilon) \cdot k| = |W_i|$ and $|k \cdot y(W_j^\eta)| = |W_j|$; $x(W_i^\varepsilon) \cdot k \cdot y(W_j^\eta)$ then is called a 2-*product with catalyst*, likewise in the case $k = 1$. $x(W_i^\varepsilon) \cdot k_1 \cdot W_j \cdot k_2 \cdot y(W_i^\eta)$, $\varepsilon, \eta \in \{-1, +1\}$ is called a 3-*product with catalysts*, if $x(W_i^\varepsilon) \cdot k_1 \cdot W_j \neq 1$ as well as $W_j \cdot k_2 \cdot y(W_i^\eta) \neq 1$ are 2-products with catalysts and $x(W_i^\varepsilon) \cdot k_1 \cdot W_j \cdot k_2 \cdot y(W_i^\eta)$ is not a 2-product with catalyst $k_1 \cdot W_j \cdot k_2$, $W_j \in K_\beta$, each of $k_1, k_2, k_1 \cdot W_j \cdot k_2$ being a critical element of $\overline{\mathrm{Gp}(K_\beta)}$ which is trivial or has the essential part A_β. All products in these definitions are assumed to be reduced words in the $z(W_\nu)^{\pm 1}$.

A nontrivial catalyst $k \in \overline{\mathrm{Gp}(K_\beta)}$ is critical with essential part A_β. In a 2-product with catalyst $x(W_i^\varepsilon) \cdot k \cdot y(W_j^\eta)$ both halves of such a k react with one of the remaining factors as in Example 2. Because $k \neq 1$ we may assume that k is given as a product $\prod$ in the $z(W_\nu)^{\pm 1}$, $W_\nu \in K_\beta$, such that no subproduct has the value 1. Then adjacent factors in $\prod$ cancel exactly half of each other. Together with $|x(W_i^\varepsilon) \cdot k| = |W_i|$ and $|k \cdot y(W_j^\eta)| = |W_j|$ this implies that

(14) the transitions $x(W_i^\varepsilon) \to x(W_i^\varepsilon) \cdot k$ and $y(W_j^\eta) \to k \cdot y(W_j^\eta)$ can be performed in length-preserving elementary steps.

We can now state the

Main Theorem. *Let* $(W_1, \ldots, W_m)$ *be an* m*-tuple of elements* $W_i \in F$. *Then there exists a relative Nielsen transformation*

$$\Phi\colon (W_1, \ldots, W_m) \to (W'_1, \ldots, W'_{m'})$$

consisting of length-reducing or length-preserving elementary transformations such that the W'_i *fulfill*

(15) (RO) $W'_i \neq 1$,

(*) *The operator parts of the* W'_i *in each* K'_β *are freely independent.*

(16) *For each* 2-*product with catalyst either*

(R1) $|x(W_i'^{\varepsilon}) \cdot k \cdot y(W_j'^{\eta})| \geqslant \max\{|W'_i|, |W'_j|\}$

or

(+) $x(W_i'^{\varepsilon}) \cdot k \cdot y(W_j'^{\eta}) = 1$, $W'_i, W'_j \in K'_\beta$, $k \in \overline{\mathrm{Gp}(K'_\beta)}$

holds.

(17) *For each* 3-*product with catalysts*

(R2) $|x(W_i'^{\varepsilon}) \cdot k_1 \cdot W'_j \cdot k_2 \cdot y(W_i'^{\eta})| > |W'_i| - |W'_j| + |W'_i|$

holds.

Remarks. (RO), (R1) and (R2) correspond to the Nielsen properties in the absolute case. (R1) states that, even after inserting an admissable catalyst, neither $x(W_i'^{\varepsilon})$ nor $y(W_j'^{\eta})$ cancel more than half. (R2) states that in any 3-product with catalysts,

W_j does not cancel totally, no matter how k_1 and k_2 have been chosen. (*) and (+) are new properties in the relative case.

In every 2-product with catalyst of type (+), k is trivial or — compare the argument preceding (14) — can be given as a product of critical elements, such that no subproduct has value 1. If x'_ν is the operator part of $W'_\nu \in K'_\beta$, (+) thus yields a reduced nontrivial word $w(x'_\nu)$ of value 1 in G. Such a word raises the question, whether the x'_ν are freely independent or not, hence of a possible application of (12) including a deletion.

If in (5) we collect all but the first and the last factor to a catalyst, Example 1 gives a case of (+), where (*) is fulfilled. In fact, all of the properties (15), (16), (17) of the Main theorem hold.

Examples of this type can be constructed from (6) and (7), whenever G has a finitely generated nonfree subgroup.

On the other hand, if G is (locally) free, then the existence of $w(x'_\nu)$ is incompatible with (*). Hence

(18) the case (+) does not occur in the Main Theorem, if G is (locally) free.

The Main Theorem will be proved by the construction of a reduction process. Two of its steps parallel the "Nielsen modifications of type 1 and 2" of the absolute case and will be given similar names. But between these there is an important "intermediate modification", which does not change lengths but increases the number of critical elements of a generating system. These steps are the subject of the following lemmata.

Lemma 1. *If* $(W_1, \ldots, W_m)$ *violates* (16) *for some* 2-*product with catalyst* $x(W_i^\varepsilon)\cdot k\cdot y(W_j^\eta)$ *and* $i \neq j$, *then there exists a* RNT *of length-reducing or length-preserving elementary steps which replaces* W_i *or* W_j *by a shorter element.* (This transition is called a *modification of type* 1.)

