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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Classification of weakly infinite-dimensional spaces Part I: A transfinite extension of the covering dimension *

by

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Abstract. A classification of weakly infinite-dimensional spaces is given by introducing a transfinite extension of the covering dimension, dim. The classification provided by dim is the same as the one given by R. Pol's [P2] index for weakly infinite-dimensional compact metric spaces. Several invariants of this dimension function are studied.

In Part II we will prove several results concerning the relation between dim and essential mappings onto D. W. Henderson's [He] cubes J^{α}, where α is a countable ordinal number.

Chapter I. Infinite-dimensional spaces

A space is called *infinite-dimensional* if it is not finite-dimensional. Hurewicz [Hu] mentioned the possibility to somehow topologically classify infinite-dimensional spaces. In the last twentyfive years quite an extensive theory was developed and most of the important results were recordedin [N], [E3] and [E+P].

In this chapter we shall define several important notions in the theory of infinite-dimensional spaces.

Preliminaries. Let us first establish some notational conventions. As far as standard notions from general topology and (finite) dimension theory are concerned, we mostly follow [E1] and [E2].

In particular, we note that the *boundary* of a subset A of a space X is denoted by $\mathrm{Fr}A$. A subset A of a space X is called *clopen* iff A is both open and closed in X. By C we denote the Cantor set.

The first infinite ordinal number is denoted by ω_0 and ω_1 is the first uncountable ordinal number. Moreover, for an ordinal number α, we let $\lambda(\alpha)$ be a limit ordinal or zero and $n(\alpha)$ be finite such that $\alpha = \lambda(\alpha)+n(\alpha)$.

1.1. Weakly infinite-dimensional spaces. Let us start with one of the most fundamental definitions is this treatise:

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1.1.1. DEFINITION A (finite) sequence $\{(A_i, B_i)\}_{i=1}^{\infty(m)}$ of pairs of disjoint closed sets in a space X is called *inessential* if for some $n \in N$ $(n \leqslant m)$ we can find open sets O_i $i = 1, \ldots, n$ such that

$$A_i \subset O_i \subset \overline{O}_i \subset X - B_i \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Fr} O_i = \emptyset .$$

Otherwise it is called *essential.*

We have the following characterization of the covering dimension, dim ([E2; 3.2.6]).

1.1.2. THEOREM. $\dim X \leqslant n$ *if and only if every sequence* $\{(A_i, B_i)\}_{i=1}^{n+1}$ *of pairs of disjoint closed sets in* X *is inessential.* ■

Twenty years ago the characterization of covering dimension mentioned in Theorem 1.1.2 led to the definition of weakly infinite-dimensional spaces. Both P. S. Alexandrov and Y. Smirnov used this characterization to obtain an interesting new way of classifying infinite-dimensional spaces.

1.1.3. DEFINITION. A space X is called *weakly infinite-dimensional in the sense of Alexandrov* (*Smirnov*), abbreviated A-w.i.d. (S-w.i.d.), if for every sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint closed sets in X there exist open sets V_i, $i = 1, 2, \ldots$ such that

$$A_i \subset V_i \subset \overline{V}_i \subset X - B_i \quad \text{and} \quad \bigcap_{i=1}^{\infty} \operatorname{Fr} V_i = \emptyset \quad \Big(\bigcap_{i=1}^{n} \operatorname{Fr} V_i = \emptyset \text{ for some } n\Big).$$

Observe that if X is compact the notions A-w.i.d. and S-w.i.d. coincide. We then call the space X w.i.d.

In this treatise we are interested mainly in these types of spaces. The following lemma will be very useful.

1.1.4. LEMMA. *Let* F *be a closed subspace of a space* X *and let* $\{(A_i, B_i)\}_{i=1}^{\infty(m)}$ *be a (finite) sequence of pairs of disjoint closed sets in* X *such that* $\{(A_i \cap F, B_i \cap F)\}_{i=1}^{\infty(m)}$ *is inessential in* F.

Then for some $n \in N$ $(n \leqslant m)$ *we can find open sets* O_i *in* X *for* $i = 1, \ldots, n$ *such that*

$$A_i \subset O_i \subset \overline{O}_i \subset X - B_i \quad \textit{and} \quad F \cap \bigcap_{i=1}^{n} \operatorname{Fr} O_i = \emptyset .$$

Proof. Since $\{(A_i \cap F, B_i \cap F)\}_{i=1}^{\infty(m)}$ is inessential in F, for some $n \in N$ $(n \leqslant m)$ we can find open sets W_i in F for $i = 1, \ldots, n$ such that

$$A_i \cap F \subset W_i \subset \overline{W}_1^F \subset F - B_i \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Fr}_F W_i = \emptyset .$$

(Here Fr_F denotes the boundary operator w.r.t. the subspace F; similarly ${}^{-F}$ denotes the closure operator w.r.t. F.)

By virtue of [E2; 3.1.2] there exist open sets V_i in X for $i = 1, \ldots, n$ such that

$$A_i \cap V_i = \emptyset = B_i \cap V_i, \qquad \operatorname{Fr}_F W_i \subset V_i \qquad \text{and} \qquad \bigcap_{i=1}^{n} V_i = \emptyset .$$

Put $G_i = \overline{W}_i^F - V_i$ and $H_i = (F - W_i) - V_i$ for $i = 1, \ldots, n$.

Observe that G_i and H_i are disjoint closed sets in X, both contained in F. For $i = 1, \ldots, n$ the sets $A_i \cup G_i$ and $B_i \cup H_i$ are also disjoint closed sets in X. Now let O_i for $i = 1, \ldots, n$ be open sets in X such that

$$A_i \cup G_i \subset O_i \subset \overline{O}_i \subset X - (B_i \cup H_i) .$$

Then $\operatorname{Fr} O_i \cap F \subset V_i$ for each $i = 1, \ldots, n$. Consequently,

$$\bigcap_{i=1}^{n} \operatorname{Fr} O_i \cap F \subset \bigcap_{i=1}^{n} V_i = \emptyset . \quad ■$$

1.2. Other infinite-dimensional notions.

1.2.1. DEFINITION [Sm]. Let X be a space and let α be an ordinal number. Then we define

$\operatorname{Ind} X = -1$ iff $X = \emptyset$,

$\operatorname{Ind} X \leqslant \alpha$ iff for every pair (A, B) of disjoint closed sets in X we can find an open set V in X such that $A \subset V \subset \overline{V} \subset X - B$ and $\operatorname{Ind} \operatorname{Fr} V < \alpha$,

$\operatorname{Ind} X = \alpha$ iff $\operatorname{Ind} X \leqslant \alpha$ and $\operatorname{Ind} X < \alpha$ does not hold,

$\operatorname{Ind} X = \infty$ iff $\operatorname{Ind} X > \alpha$ for every ordinal number α.

If for some ordinal number α, $\operatorname{Ind} X \leqslant \alpha$ holds, then we say that X has *large transfinite dimension* or Ind.

One of the oldest notions of infinite dimension theory is also the notion of countable dimensionality:

1.2.2. DEFINITION [Hu]. A space X is called (*strongly*) *countable dimensional*, abbreviated c.d. (s.c.d.) if X can be written as $\bigcup_{n=1}^{\infty} X_n$ with $\operatorname{Ind} X_n$ finite (and X_n closed in X) for every n.

1.2.3. DEFINITION. For each subspace A of a space X we let for each $n \geqslant 0$

$$P_n(A) = \bigcup \{U\colon U \text{ open in } A \text{ and } \dim \overline{U}^A \leqslant n\} .$$

We let $A_0 = X$ and $P_0 = P_0(X)$. For all ordinals η we will let $A_\eta = X - (\bigcup_{\xi<\eta} P_\xi)$. Then for arbitrary ξ we put $P_\xi = P_{n(\xi)}(A_{\lambda(\xi)})$. Observe that each A_ξ is closed in X and $A_\xi \subset A_{\xi'}$ whenever $\xi > \xi'$.

By Baire's Category Theorem, [E1; 3.9.3], we have

1.2.4. LEMMA. *Every topologically complete strongly countable dimensional space* X *contains an open set* U *such that* $\dim \overline{U} < \infty$. ■

Using this Lemma Z. Shmuely proved in [Sh]

1.2.5. THEOREM. *Every complete* s.c.d. *space X has $A_\alpha = \emptyset$ for some ordinal number α.* ■

The set A_{ω_0} is very useful in infinite-dimension theory. We will illustrate this by a characterization of S-w.i.d. spaces in terms of properties of A_{ω_0}. This result is due to E. G. Sklyarenko [Sk]. Similar characterizations exist for spaces having Ind (see [E3]) and spaces having both sind and Ind (see [Ha]).

The characterizations reveal the "compact" nature of such spaces.

1.2.6. THEOREM. *A space X is* S-w.i.d. *if and only if*

(1) *A_{ω_0} is* S-w.i.d.,

(2) *A_{ω_0} is compact, and*

(3) *For every closed set F disjoint from A_{ω_0} there is an $n \in N$ with $F \subset P_n$.* ■

Chapter II. Classifications of collections of finite subsets

In this chapter we will introduce a classification of collections of finite subsets by defining the ordinal number Ord for such a collection. This ordinal number will be used to introduce the transfinite dimension function dim in the next chapter.

We prove all set-theoretic results needed when dealing with the dimension function dim. The second section of this chapter will be devoted to a few sum theorems on Ord and in the third one we will investigate the relation between Ord and the classification used by R. Pol [P2] to define his index for w.i.d. compact spaces.

2.1. The ordinal number Ord. Let L be an arbitrary set. By $\mathrm{Fin}\,L$ we shall denote the collection of all finite, non-empty subsets of L.

Let M be a subset of $\mathrm{Fin}L$. For $\sigma \in \{\emptyset\} \cup \mathrm{Fin}L$ we put

$$M^\sigma = \{\tau \in \mathrm{Fin}L\colon \sigma \cap \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

M^a abbreviates $M^{\{a\}}$.

2.1.1. DEFINITION. Define the *ordinal number* $\mathrm{Ord}M$ inductively as follows

$\mathrm{Ord}M = 0$ iff $M = \emptyset$,

$\mathrm{Ord}M \leqslant \alpha$ iff for every $a \in L$, $\mathrm{Ord}M^a < \alpha$,

$\mathrm{Ord}M = \alpha$ iff $\mathrm{Ord}M \leqslant \alpha$ and $\mathrm{Ord}M < \alpha$ is not true, and

$\mathrm{Ord}M = \infty$ iff $\mathrm{Ord}M > \alpha$ for every ordinal number α.

The proof of the following easy result is left as an exercise to the reader.

2.1.2. LEMMA. *Let L be a set and let M, $N \subset \mathrm{Fin}\,L$.*

(1) *If $\sigma, \tau \in \mathrm{Fin}L$ and $\sigma \cap \tau = \emptyset$ then $(M^\sigma)^\tau = M^{\sigma\cup\tau}$.*

(2) *If $\sigma, \tau \in \mathrm{Fin}L$ are both non-empty then $\sigma \in M^\tau$ if and only if $\tau \in M^\sigma$.*

(3) *If $N \subset M$ and $\mathrm{Ord}M$ exists then $\mathrm{Ord}\,N$ exists and $\mathrm{Ord}\,N \leqslant \mathrm{Ord}M$.*

We call a subset M of $\mathrm{Fin}L$ *inclusive* if: for every $\sigma, \sigma' \in \mathrm{Fin}L$ such that $\sigma \in M$ and $\sigma' \subset \sigma$ also $\sigma' \in M$.