Proof. By inversion and/or conjugation of the product we may obtain the case $y = 1$, $|W_i| \geqslant |W_j|$. Then $|x(W_i^\varepsilon)\cdot k\cdot W_j^\eta| < |W_i|$ for some product $k \in \overline{\mathrm{Gp}(K_\beta)}$.

(a) If k does not contain a $z(W_i)^{\pm 1}$ — in particular, if $k = 1$ — then we apply the transformation $W_i \to x(W_i^\varepsilon)\cdot k\cdot W_j^\eta$.

(b) If some $z_1(W_i)^{\pm 1}$ is contained in k, but no $z_2(W_j)^{\pm 1}$, then we apply the transformation

$$W_i,\ W_j \to x(W_i^\varepsilon),\qquad k\cdot W_j^\eta \to x(W_i^\varepsilon)\cdot k\cdot W_j^\eta,\ k\cdot W_j^\eta.$$

In both cases, (14) guarantees that we can do without length-increasing elementary steps.

(c) The case that k contains G-conjugates of $W_i^{\pm 1}$ and $W_j^{\pm 1}$ would yield that both are nontrivial elements in the K_β given by k. Then, because of (10), $|W| < |W_i| = 2|A_\beta|$ implies $|W| = 0$. Thus (+) would be fulfilled, contrary to the assumption that (16) is violated. Hence this case cannot occur. ∎

In ordinary Nielsen reduction the products $W_i^\varepsilon W_i^\eta$ need no special attention. This is not so in the relative case, which leads to a new type of modification:

Lemma 2. *Let* W_i, $i = 1, \ldots, m$ *be nontrivial elements of* $\bar{F}$ *fulfilling* (16) *for* $i \neq j$. *Then for each* 2-*product with catalyst the following holds*:

(18) $|x(W_i^\varepsilon)\cdot k\cdot y(W_i^{-\varepsilon})| \geqslant |W_i|$, *if* W_i *is noncritical,*

(19) (R1′) *if* W_i *is noncritical,* $|x(W_i^\varepsilon)\cdot k\cdot y(W_i^\varepsilon)| > |W_i|$, *or* $W_i \to x(W_i^\varepsilon)\cdot k$ *is a* RNT *of length-preserving steps,* $x(W_i^\varepsilon)\cdot k$ *being critical.* (This transition is called an *intermediate modification.*)

(20) *If* W_i *is critical, then* $|x(W_i^\varepsilon)\cdot k\cdot y(W_i^\eta)| \geqslant |W_i|$, *or* $\varepsilon = \eta$ *and the given product is of type* (+)

Proof. We may assume $y = 1$. *First we prove* (18):

The case $k = 1$ is trivial: As 2-products with catalysts are assumed to be reduced as words in the $z(W_\nu)^{\pm 1}$, we have $x \neq 1$. Because of $W_i \neq 1$ there is no cancellation at all in $x(W_i)^\varepsilon\cdot W_i^{-\varepsilon}$. We even get $|x(W_i^\varepsilon)\cdot W_i^{-\varepsilon}| > |W_i|$.

$k \neq 1$ implies $|W_i| \geqslant |k|$: If $z_0(W_{\nu_0}^{\eta_0})$ is the last (critical) factor of a reduced $z(W_\nu)^{\pm 1}$ factorization of k, i.e. $k = k'\cdot z_0(W_{\nu_0}^{\eta_0})$, $|W_i| < |k|$ would imply

$$|x(W_i^\varepsilon)\cdot k'\cdot z_0(W_{\nu_0}^{\eta_0})| = |x(W_i^\varepsilon)\cdot k| = |W_i| < |k| = |W_{\nu_0}|\,, \tag{21}$$

violating (16) with $i \neq \nu_0$ (since W_{ν_0} is critical and W_i not). If $|W_i|$ is odd, then $W_i^\varepsilon = L\cdot z_1(a_{\nu_1}^{\eta_1})\cdot R$, $|L| = |R| = \frac{1}{2}(|W_i|-1)$. A violation of (18) would imply $x(R)\cdot k\cdot R^{-1} = 1$ and $xz_1(a_{\nu_1}^{\eta_1})\cdot z_1(a_{\nu_1}^{-\eta_1}) = 1$. x would then be trivial and therefore also $k = 1$. The given product in (18) would not be reduced in the $z(W_\nu)^{\pm 1}$.

If $|W_i|$ is even, let $L\cdot R$ be the decomposition into halves. A violation of (18) would imply $x(R)\cdot k\cdot R^{-1} = 1$ and $|x(L)\cdot L^{-1}| < |W_i|$. As in the "odd case", x and k would be trivial and again the given product not be reduced.

Proof of (19). We assume that the inequality is violated. The case $k = 1$ cannot occur because for $|W_i|$ odd the middle letter of W_i would have to cancel against a G-conjugate of itself, which is impossible, and for even $|W_i|$ the cancellation would establish W_i as critical. So we have $k \neq 1$ and by (21): $|W_i| \geqslant |k|$.

Once more we conclude "$|W_i|$ even" by the middle letter argument, as above. Let $W_i^\varepsilon = L\cdot R$ with $|L| = |R| = \frac{1}{2}|W_i|$. The violation of the inequality in (19) requires so much cancellation as to yield $x(R)\cdot k\cdot L = 1$. Then $x(W_i^\varepsilon)\cdot k = x(L)\cdot L^{-1}$ is critical. W_i was not critical and hence does not occur in the given factorization of k. (14) implies that $W_i \to x(W_i^\varepsilon)\cdot k$ can be performed in length-preserving elementary steps.

Proof of (20). In addition to $y = 1$ we may assume that W_i^ε is normalized, i.e. $W_i^\varepsilon = A_\beta\cdot z(A_\beta)^{-1}$.

The case $k = 1$, $\varepsilon = -\eta$ is treated as in the proof of (18).

For $k = 1$, $\varepsilon = \eta$ the given product is equal to $x(A_\beta)\cdot xz(A_\beta)^{-1}\cdot A_\beta\cdot z(A_\beta)^{-1}$. The length of it is smaller than $|W_i|$ if and only if $xz = 1$ and $x = z$, but then (G has 2-torsion and) we clearly have a case of (+).

If $k \neq 1$, the case $|W_i| < |k|$ can be ruled out similarly as in (21): the only modification is that we may assume $|W_{v_0}| = |k|$; hence $i \neq v_0$ holds as before.

For $k \neq 1$, $\varepsilon = -\eta$, $|W_i| \geq |k|$, $|x(W_i^\varepsilon)\cdot k\cdot W_i^{-\varepsilon}| < |W_i|$ would require cancellation of amount $xz(A_\beta)^{-1}\cdot k\cdot z(A_\beta) = 1$, $|x(A_\beta)\cdot A_\beta^{-1}| < 2|A_\beta|$, hence $x = 1$ and $k = 1$, being a contradiction. Thus $|x(W_i^\varepsilon)\cdot k\cdot W_i^{-\varepsilon}| \geq |W_i|$ holds.

In the case $k \neq 1$, $\varepsilon = \eta$, $|W_i| \geq |k|$, $|x(A_\beta)\cdot xz(A_\beta)^{-1}\cdot k\cdot A_\beta\cdot z(A_\beta)^{-1}| < |W_i| = 2|A_\beta|$ similarly requires $xz(A_\beta)^{-1}\cdot k\cdot A_\beta = 1$ and some further cancellation in $x(A_\beta)\cdot z(A_\beta)^{-1}$. This implies $x = z$, $k = z^2(A_\beta)\cdot A_\beta^{-1} \in \overline{\mathrm{Gp}(K_\beta)}$, (thus $z^2 \neq 1$), and establishes this case as one of type (+). ■

Given $(W_1, \ldots, W_m)$, we define for each $j \in \{1, \ldots, m\}$ M_j to be the system of all W_i^ε such that, for appropriate choices of the remaining factors, (R2) is violated with W_j as middle factor and some $x(W_i^\varepsilon)$ as first factor:

$$|x(W_i^\varepsilon)\cdot k_1\cdot W_j\cdot k_2\cdot y(W_l^\eta)| \leq |W_i| - |W_j| + |W_l| .$$

We first collect criteria which will be used to guarantee that M_j is empty:

(22) If (16) is fulfilled and $|W_j|$ is odd, then $M_j = \emptyset$.

Proof. In a 3-product with catalyst $x(W_i^\varepsilon)\cdot k_1\cdot W_j\cdot k_2\cdot y(W_l^\eta)$ (16) guarantees that W_j can be reduced from right and left not more than half. Hence at least the middle letter of W_j remains uncancelled and (R2) is valid. ■

(23) Let (16) and (R1′) be fulfilled and $W_j \neq 1$ be a noncritical element of even length. If then $|x(W_i^\varepsilon)\cdot k\cdot W_j| > |W_i|$ holds for all 2-products with catalysts $x(W_i^\varepsilon)\cdot k\cdot W_j$, $i \neq j$, then $M_j = \emptyset$.

(In this case the left half of W_j is called *G-isolated.*)

Proof. In $x(W_i^\varepsilon)\cdot k_1\cdot W_j\cdot k_2\cdot y(W_l^\eta)$, (a) *$W_j$ is cancelled from the left less than a half*, if $i \neq j$, because of the inequality of the assumption. But the same holds for $i = j$: If $\varepsilon = 1$, this follows from (R1′). If $\varepsilon = -1$, we argue as in the beginning of the proof of (18): the case $k = 1$ is trivial and otherwise we get $|W_j| \geq |k|$. More than the asserted cancellation would yield $x(L^{-1})\cdot k\cdot L = 1$, $W_j = L\cdot R$ being the decomposition of W_j into halves. Then, as W_j is not contained in the K_β given by k, $k\cdot W_j = x(L)\cdot R$ itself would violate the inequality of the assumption. (b) *W_j is cancelled from the right at most half* because of (16).

(a) and (b) together imply that W_j is not cancelled totally. ■

(24) Let (16) be fulfilled and all W_i be nontrivial, and let $W_j \in K_\beta$ be a critical element. Then the following are equivalent:

(a) for each 2-product with catalyst $|x(W_i^\varepsilon)\cdot k\cdot W_j| = |W_i|$ implies $k = 1$ or $k \in \overline{\mathrm{Gp}(K_\beta)}$ is critical with essential part A_β.

(b) $M_j = \emptyset$.

(In this case W_j is called *anti-G-isolated.*)

Proof. If (a) is violated, i.e. $|x(W_i^\varepsilon)\cdot k\cdot W_j| = |W_i|$ holds for some $k \neq 1$ and either $k \notin \overline{\mathrm{Gp}(K_\beta)}$ or k has essential part $\neq A_\beta$, then $x(W_i^\varepsilon)\cdot k\cdot W_j\cdot z(k^{-1})\cdot zx(W_i^\varepsilon)$ (for $W_j = A\cdot z(A)^{-1}$) is a 3-product with catalyst, in which W_j is cancelled totally. Hence $M_j \neq \emptyset$. If (a) holds, then each hypothetical 3-product with catalysts and middle factor W_j, in which W_j is cancelled totally, turns out to be a 2-product with catalyst $k_1\cdot W_j\cdot k_2$ of the form which is excluded by definition. ■

Remark. In (23) one can give a symmetric condition for the right half of W_j; it holds if and only if the condition for the left half of W_j^{-1} holds. (a) of (24) holds if and only if the corresponding criterion for $W_j\cdot k\cdot y(W_i)^\eta$ holds; thus we do not distinguish between right and left in the definition.