2.1.3. LEMMA. *Let L be a set and let M be an inclusive subset of $\mathrm{Fin}L$. Then $\mathrm{Ord}M = \infty$ if and only if there exists a sequence $\{a_i\}_{i=1}^\infty$ of distinct elements of L such that $\sigma_n = \{a_i\}_{i=1}^n \in M$ for each $n \in N$.*

Proof. $\Rightarrow$: For some $a_1 \in L$, $\mathrm{Ord}M^{a_1} = \infty$. Then also for some $a_2 \in L-\{a_1\}$, $\mathrm{Ord}M^{\{a_1,a_2\}} = \infty$, etc. We see that we can find a sequence $a_1, a_2, \ldots$ of distinct elements of L such that for $\sigma_n = \{a_i\}_{i=1}^n$, $\mathrm{Ord}M^{\sigma_n} = \infty$; consequently $M^{\sigma_n} \neq \emptyset$. Since $\sigma_n \cup \sigma \in M$ for all $\sigma \in M^{\sigma_n}$ and M is inclusive we have $\sigma_n \in M$ for each n.

$\Leftarrow$: To the contrary, assume that $\mathrm{Ord}M$ exists. Since $\sigma_{n+1} = \sigma_n \cup \{a_{n+1}\} \in M$ and $a_{n+1} \notin \sigma_n$ we have $\mathrm{Ord}M^{\sigma_n} > 0$ for every n. By definition,

$$\mathrm{Ord}M^{\sigma_n} > \mathrm{Ord}M^{\sigma_n \cup \{a_{n+1}\}} = \mathrm{Ord}M^{\sigma_{n+1}}.$$

Consequently, we proved the existence of an infinite decreasing sequence of ordinal numbers, which is impossible. ■

2.1.4. LEMMA. *Let L be a set and let M be a subset of $\mathrm{Fin}L$. In addition, let $n \in \omega$. Then $\mathrm{Ord}M \leqslant n$ if and only if $|\sigma| \leqslant n$ for every $\sigma \in M$.*

Proof. $\Rightarrow$: By induction on the number n. If $n = 0$ then $M = \emptyset$ so there is nothing to prove. Assume that the implication "$\Rightarrow$" is true for $n-1$. Let M be such that $\mathrm{Ord}M \leqslant n$. Let $\sigma \in M$ and take $a \in \sigma$. Then $\mathrm{Ord}M^a \leqslant n-1$ so $|\sigma-\{a\}| \leqslant n-1$, i.e. $|\sigma| \leqslant n$.

$\Leftarrow$: Also by induction on n. If $n = 0$ then $M = \emptyset$ so $\mathrm{Ord}M = 0$. Assume that the implication "$\Leftarrow$" is true for $n-1$. Let M be such that $|\sigma| \leqslant n$ for every $\sigma \in M$. Pick $a \in L$. Then if $\sigma' \in M^a$ we have $\sigma' = \sigma-\{a\}$ for some $\sigma \in M$ with $a \in \sigma$. So $|\sigma'| \leqslant n-1$ for every $\sigma' \in M^a$. Therefore by our inductive hypothesis we have $\mathrm{Ord}M^a \leqslant n-1$ for every $a \in L$. Consequently, $\mathrm{Ord}M \leqslant n$. ■

2.1.5. LEMMA. *Let L be a set and let M be a subset of $\mathrm{Fin}L$. If $\mathrm{Ord}M^\gamma \geqslant \alpha+p$ for some $\gamma \in \{\emptyset\} \cup \mathrm{Fin}L$, some ordinal number α and integer $p \geqslant 0$, then $\mathrm{Ord}M^{\gamma\cup\sigma} \geqslant \alpha$ for some $\sigma \in \{\emptyset\} \cup \mathrm{Fin}L$ with $|\sigma| = p$ and $\gamma \cap \sigma = \emptyset$.*

Proof. By induction on p. If $p = 0$ then $\sigma = \emptyset$ is as required. Assume the lemma holds for $p-1$. Then, by definition, if

$$\mathrm{Ord}M^\gamma \geqslant \alpha+p = \alpha+1+p-1$$

then we can find some $\sigma' \in \{\emptyset\} \cup \mathrm{Fin}L$ such that $\mathrm{Ord}M^{\gamma\cup\sigma'} \geqslant \alpha+1$, $|\sigma'| = p-1$ and $\sigma' \cap \gamma = \emptyset$. Then by definition we can find some $a \in L$ with $a \notin \gamma \cup \sigma'$ and $\mathrm{Ord}M^{\gamma\cup\sigma'\cup\{a\}} \geqslant \alpha$. Put $\sigma = \sigma' \cup \{a\}$. ■

2.1.6. LEMMA. *Let $\Phi\colon L \to L'$ be a function from a set L to a set L' and let $M \subset \mathrm{Fin}L$ and $M' \subset \mathrm{Fin}L'$ be such that for every $\sigma \in M$ we have*

(1) *$\Phi(\sigma) \in M'$, and*

(2) *$|\Phi(\sigma)| = |\sigma|$.*

Then

$$\mathrm{Ord}M \leqslant \mathrm{Ord}M',$$

Proof. By transfinite induction on $\operatorname{Ord} M' = \alpha$. $\alpha = 0$: then $M' = \emptyset$ and hence $M = \emptyset$ so $\operatorname{Ord} M = 0$. Assume that the lemma holds for all ordinal numbers less than a given ordinal number α. Let M and M' be as in the lemma and let $\operatorname{Ord} M' = \alpha$. Consider M^a for a given $a \in L$. Let $\sigma \in M^a$. Then $\sigma \cup \{a\} \in M$ and $a \notin \sigma$. Let $b = \Phi(a)$ and $\sigma' = \Phi(\sigma)$. By (1): $\Phi(\sigma \cup \{a\}) = \sigma' \cup \{b\} \in M'$, and by (2) we have $\sigma' \cap \{b\} = \emptyset$. Consequently,

(1′) $\Phi(\sigma) = \sigma' \in (M')^b$.

By (2) we also have

(2′) $|\Phi(\sigma)| = |\sigma|$.

Then (1′) and (2′) imply that the properties (1) and (2) of Φ are also satisfied w.r.t. M^a and $(M')^b$. Observe that by definition,

$$\operatorname{Ord}(M')^b < \operatorname{Ord} M' = \alpha .$$

Hence by our inductive assumptions $\operatorname{Ord} M^a \leqslant \operatorname{Ord}(M')^b < \alpha$. Since $a \in L$ was arbitrary, $\operatorname{Ord} M \leqslant \operatorname{Ord} M' = \alpha$. ■

An alternative way of defining Ord comes from descriptive set theory, cf. [Mo].