(25) Let (16) be fulfilled and $W_i^\varepsilon \in M_j$. Then there exists a 2-product with catalyst such that $|x(W_i)^\varepsilon\cdot k\cdot W_j| = |W_i|$ and $|k| < |W_j|$.

Proof. Let $x(W_i^\varepsilon)\cdot k_1\cdot W_j\cdot k_2\cdot y(W_l^\eta)$ be a 3-product with catalysts in which W_j cancels totally. By (22) $|W_j|$ is even. Because of (16) W_j is cancelled half from both sides and we get: $|x(W_i^\varepsilon)\cdot k_1\cdot W_j| = |W_i|$, $|W_j\cdot k_2\cdot y(W_l^\eta)| = |W_l|$ and $|W_i| \geq |W_j| \geq |k_1|$, $|W_l| \geq |W_j| \geq |k_2|$. We are finished if $|W_j| > |k_1|$. If $|W_j| = |k_1|$, let $W_j = L\cdot R$ be the decomposition of W_j into halves and $k_1 = A\cdot z(A)^{-1}$. Then $z(A)^{-1} = L$ and thus $x(W_i)^\varepsilon$ ends with $z^{-1}(L^{-1}) = A$. It follows that $zx(W_i^\varepsilon)\cdot W_j$ fulfills the conclusion with $|k| = 0 < |W_j|$. ■

For the following lemma we assume that the W_i are ordered so that $|W_i| \leq |W_{i+1}|$, $i = 1, \ldots, m-1$ and that for all W_i of equal length the noncritical ones are listed prior to the critical ones.

Lemma 3. *Let* (R0), (16) *and* (R1′) *be fulfilled*, $M_\nu = \emptyset$ *for* $\nu \leq j-1$ *and* $M_j \neq \emptyset$. *Then there exists a* RNT Φ *in which every elementary step preserves the length and which transforms critical elements into critical ones, such that afterwards (at least) one of the following cases holds*: (i) $M_\nu^{(\prime)} = \emptyset$ *for* $\nu \leq j$, (ii) (16) *and* (R1′) *are not both valid any more*, (iii) *the number of critical elements has increased.* (This transition is called a *modification of type* 2.)

Proof. Because of $M_\nu = \emptyset$ for $\nu \leq j-1$, at least one half of a noncritical W^ν of even length, $\nu \leq j-1$ is G-isolated. Applying inversions, we thus may assume

(26) the left half of a noncritical W_ν of even length, $\nu \leq j-1$, is G-isolated.

By (22) $|W_j|$ is even. Let W_i^ε be in M_j. Because of (25) this yields a 3-product with catalysts k_1, k_2 and $|k_1| < |W_j|$, in which W_j is cancelled totally. Thus we have

(27)
$$|x(W_i^\varepsilon)\cdot k_1\cdot W_j\cdot k_2\cdot y(W_l^\eta)| \leq |W_i| - |W_j| + |W_l| ,$$
$$|x(W_i^\varepsilon)\cdot k_1\cdot W_j| = |W_i| , \quad |W_j\cdot k_2\cdot y(W_l^\eta)| = |W_l|$$

(28)
$$|W_i| \geq |W_j| > |k_1| .$$

If $W_i^\varepsilon \in M_j$, $i \neq j$, and W_i is noncritical, we define a RNT $\varphi_{i,\varepsilon}^{(j)}$ by

(29)
$$\varphi_{i,\varepsilon}^{(j)}\colon W_i \to (x(W_i^\varepsilon)\cdot k_1\cdot W_j)^\varepsilon .$$

If $W_i^\varepsilon \in M_j$, $i \neq j$, and W_i is critical, then $W_i^{-\varepsilon} \in M_j$ holds also, which implies $W_i \in M_j$. Let $W_i = z(L_i) \cdot L_i^{-1}$ be the decomposition of W_i. Then we define a RNT $\psi_i^{(j)}$ by

(30) $$\psi_i^{(j)}\colon\ W_i \to xzx^{-1}(W_j^{-1} \cdot k_1^{-1}) \cdot x(W_j) \cdot k_1 \cdot W_j .$$

$\varphi_{i,\varepsilon}^{(j)}$ and $\psi_i^{(j)}$ are well defined and, because of (14), can be performed without length-increasing elementary steps. Define Φ to be the product of these transformations in any order. This makes sense as one transformation does not block the application of another; in particular, $\varphi_{i,\varepsilon}^{(j)}$ and $\varphi_{i,-\varepsilon}^{(j)}$ do not conflict with each other: Because of (28) only W_i of at least the length of W_j are modified; the necessary catalysts are shorter than W_j, and the critical generators to form these catalysts have not been altered.

The elementary transformations of Φ are even length-preserving: A length-reduction is impossible for $\varphi_{i,\varepsilon}^{(j)}$ by (16). Critical W_i are transformed into critical ones throughout, and the essential part of the result is not shorter than the one of W_i.

The result of the transformation is denoted by W_i' with corresponding M_i', etc. We assume that (ii) is not valid and show this to imply $M_\nu' = \emptyset$, $\nu \leqslant j$: $M_j' = \emptyset$:

Case 1. $W_j' = W_j$ is noncritical. The left half of W_j' is G-isolated: If $W_i^\varepsilon \in M_j$, $i \neq j$, then W_i^ε has been transformed to end with a G-conjugate of the right half of $W_j^{(\prime)}$. Any 2-product with catalyst and $|x(W_i'^\varepsilon) \cdot k' \cdot W_j'| = |W_i'|$ thus would yield $|y(W_j') \cdot k' \cdot W_j'| = |W_j'|$ for some 2-product with the same catalyst, violating (R1$'$). Hence, because of (16),

(31) $$|x(W_i'^\varepsilon) \cdot k' \cdot W_j'| > |W_i'|$$

holds for $W_i^\varepsilon \in M_j$, $i \neq j$ and all (2-products with catalyst) $x(W_i'^\varepsilon) \cdot k' \cdot W_j'$. But this is also true for $W_i^\varepsilon \notin M_j$, $i \neq j$: The fact that no W_μ with $|W_\mu| < |W_j|$ has been changed has the consequence that the necessary catalysts for a violation of (31) do not exist. By (23) M_j' is void.

Case 2. $W_j' = W_j$ is critical. In this case, W_j' turns out to be anti-G-isolated: Let us consider a 2-product with catalyst and $|x(W_i')^\varepsilon \cdot k' \cdot W_j'| = |W_i'|$. For $W_i^\varepsilon \in M_j$, $W_i'^\varepsilon$ ends with a G-conjugate of L_j^{-1}, $W_j = L_j \cdot y(L_j)^{-1}$ being the decomposition of W_j. As half of W_j has to cancel in the given product, there is a relation $v(L_j)^{-1} \cdot k' \cdot L_j = 1$, which implies $k' = 1$ or: *k' and $W_j^{(\prime)}$ have the same essential part*. But this is also true for $W_i^\varepsilon \notin M_j$, as no new relevant catalysts have been produced. By (24) we thus get $M_j' = \emptyset$.

$M_\nu' = \emptyset$, $\nu \leqslant j-1$: This is evident for $|W_\nu| < |W_j|$: Because of (28) these W_ν have neither been transformed nor have relevant new catalysts appeared. Even in the case $|W_\nu| = |W_j|$ there are no new catalysts. It might only be that W_ν itself has been transformed to yield $M_\nu' \neq \emptyset$. This is excluded (for $|W_\nu| = |W_j|$) as follows:

Let W_ν be noncritical. $|W_\nu|$ is even (like $|W_j|$) and, because of (26), has a G-isolated left half. By (23) $|x(W_j^{-1}) \cdot k \cdot W_\nu| > |W_j| = |W_\nu|$ holds for every (2-product with catalyst) $x(W_j^{-1}) \cdot k \cdot W_\nu$.

The inequality can be rewritten as $|x^{-1}(W_\nu^{-1}) \cdot x^{-1}(k^{-1}) \cdot W_j| > |W_\nu|$, implying $W_\nu^{-1} \notin M_j$ (by (25)). The right half of W_ν^{-1} thus has not been changed which implies that the left half of W_ν' has remained G-isolated. By (23) we get $M_\nu' = \emptyset$.

If W_ν is critical, W_ν is anti-G-isolated. Then the transformation of W_ν must have been performed with a trivial catalyst because of (24), (27) and (28). The order convention preceding the formulation of Lemma 3 and (30) together now yield W_j to be critical with the same essential part as W_ν. But then $M_\nu = \emptyset$ implies $M_j = \emptyset$ which is a contradiction. ■

We are now ready for the

Proof of the Main Theorem. We consider finite systems of $(W_1, \ldots, W_m)$, in which the W_i are ordered as in Lemma 3. Two such systems will be compared anew with respect to: (α) the number of elements, (β) the length sum, (γ) the number of critical elements, (δ) the first M_j (from left to right) with $M_j \neq \emptyset$. If all these coincide, then the systems are called equivalent. Otherwise we let the first different measure of (α), (β), (γ), (δ) from left to right determine the order. (In the case of (γ) $(W_1, \ldots, W_m) < (W_1', \ldots, W_m')$, if there are more critical W_i than W_i'; in the case of (δ) $(W_1, \ldots, W_m) < (W_1', \ldots, W_m')$, if the first nontrivial M_j occurs later than in $(W_1', \ldots, W_m')$, or, if all M_j are trivial but not all M_j'.) If S is a given equivalence class and l an upper bound for the length sum, then

(32) there are only finitely many classes $R < S$ such that the length sum of R does not exceed l.

The reduction procedure is the following: Start with a given system $(W_1, \ldots, W_m)$ and consider whether all of the criteria (15), (16) for $i \neq j$, (16) and (R1$'$), (R2) are fulfilled. If so, then the conclusion of the main theorem holds; if not, then we focus on the first of these criteria (from left to right) which is not valid and replace $(W_1, \ldots, W_m)$ by a smaller system (according to the order defined above) as follows:

Case 1. If (15) is violated then we can apply deletions or elementary operations of type (12) which preserve the number of elements, do not increase the length, and eventually also make possible a deletion. Reorder the result according to Lemma 3. $(W_1, \ldots, W_m)$ is replaced by a smaller system. (This may be called a *modification of type* 0.)