Given a set L let $\operatorname{seq}(L)$ denote the set of finite sequences of elements of L. A *tree* on L is a subset T of $\operatorname{seq}(L)$ such that if $s \in T$ and if t is an initial segment of s then $t \in T$. A tree T is *well-founded* iff there is no sequence $\{s_n\colon n \in N\}$ in T such that for all n, $\operatorname{dom} S_n = n$ and $s_n \subset s_{n+1}$, i.e. T has no infinite branch.

Next given T and $a \in L$ we set

$$T(a) = \{s \in \operatorname{seq}(L)\colon \{a\}^\frown s \in T\} .$$

One then defines the *rank* of well-founded trees as follows:

$\operatorname{Rank}(\emptyset) = 0$;
$\operatorname{Rank}(T) \leqslant \alpha$ iff for all $a \in L$ $\operatorname{Rank}(T(a)) < \alpha$;
$\operatorname{Rank}(T) = \alpha$ iff $\operatorname{Rank}(T) \leqslant \alpha$ and not $\operatorname{Rank}(T) < \alpha$;
$\operatorname{Rank}(T) = \infty$ iff T is not well-founded;

To connect Rank and Ord we define for $M \subset \operatorname{Fin} L$ a tree T_M by

$$T_M = \{s \in \operatorname{seq}(L)\colon s \text{ is 1-1 and for some } \sigma \in M \ \operatorname{ran}(s) \subset \sigma\} .$$

One can show that $\operatorname{Ord} M = \operatorname{Rank}(T_M)$.

2.2. Sum theorems. Let us now prove some sum theorems on Ord.

2.2.1. LEMMA. *Let L be a set and let M, M_1 and M_2 be subsets of* $\operatorname{Fin} L$ *such that $M \subset M_1 \cup M_2$. Then*

$$\operatorname{Ord} M \leqslant \max(\operatorname{Ord} M_1, \operatorname{Ord} M_2) .$$

Proof. Induction on $\alpha = \max(\operatorname{Ord} M_1, \operatorname{Ord} M_2)$. If $\alpha = 0$ then $M_1 = M_2 = \emptyset$; hence $M = \emptyset$. If $\alpha > 0$ observe that for every $a \in L$,

$$M^a \subset M_1^a \cup M_2^a ,$$

so that $\operatorname{Ord} M^a \leqslant \max(\operatorname{Ord} M_1^a, \operatorname{Ord} M_2^a) < \alpha$ by the inductive assumption it follows that $\operatorname{Ord} M \leqslant \alpha$. ■

For the next sum theorem we need the notion of a lower sum for ordinal numbers as introduced by G. H. Toulmin [T].

2.2.2. DEFINITION [T], [Pe]. Let α and β be ordinal numbers such that $\alpha = \alpha' + p$ and $\beta = \beta' + q$, where α' and β' are limit ordinals and p and q are integers. Then the *lower sum* $\alpha \oplus \beta$ of α and β is defined as follows:

$$\alpha \oplus \beta = \begin{cases} \alpha & \text{if } \alpha' > \beta' , \\ \alpha + q = \beta + p & \text{if } \alpha' = \beta' , \\ \beta & \text{if } \alpha' < \beta' . \end{cases}$$

Let α and β be ordinal numbers. If $\alpha \geqslant \beta$ then let $\Phi(\alpha, \beta) = \alpha \oplus (\beta + 1)$. Similarly, if $\alpha \leqslant \beta$ then let $\Phi(\alpha, \beta) = \beta \oplus (\alpha + 1)$. For this function the following holds:

2.2.3. LEMMA. *Let α and β be two ordinal numbers. Then*

(1) $\Phi(\alpha, \beta) = \Phi(\beta, \alpha)$,
(2) *if $\beta \leqslant \alpha$ then $\Phi(\alpha, \beta) > \beta$, and*
(3) *if $\beta \leqslant \alpha$ and $\alpha' < \alpha$ then $\Phi(\alpha, \beta) > \Phi(\alpha', \beta)$.*

Proof. (1) if $\alpha \geqslant \beta$ then $\Phi(\alpha, \beta) = \alpha \oplus \beta + 1$ and $\Phi(\beta, \alpha) = \alpha \oplus \beta + 1$.

(2) if $\beta = \alpha$ then $\Phi(\alpha, \alpha) = \alpha \oplus \alpha + 1 > \alpha$,
if $\beta < \alpha$ then $\Phi(\alpha, \beta) \geqslant \alpha > \beta$,

(3) if $\lambda(\alpha') < \lambda(\beta)$ then $\Phi(\alpha', \beta) = \beta < \Phi(\alpha, \beta)$,
if $\lambda(\alpha') = \lambda(\beta)$ then $\Phi(\alpha', \beta) = \lambda(\alpha') + n(\alpha') + n(\beta) + 1 < \Phi(\alpha, \beta)$,
if $\lambda(\alpha') > \lambda(\beta)$ then $\Phi(\alpha', \beta) = \alpha' < \alpha = \Phi(\alpha, \beta)$. ■

2.2.4. LEMMA. *Let L be a set and let M, M_1 and M_2 be inclusive subsets of* $\operatorname{Fin} L$, *such that whenever $\sigma \in M$ is indexed as $\{a_1, \ldots, a_n\}$ there is an $i \in \{1, \ldots, n\}$ such that*

$$\{a_j\colon j < i\} \in M_1 \cup \{\emptyset\} \quad \textit{and} \quad \{a_j\colon j > i\} \in M_2 \cup \{\emptyset\} .$$

Then whenever $\operatorname{Ord} M_1 \leqslant \alpha$ *and* $\operatorname{Ord} M_2 \leqslant \beta$,

$$\operatorname{Ord} M \leqslant \Phi(\alpha, \beta) .$$

Proof. The relations $\leqslant$ on the class of pairs of ordinal numbers defined by

$$\langle \alpha, \beta \rangle \leqslant \langle j, \delta \rangle \quad \text{iff} \quad \alpha \leqslant j \text{ and } \beta \leqslant \delta$$

is well founded, so it suffices to prove the lemma inductively w.r.t. this relation.

If $\langle \alpha, \beta \rangle = \langle 0, 0 \rangle$ then $M_1 = M_2 = \emptyset$ and hence for every $\sigma \in M$ we have $|\sigma| \leqslant 1$. Consequently by Lemma 2.1.4, $\operatorname{Ord} M \leqslant 1 = \Phi(0, 0)$.

Let $\langle \alpha, \beta \rangle > \langle 0, 0 \rangle$ and assume that the lemma holds for all pairs $\langle \gamma, \delta \rangle$ with $\langle \gamma, \delta \rangle < \langle \alpha, \beta \rangle$.

CLAIM. *Let $a \in L$ and $\sigma = \{a_1, \ldots, a_n\} \in M^a$. Then there are i_1 and i_2 in $\{1, \ldots, n\}$ such that*

(1) $\{a_j\colon 1 \leqslant j < i_1\} \in M_1^a \cup \{\emptyset\}$ *and* $\{a_j\colon j > i_1\} \in M_2 \cup \{\emptyset\}$, *and*
(2) $\{a_j\colon 1 \leqslant j < i_2\} \in M_1 \cup \{\emptyset\}$ *and* $\{a_j\colon j > i_2\} \in M_2^a \cup \{\emptyset\}$.

Proof of the claim. To find i_1 put $a = a_0$ and find $i \in \{0, 1, \ldots, n\}$ such that $\{a_j\colon 0 \leqslant j < i\} \in M_1 \cup \{\emptyset\}$ and $\{a_j\colon j > i\} \in M_2 \cup \{\emptyset\}$. Now put $i_1 = \max(i, 1)$. Then $\{a_j\colon j > i_1\} \in M_2 \cup \{\emptyset\}$ because M_2 is inclusive and

$$\{a_j\colon 1 \leqslant j < i_1\} = \{a_j\colon 1 \leqslant j < i\} \in M_1^a \cup \{\emptyset\}.$$

To find i_2 put $a_{n+1} = a$ and do the same thing.

Case 1. $\alpha \geqslant \beta$. Let $a \in L$. By the claim M^a, M_1^a and M_2 satisfy the conditions of the lemma, while moreover

$$\operatorname{Ord} M_1^a < \alpha' < \alpha \quad \text{for some } \alpha'.$$

Then $\langle \alpha', \beta \rangle < \langle \alpha, \beta \rangle$ so by assumption and Lemma 2.2.3(3)

$$\operatorname{Ord} M^a \leqslant \Phi(\alpha', \beta) < \Phi(\alpha, \beta).$$

Case 2. $\alpha \leqslant \beta$. Then for $a \in L$ the sets M^a, M_1 and M_2^a are as in the lemma and

$$\operatorname{Ord} M_2^a \leqslant \beta' < \beta \quad \text{for some } \beta'.$$

From our assumption and Lemma 2.2.3(1) and (3) we then obtain

$$\operatorname{Ord} M^a \leqslant \Phi(\alpha, \beta') < \Phi(\alpha, \beta).$$

In both cases $\operatorname{Ord} M^a < \Phi(\alpha, \beta)$. Consequently, $\operatorname{Ord} M \leqslant \Phi(\alpha, \beta)$. ■

2.3. The Brouwer-Kleene order and Ord. In [P2], R. Pol introduced a transfinite classification of w.i.d. compact metric spaces by means of topological invariant called index. For this he uses the *Brouwer-Kleene order* on $\operatorname{Fin} N$, which is defined as follows:

2.3.1. DEFINITION. For every $\sigma, \tau \in \operatorname{Fin} N$ let $\sigma < \tau$ if there is an $n \in N$ such that

$$\sigma \cap \{1, \ldots, n-1\} = \tau \cap \{1, \ldots, n-1\} \quad \text{and} \quad n \in \sigma - \tau.$$

When a subset M of $\operatorname{Fin} N$ is well-ordered w.r.t. the Brouwer–Kleene order we denote by *type* M the ordertype of M.

2.3.2. DEFINITION. A subset M of $\operatorname{Fin} N$ satisfies *property* (*) if:

(a) M is inclusive,
(b) for every $\sigma \in M$ we can find infinitely many pairwise disjoint $\sigma' \in M$ such that

(b1): $\sigma \cap \sigma' = \emptyset$,
(b2): $|\sigma| = |\sigma'|$, and
(b3): $M^\sigma \subset M^{\sigma'}$

The following lemma is useful in transfinite induction proofs involving property (*).

2.3.3. LEMMA. *Let M be a subset of $\operatorname{Fin} N$. If M has property (*) then for each $a \in N$, M^a has property (*).*

Proof. Clearly M^a is inclusive, so (a) holds. For (b) let $\sigma \in M^a$. Observe that $M^a \subset M$ (M is inclusive) so that $\sigma \in M$. Find infinitely many pairwise disjoint $\sigma' \in M$ satisfying (b1), (b2) and (b3). For such a $\sigma' \in M^a$ for $\sigma \in M^a$ so $\{a\} \in M^\sigma \subset M^{\sigma'}$ and so $\sigma' \in M^a$. Next, (b1) and (b2) have nothing to do with a and for (b3) simply observe that

$$(M^a)^\sigma = (M^\sigma)^a \subset (M^{\sigma'})^a = (M^a)^{\sigma'}. \ ■$$

2.3.4. LEMMA. *Let M be a subset of $\operatorname{Fin} N$ having property (*) and let B be a cofinite subset of N. Put $M_B = M \cap \operatorname{Fin} B$.*

Then M_B has property (*) and if $\operatorname{Ord} M$ exists then $\operatorname{Ord} M_B$ exists and $\operatorname{Ord} M_B = \operatorname{Ord} M$.