Case 2. If (15) holds, but (16) is violated for some $i \neq j$, then by Lemma 1 we can apply a modification of type 1 to replace $(W_1, \ldots, W_m)$ by a system of an equal number of elements and smaller length which (after reordering) is a smaller system.

Case 3. If (15) holds and (16) for $i \neq j$, but (16) and (R1$'$) are not both fulfilled, then by Lemma 2 we can apply an intermediate modification preserving the number of elements and the length but increasing the number of critical elements. Reorder. $(W_1, \ldots, W_m)$ thus is replaced by a smaller system.

Case 4. (15), (16) and (R1$'$) hold, but (R2) is violated. By Lemma 3 we can apply a modification of type 2 yielding a system of equal number of elements and equal length sum. If (iii) is not valid but (i) is, then the new system is already ordered properly and smaller than the old one; if (iii) is valid, then it becomes smaller (after reordering the W_i'); if (ii) is valid, then a further application of a modification of type 1 or an intermediate modification yield a smaller system.

As (even in case 1) the length is not increased, it is bounded by the initial value l. Hence by (32), the reduction must terminate after a finite number of steps yielding a system $(W_1', \ldots, W_{m'}')$ which fulfills (15), (16) and (17). ■

Remark. Under suitable decision hypotheses on G the reduction process can be turned into an algorithm. They hold, in particular, if G is free of finite rank (compare the discussion in the concluding § 3 of [3]).

§ III. Consequences and applications.

(a) Let $W_1, \ldots, W_m \in \bar{F}$ be nontrivial elements fulfilling (16) and (17), and let

(33) $$W = \prod_{\nu=1}^{s} u_\nu(W_{i_\nu}^{\varepsilon_\nu}), \quad \varepsilon_\nu = \pm 1,\ 1 \leqslant i_\nu \leqslant m$$

be a reduced word in the $u(W_i)^{\pm 1}$ in which no subproduct of type (+) occurs. Then

(34) $$|W| \geqslant \#\{\nu \in \{1, \ldots, s\} \text{ with: } W_{i_\nu} \text{ noncritical}\}$$

and

(35) $$|W| \geqslant \max\{|W_{i_\nu}|\}.$$

Proof. From left to right we collect factors to catalysts (possibly trivial), such that (33) is rewritten as

(36) $$W = V_1 \cdot k_1 \cdot V_2 \cdot k_2 \cdot \ldots \cdot k_{t-1} \cdot V_t,$$

where each V_μ is one of the $u_\nu(W_{i_\nu}^{\varepsilon_\nu})$, $V_\mu \cdot k_\mu \cdot V_{\mu+1}$ is a 2-product with catalyst and the k_μ cannot be enlarged with respect to these requirements. As (+) does not occur, cancellation in (36) leaves (at least) the left half of V_1, the right half of V_t and (at least) one letter of each V_μ. Hence we get

(37) $$|W| \geqslant \tfrac{1}{2}|W_{i_1}| + \tfrac{1}{2}|W_{i_s}| + (t-2),$$

(38) $$|W| \geqslant \tfrac{1}{2}|W_{i_1}| + \tfrac{1}{2}|W_{i_s}| + \#\{\nu \in \{2, \ldots, s-1\} \text{ with: } W_{i_\nu} \text{ noncritical}\}$$

and (34).

For the proof of (35) we note

(α) $$\max\{|W_{i_\nu}|\} = \max\{|V_\mu|\}.$$

(36) implies

(β) $$|W| \geqslant \max\{|V_\mu|\},$$

as in the absolute case, see [8], Proposition 2.13. (α) and (β) together yield the desired result. ■

(39) If (RO), (16) and (17) are fulfilled, then every $W \in \overline{\mathrm{Gp}(W_1, \ldots, W_m)}$ can be written as $W = \prod_{\nu=1}^{s} u_\nu(W_{i_\nu}^{\varepsilon_\nu})$, in such a way that (34) and (35) hold,

because in an arbitrary product of G-conjugates of the $W_i^{\pm 1}$ we may cancel subproducts of type (+).

Proof of Theorem 1. Apply the main theorem to transform $(W_1, \ldots, W_m)$ to $(W_1', \ldots, W_{m'}')$. As $\overline{\mathrm{Gp}(W_1', \ldots, W_{m'}')} = \bar{F}$, each a_j can be written according to (39):

$$a_j = \prod_{\nu=1}^{s} u_\nu(W_{i_\nu}'^{\varepsilon_\nu}).$$

Because of (35), all W_{i_ν}' in this product are of length 1, hence noncritical. (34) then implies $s = 1$, which yields that the final system $(W_1', \ldots, W_{m'}')$ contains some G-conjugate of $a_j^{\pm 1}$. We may assume without loss of generality $(W_1', \ldots, W_{m'}') = (a_1, \ldots, a_n, W_{n+1}', \ldots, W_{m'}')$. The case $m' > n$ now can be excluded, as (16) would be violated. ■

THEOREM 2. *Let G be (locally) free, $W_1, \ldots, W_m \in \bar{F}$. Then $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$ is G-free.*

Proof. Apply the Main Theorem to transform $(W_1, \ldots, W_m)$ to $(W_1', \ldots, W_{m'}')$. Because of (18), there is no product of type (+) for the final system $(W_1', \ldots, W_{m'}')$. (35) then yields that every nontrivial reduced word in the $x(W_i')^{\pm 1}$ determines a nontrivial element of $\bar{F}$. Hence $(W_1', \ldots, W_{m'}')$ is a G-basis of $\overline{\mathrm{Gp}(W_1, \ldots, W_m)}$. ■