Proof. It is readily seen that M_B has property (*). As $M_B \subset M$, $\operatorname{Ord} M_B \leqslant \operatorname{Ord} M$. Next if $\operatorname{Ord} M = 0$ then $\operatorname{Ord} M_B = 0$ as well. So assume $\operatorname{Ord} M > 0$ and assume that the lemma holds for all subsets $N \subset \operatorname{Fin} N$ with $\operatorname{Ord} N < \operatorname{Ord} M$. It suffices to show that

$$\text{for every } a \in N \text{ there is some } b \in N \text{ such that } \operatorname{Ord} M^a \leqslant \operatorname{Ord}(M_B)^b.$$

If $M^a = \emptyset$ then any $b \in B$ will suffice. If $M^a \neq \emptyset$ then $\{a\} \in M$ for M is inclusive. So find infinitely many $b \in N$ with $M^a \subset M^b$. As B is cofinite we can pick such a b in B. Then for this b:

$$\operatorname{Ord} M^b \geqslant \operatorname{Ord} M^a.$$

Also because $b \in B$, $(M_B)^b = M^b \cap \operatorname{Fin} B$ (if $c \notin B$ then $(M_B)^c = \emptyset$) so by Lemma 2.3.3 and our inductive assumption

$$\operatorname{Ord}(M_B)^b = \operatorname{Ord} M^b \geqslant \operatorname{Ord} M^a. \ ■$$

2.3.5. LEMMA. *Let $M \subset \operatorname{Fin} N$ be inclusive.*

Then $\operatorname{Ord} M$ exists iff M is well-ordered w.r.t. the Brouwer-Kleene order.

Proof. We prove the contrapositive. (i) If $\operatorname{Ord} M = \infty$ then, by Lemma 2.1.3, there is a sequence $\{a_n\colon n \in N\}$ in N such that for every n $\{a_1, \ldots, a_n\} \in M$. Using the fact that M is inclusive, we may assume that $a_1 < a_2 < \ldots$. But then $\langle \{a_1, \ldots, a_n\}\colon n \in N \rangle$ is a strictly decreasing sequence in the Brouwer–Kleene order.

(ii) In what follows whenever $\sigma \in \operatorname{Fin} N$ is written as $\{a_1, \ldots, a_n\}$ it is implicitly assumed that $a_1 < a_2 < \ldots < a_n$. Now observe that if $\sigma = \{a_1, \ldots, a_n\}$ and $\tau = \{b_1, \ldots, b_m\}$ then

$\sigma < \tau$ iff there is an i such that for $j < i$ $a_j = b_j$ and either $i \leqslant m$, n and $a_i < b_i$ or $m = i-1 < n$.

If M is not well-ordered by the Brouwer Kleene order we can find $\langle\sigma_n\colon n\in N\rangle$ in M with $\sigma_1>\sigma_2>\sigma_3>\ldots$ Write $\sigma_i=\{a_1^i,\ldots,a_{m_i}^i\}$. It follows that $a_1^1\geqslant a_1^2\geqslant a_1^3\geqslant\ldots$ so there is an i_1 such that $a_1^i=a_1^{i_1}$, for $i\geqslant i_1$.

Then $a_2^{i_1}\geqslant a_2^{i_1+1}\geqslant\ldots$ so there is an $i_2\geqslant i_1$ such that $a_2^i=a_2^{i_2}$ for $i\geqslant i_2$. And so on.

Then $\{a_1^{i_1},\ldots,a_b^{i_n}\}\in M$ for every n because M is inclusive.

Consequently by Lemma 2.1.3, $\operatorname{Ord}M=\infty$. ■

2.3.6. DEFINITION. Let A and B be sets well ordered by the order $<_A$ and $<_B$ respectively. On $A\times B$ we define the *lexicografic order* $<_l$ as follows:

$$(a_1,b_1)<_l(a_2,b_2)\quad\text{iff}\quad a_1<_A a_2\ \text{or}\ a_1=a_2\ \text{and}\ b_1<_B b_2.$$

Observe that $<_l$ well orders $A\times B$.

For each ordinal α let $A_\alpha=\{\beta\colon\beta<\alpha\}$. Clearly A_α is well ordered by $<$. We define for each pair of ordinal numbers α and β the *product* $\alpha\times\beta$ as follows:

$$\alpha\times\beta=\text{the ordertype of } A_\beta\times A_\alpha \text{ with respect to the lexicografic order.}$$

For each ordinal number α we define ω_0^α as follows:

$\alpha=0$: $\omega_0^\alpha=1$,

$\alpha=\beta+1$: $\omega_0^\alpha=\omega_0^\beta\omega_0$,

α limit ordinal: $\omega_0^\alpha=\sup\{\omega_0^\beta\colon\beta<\alpha\}$.

Let $M\subset\operatorname{Fin}N$ be endowed with the Brouwer-Kleene order.

Then we put $M(i)=\{\sigma\in M\colon\{i-1\}<\sigma\leqslant\{i\}\}$ $(=\{\sigma\in M\colon i=\min\sigma\})$ and $M(\leqslant i)=\{\sigma\in M\colon\sigma\leqslant\{i\}\}$.

Observe that $M(i)\cap M(j)=\emptyset$ if $i\neq j$ and also that $M(\leqslant i)\subset M(\leqslant j)$ if $i\leqslant j$.

2.3.7. Remark. Let M be well ordered. Observe that if type $M(i)\geqslant\alpha$ for infinitely many i then

$$\operatorname{type}M\geqslant\alpha\cdot\omega_0.$$

2.3.8. Remark. Let M be well ordered. Then

$$\operatorname{type}M=\sup\{\operatorname{type}M(\leqslant i)\colon i\in N\}.$$

2.3.9. THEOREM. *Let M be a non-empty subset of* $\operatorname{Fin}N$ *with property* $(*)$. *Then* (i) $\operatorname{Ord}M$ *exists iff M is well ordered by the Brouwer–Kleene order and* (ii) *in that case*

$$\operatorname{type}M=\omega_0^{\operatorname{Ord}M}.$$

Proof. Lemma 2.3.5 gives (i).

We prove (ii) by induction on $\operatorname{Ord}M$ (note that (ii) is false for $M=\emptyset$: $0\neq 1$). In case $\operatorname{Ord}M=1$, M consists of infinitely many singletons, Lemma 2.1.4. Noting that $i<j$ iff $\{i\}<\{j\}$ (in the B–K order) we obtain

$$\operatorname{type}M=\omega_0=\omega_0^1.$$

Let $\operatorname{Ord}M=\alpha>1$ and assume that type $N=\omega_0^{\operatorname{Ord}N}$ whenever N has property $(*)$ and $\operatorname{Ord}N<\alpha$.

(a) $\omega_0^{\operatorname{Ord}M}\leqslant\operatorname{type}M$.

It suffices to show that for each $\beta<\alpha$: $\omega_0^{\beta+1}\leqslant$ type M. Let β be an ordinal number such that $\beta<\alpha$ and $\beta>0$. Then by the definition of $\operatorname{Ord}M$ we can find an $i\in N$ such that

$$\operatorname{Ord}M^i\geqslant\beta>0.$$

By property $(*)$, we can find an increasing sequence $j(1)<j(2)<\ldots$ of natural numbers such that for every $k\in N$ we have $M^i\subset M^{j(k)}$. This yields $\operatorname{Ord}M^{j(k)}\geqslant\beta>0$ for $k=1,2,\ldots$ Set $\Gamma_k=M^{j(k)}\cap\{\operatorname{Fin}(N-\{1,\ldots,j(k)\})\}$. Observe that $M^{j(k)}$ has property $(*)$ by Lemma 2.3.3, and hence so does Γ_k by Lemma 2.3.4. According to Lemma 2.3.4, $\operatorname{Ord}\Gamma_k=\operatorname{Ord}M^{j(k)}\geqslant\beta>0$. Define $\Phi\colon\Gamma_k\to M(j(k))$ as follows $\Phi(\sigma)=\sigma\cup\{j(k)\}$.

Then Φ is an order-isomorphic imbedding of Γ_k into $M(j(k))$ when we consider Γ_k endowed with the Brouwer–Kleene order.

This situation and the inductive hypothesis give

$$\omega_0^\beta\leqslant\operatorname{type}\Gamma_k=\operatorname{type}\Phi(\Gamma_k)\leqslant\operatorname{type}M(j(k))\quad\text{for every } k=1,2,\ldots$$

Considering the sequence of all $M(j(k))$ within M we obtain

$$\omega_0^\beta\cdot\omega_0=\omega_0^{\beta+1}\leqslant\operatorname{type}M.$$

(b) $\operatorname{type}M\leqslant\omega_0^{\operatorname{Ord}M}$.

By definition, for all $i\in N$, $\operatorname{Ord}M^i=\beta_i<\alpha$.

Hence by our inductive hypothesis type $M^i\leqslant\omega_0^{\operatorname{Ord}M^i}=\omega_0^{\beta_i}$.

Put $\Gamma_i=M(i)-\{i\}$.

Define $\Phi\colon\Gamma_i\to M^i$ as follows: $\Phi(\sigma)=\sigma-\{i\}$. Then Φ is an order-isomorphic imbedding of Γ_i into M^i. Hence type $\Gamma_i\leqslant$ type $M^i=\omega_0^{\beta_i}$ and

(1) $\operatorname{type}M(i)\leqslant\operatorname{type}\ \Gamma_i+1\leqslant\omega_0^{\beta_i}+1$.

For every $i\in N$ let

(2) $\gamma_i=\max\{\beta_j\colon j\leqslant i\}$.

Then

(3) $\alpha=\sup_i(\gamma_i+1)$.

Then by the way the subsets $M(i)$ are ordered in $M(\leqslant i)$ and $M(\leqslant i)=\bigcup_{j=1}^{i}M(i)$

$$\begin{aligned}\operatorname{type}\ M(\leqslant i)&\leqslant\max\{\operatorname{type}\ M(j)\colon j\leqslant i\}\cdot i\\&\leqslant\max\{\omega_0^{\beta_j}+1\colon j\leqslant i\}\cdot i&&\text{(by (1))}\\&\leqslant(\omega_0^{\gamma_i}+1)\cdot i&&\text{(by (2))}\\&\leqslant(\omega_0^{\gamma_i}+\omega_0^{\gamma_i})\cdot i=\omega_0^{\gamma_i}\cdot 2i\\&\leqslant\omega_0^{\gamma_i}\cdot\omega_0=\omega_0^{\gamma_i+1}.\end{aligned}$$

Consequently, (3) and type $M = \sup\{\text{type}\, M(\leqslant i)\colon i \in N\}$ yield

$$\text{type } M \leqslant \sup_i \{\omega_0^{\gamma_i+1}\} = \omega_0^{\alpha},$$

which proves our theorem. ■

Chapter III. A transfinite extension of the covering dimension

Using the ordinal invariant Ord discussed in Chapter II, we shall define a transfinite extension of the covering dimension dim. We show that for a space X $\dim X$ exists if and only if X is S-w.i.d.

In [P2] R. Pol developed a classification of w.i.d. compact spaces by assigning to such a space X the ordinal number index X. We will prove that for these spaces dim and index give the same classification; in fact the following equality holds for a w.i.d. compact space X with $\dim X \geqslant 1$:

$$\operatorname{index} X = \omega_0^{\dim X} \quad \text{(ordinal exponentiation)}.$$

We will also extend in this chapter some theorems from the finite case; e.g. the subspace theorem and the inequality $\dim X \leqslant \operatorname{Ind} X$. We will give an example of a compact space X such that $\dim X = \omega_0$ and $\operatorname{Ind} X = \omega_0 + 1$ in Section 5.1.

We shall prove a sum theorem for dim which is similar to the one for Ind proved by A. R. Pears [Pe]. We shall also prove a product theorem. Both theorems will be needed in the next chapter on essential mappings.

3.1. Transfinite covering dimension. In this section we introduce the transfinite covering dimension and derive some of its basic properties.