(b) Let $\overline{G_1 * \ldots * G_n}$ be the normal closure of $G_1 * \ldots * G_n$ in $G * G_1 * \ldots * G_n$. With notations analogous to those in § I we get

THEOREM 3. *Any G-generating system $(W_1, \ldots, W_m)$ of $\overline{G_1 * \ldots * G_n}$ can be transformed by a relative Nielsen transformation such that each element of the final system is contained in some G_i.*

Proof. (i) *We first assume G to be finitely generated.* Then there is an epimorphism $\psi\colon \hat{G} \to G$, where $\hat{G}$ is a free group of finite rank and an epimorphism $\varphi\colon \hat{G} * F(a_1, \ldots, a_m) \to G * G_1 * \ldots * G_n$, which maps a_i to W_i and coincides with ψ on $\hat{G}$. Stallings' proof of the Grushko–Neumann Theorem [16] yields a new product decomposition $\hat{G} * F(a_1, \ldots, a_m) = \hat{G} * F_0 * F_1 * \ldots * F_n$ such that the factor $\hat{G}$ remains unchanged (the corresponding loops are not subdivided in Stallings' process), F_0 is mapped by φ to G and F_j to G_j, $j = 1, \ldots, m$. We may assume without loss of generality that F_0 vanishes, for, as the image of $\hat{G}$ already is G, we can apply ordinary Nielsen transformations to yield a situation in which F_0 is mapped trivially and thus can be thrown to the remaining F_j. φ can now be factored as

$$\varphi\colon \hat{G} * F_1 * \ldots * F_n \xrightarrow{\varphi_1} G * F_1 * \ldots * F_n \xrightarrow{\varphi_2} G * G_1 * \ldots * G_n,$$

the factorization arising from ψ. The $\varphi_1(a_i)$ yield a G-generating system of $\overline{F_1 * \dots * F_n} \subseteq G * F_1 * \dots * F_n$, which, by Theorem 1, can be transformed into an assorted one. By means of φ_2, this RNT projects to a transformation as required by the assertion.

(ii) *If G is not finitely generated*, we note that nevertheless $G_1 * \dots * G_n$ is finitely generated. In expressing a finite system of generators of $G_1 * \dots * G_n$ as G-conjugate products of the W_i, only finitely many elements of G occur. These constitute a finitely generated subgroup G_0 of G which can be used to replace G in (i). ■

Remark. In [5], Theorem 3 is deduced directly from the (ordinary) Grushko–Neumann Theorem.

(c) The case $m = n$ of Theorem 1 was used in [11] to treat simple homotopy equivalences of relatively 1-dim CW-complexes:

(40) If $f\colon K \to K'$ is a map of connected CW-complexes inducing the identity on the common connected subcomplex L and an isomorphism $\pi_1(K) \to \pi_1(K')$, and if $K - L$ as well as $K' - L$ are finite and of dim $\leqslant 1$, then f if a simple homotopy equivalence and homotopic rel. L to a formal deformation which involves expansions of dimension at most 2.

The operator group for the proof of (40) is $G = \pi_1(L)$. Wall [17] subsequently removed the connectivity hypothesis on L.

In [11] the dimension problem of simple homotopy equivalences between (relatively) 2-dimensional complexes was also treated, i.e. the question, whether moves of dim $\leqslant 3$ suffice ((generalized) Andrews–Curtis problem). By means of relative Nielsen transformations it was shown to be equivalent to a problem about identities of group presentations. These are defined as follows:

If $\mathfrak{p} = \langle x_1, \dots, x_m | R_1, \dots, R_n \rangle$ is a finite presentation, then we consider the projection $p\colon F(x_1, \dots, x_m) * F(y_1, \dots, y_n) \to F(x_1, \dots, x_m)$ given by $x_i \to x_i$, $y_j \to R_j(x_i)$. It induces a map $q\colon \overline{F} \to F(x_i)$ of the normal closure $\overline{F}$ of $F(y_j)$ in $G * F(y_i)$ with $G = F(x_i)$. The kernel of q is the *group of identities* of $\mathfrak{p}$. Of particular relevance for the Andrews–Curtis problem and the Whitehead-asphericity problem are *Peiffer identities*, i.e. identities of type

$$(r, s) = r \cdot s \cdot r^{-1} \cdot q(r) \cdot s^{-1} \cdot q(r)^{-1}, \quad r, s \in \overline{F}.$$

Theorem 2 immediately implies

THEOREM 4. *Every $F(x_i)$-finitely generated $F(x_i)$-subgroup of identities of $\mathfrak{p}$ is $F(x_i)$-free.* ■

By the remark after the proof of the Main Theorem, the reduction to a G-basis in Theorem 4 can be performed effectively.

Peiffer elements may be critical (for instance, (r, r)), and there is no obvious deduction of Theorem 4 from Theorem 1. Hence the study of identities was a main reason for developing a relative Nielsen reduction method for subgroups. Applications of Theorem 4 to the above problems are part of our current work.

The translation of the Andrews–Curtis problem into an algebraic one involves the following result of P. Wright [18]: A formal deformation $K^2 \overset{3}{\nearrow\!\searrow} L^2$ between finite, connected CW-complexes ($\overset{3}{\nearrow\!\searrow}$ indicates that only moves of dim $\leqslant 3$ occur) can be replaced by another one in which each 3-cell is collapsed immediately after its introduction (*transient moves*). The proof of P. Wright uses a composition argument on homotopies of attaching maps which can be carried over to an n-dimensional and/or simplicial version (Kreher and Metzler [9]). In dimension 3 it is a direct consequence of our concluding theorem. The transient moves result from elementary RNT's.

THEOREM 5. *Let $K^2 \overset{3}{\nearrow\!\searrow} L^2$ be a formal deformation between finite, connected CW-complexes which involves only moves of dimension 3. Then presentations $\mathfrak{p} = \langle x_1, \dots, x_m | R_1, \dots, R_n \rangle$ and $\mathcal{P} = \langle x_1, \dots, x_m | R_1', \dots, R_n' \rangle$ can be read off from K^2 resp. L^2 in the usual way which are Q-equivalent, i.e. $\mathfrak{p}$ transforms into $\mathcal{P}$ by a finite sequence of Nielsen transformations and conjugations ($R_{j_0} \to w R_{j_0} w^{-1}$, $w \in F(x_i)$, for some j_0) of the relators.*

Proof. Consider a formal deformation $K^2 = K_1^3 \to K_2^3 \to \dots \to K_s^3 = L^2$, each step being an expansion or a collapse of a 3-cell. As the 1-skeleton is not changed during the deformation, we may assume a spanning tree, and hence a basis x_i for $G = F(x_1, \dots, x_m)$, to be given throughout. From each 2-cell e_j^2 occurring in the whole process we read off a relation R_j and form the group $F(y_j)$ with projection $p\colon G * F(y_j) \to G$ as above. Each 3-cell e_l^3 can be used to read off from its boundary an identity z_l such that

(i) in z_l only G-conjugates of those $y_j^{\pm 1}$ occur for which $e_j^2 \subseteq \bar{e}_l^3$ holds, and

(ii) for the free faces $e_{l_1}^2$ and $e_{l_2}^2$ (possibly coinciding), by which e_l^3 is introduced resp. collapsed, there is exactly one occurrence of a G-conjugate of $y_{l_1}^{\pm 1}$, $y_{l_2}^{\pm 1}$ in z_l.

(This algebraic freeness property follows either from homotopy considerations of characteristic maps or from the existence of a simultaneous characteristic map which serves for the expansion *and* the collapse of e_l^3; see Brown [1].)

For each K_ν^3 we now form the group $\overline{F}_\nu | N_\nu$, where $\overline{F}_\nu$ is the subgroup of $G * F(y_j)$ which is G-generated by those y_j with $e_j^2 \in K_\nu^3$, and N_ν is the G-invariant normal subgroup of $\overline{F}_\nu$ which is generated by the z_l with $e_l^3 \in K_\nu^3$. Thus $\overline{F}_\nu | N_\nu$ is the group given by a *relative presentation* (in the sense of [11]) with operator group G, generators given by 2-cells and relators given by 3-cells. p induces projections $q_\nu\colon \overline{F}_\nu | N_\nu \to G$.

Because of (i) and (ii), each 3-expansion resp. 3-collapse yields a *generalized prolongation* or the inverse process, i.e. the corresponding relative presentations of $\overline{F}_\nu | N_\nu$ and $\overline{F}_{\nu+1} | N_{\nu+1}$ differ by one y-symbol, and one relator in which this y-symbol is contained algebraically free. Hence we get a chain of G-isomorphisms:

$$\overline{F}(y_1, \dots, y_n) = \overline{F}_1 | N_j \to \overline{F}_2 | N_2 \to \dots \to \overline{F}_s | N_s = \overline{F}(y_1', \dots, y_n')$$

which commute with the q_ν. They constitute a G-isomorphism $\varphi\colon \bar{F}(y_j) \to \bar{F}(y'_j)$, which commutes with q_1, q_s. The $\varphi(y_j)$ and the y'_j thus are two G-bases of $\bar{F}(y'_j)$, and Theorem 1 yields elementary transformations which imply $\mathfrak{p}$ and $\mathscr{P}$ to be Q-equivalent. ■

§ IV. Some open questions. In addition to the topics of this paper it would seem worth while to treat the following questions of *Combinatorial Group Theory with Operators*:

(1) Derive a *Relative Kurosh Subgroup Theorem* for G-invariant subgroups of $\overline{G_1 * \dots * G_n}$ (as in § IIIb)). (A covering treatment may contribute to further understanding of critical elements.)

(2) For $\bar{F}(a_1, \dots, a_n)$ one has a natural homomorphism of the G-isomorphisms $\mathrm{Aut}_G(\bar{F})$ to $\mathrm{GL}(n, Z(G))$. We conjecture that its kernel is generated by the automorphisms $a_i \to x(a_j)\cdot a_i \cdot x(a_j)^{-1}$, $i \neq j$, $x \in G$, and $a_i \to a_i \cdot [x(a_j), y(a_k)]$, $i \neq j, k$; $x, y \in G$, (all variables except a_i are fixed) in analogy to Magnus [10].

(3) If $p\colon F \to F'$ is a homomorphism between two free groups of finite rank, then one can ask for those automorphisms of F which commute with p. By using Nielsen transformations we may restrict ourselves to a standard situation $p\colon F(x_1, \dots, x_m, y_1, \dots, y_n) \to F(x_1, \dots, x_m)$ given by $x_i \to x_i, y_j \to 1$. We conjecture that the automorphisms in question are generated by (i) RNT's of $\bar{F}(y_1, \dots, y_n)$ with $G = F(x_i)$ and (ii) multiplications of a y_j to an x_i from left or right. (So far we have verified the case $m = 1$.)

The question is a "dual" one to the determination of stabilizers in $\mathrm{Aut}(F)$. Its solution will be helpful for the treatment of stabilizers of presentations in the sense of [12].

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