Let X be a space. Define

$$L(X) = \{(A, B)\colon A, B \subset X, \text{ closed, disjoint}\}.$$

For arbitrary $L \subset L(X)$, we set

$$M_L = \{\sigma \in \operatorname{Fin} L\colon \sigma \text{ is essential in } X\}.$$

The following theorem inspired our definition of the transfinite covering dimension.

3.1.1. Theorem. *Let X be a space and $n \in \omega$. Then*

$$\operatorname{Ord} M_{L(X)} \leqslant n \textit{ if and only if } \dim X \leqslant n.$$

Proof. By Lemma 2.1.4 $\operatorname{Ord} M_{L(X)} \leqslant n$ iff for every $\sigma \in M_{L(X)}$ $|\sigma| \leqslant n$. By Theorem 1.1.2,

$$\dim X \leqslant n \quad \text{iff} \quad \text{for every } \sigma \in M_{L(X)}\ |\sigma| \leqslant n. ■$$

We now give our transfinite extension of the covering dimension.

3.1.2. Definition. For a space X we set

$$\dim X = \operatorname{Ord} M_{L(X)}.$$

Recall that by convention "$\dim X = \infty$" is synonimous with "$\dim X$ does not exist".

We now delineate the class of spaces for which $\dim X$ does exist.

3.1.3. Theorem. *For a space X*

$\dim X$ exists if and only if X is S-w.i.d.

Proof. We prove the contrapositive:
First observe that a sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ in $L(X)$ is essential iff $\{(A_i, B_i)\}_{i=1}^{n} \in M_{L(X)}$ for every n.

Now we see that $\dim X = \infty$

iff there is a sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ in $L(X)$ such that $\{(A_i, B_i)\}_{i=1}^{n} \in M_{L(X)}$ for every n (Lemma 2.1.3);
iff there is an essential sequence in $L(X)$;
iff X is not S-w.i.d. ■

Next we prove a subspace theorem. In the process we introduce a few notions which will be useful later on.

Let X be a space, $F \subset X$ be closed and $L \subset L(X)$. Put

$$L|F = \{(A \cap F, B \cap F)\colon (A, B) \in L\}.$$

Observe that

$$L(X)|F = L(F)$$

and consequently,

$$M_{L(X)|F} = M_{L(F)}.$$

Define $\Phi_F\colon L(X) \to L(F)$ by

$$\Phi_F(A, B) = (A \cap F, B \cap F).$$

Observe that

$$L|F = \Phi_F(L).$$

Finally, put

$$\tilde{M}_{L|F} = \{\sigma = \{(A_i, B_i)\}_{i=1}^{n} \in \operatorname{Fin} L\colon \text{for all open } O_i \subset X \text{ with}$$
$$A_i \subset O_i \subset \overline{O}_i \subset X - B_i,\ 1 \leqslant i \leqslant n, \text{ we have } F \cap \bigcap_{i=1}^{n} \operatorname{Fr} O_i \neq \emptyset\}.$$

3.1.4. Lemma. *Let $\{(A_i, B_i)\}_{i=1}^{n}$ be an essential family in a space X. Then $(A_i, B_i) \neq (A_j, B_j)$ for all distinct $i, j \in \{1, \ldots, n\}$.*

Proof. Suppose $(A_{i'}, B_{i'}) = (A_{j'}, B_{j'})$ for certain distinct $i', j' \in \{1, \ldots, n\}$. By use of normality of X we can find open sets $O_{i'}$ and $O_{j'}$ in X such that

$$A_{i'} = A_{j'} \subset O_{i'} \subset \overline{O}_{i'} \subset O_{j'} \subset \overline{O}_{j'} \subset X - B_{j'} = X - B_{i'}.$$

For each $i \in \{1, \ldots, n\} - \{i', j'\}$ let O_i be an arbitrary open set satisfying

$$A_i \subset O_i \subset \bar{O}_i \subset X - B_i .$$

Observe that for each $i = 1, \ldots, n$, $A_i \subset O_i \subset \bar{O}_i \subset X - B_i$ and

$$\bigcap_{i=1}^{n} \operatorname{Fr} O_i \subset \operatorname{Fr} O_{i'} \cap \operatorname{Fr} O_{j'} = \emptyset .$$

This implies that $\{(A_i, B_i)\}_{i=1}^n$ is inessential. Contradiction. ■

3.1.5. LEMMA. $\tilde{M}_{L|F} = \{\sigma \in \operatorname{Fin} L$: *$\sigma|F$ is essential in F and* $|\sigma|F| = |\sigma|\}$.

Proof. The inclusion $\subset$ follows from Lemma 1.1.4 and 3.1.4.

For $\supset$, note that if $\{(A_i \cap F, B_i \cap F)\}_{i=1}^n$ is essential in F and if

$$A_i \subset O_i \subset \bar{O}_i \subset X - B_i$$

for $i = 1, \ldots, n$ then

$$A_i \cap F \subset O_i \cap F \subset \overline{O_i \cap F} \subset F - B_i \quad \text{and} \quad \operatorname{Fr}_F(O_i \cap F) \subset \operatorname{Fr} O_i \cap F$$

for $i = 1, \ldots, n$, so that certainly $F \cap \bigcap_{i=1}^{n} \operatorname{Fr} O_i \neq \emptyset$.

3.1.6. THEOREM. *Let F be a closed subset of a space X. Then*

$$\dim F = \operatorname{Ord} \tilde{M}_{L(X)|F} \leqslant \dim X .$$

Proof. Note that $\tilde{M}_{L(X)|F} \subset M_{L(X)}$ from which follows that

$$\operatorname{Ord} \tilde{M}_{L(X)|F} \leqslant \dim X .$$

Clearly, $M_{L(F)} \subset \tilde{M}_{L(X)|F}$; consequently

$$\dim F \leqslant \operatorname{Ord} \tilde{M}_{L(X)|F} .$$

Define $\Phi_F\colon L(X) \to L(F)$ as above. Then for every $\sigma \in \tilde{M}_{L(X)|F}$

$$\Phi_F(\sigma) \in M_{L(F)} \quad \text{and} \quad |\sigma| = |\Phi_F(\sigma)| .$$

Consequently, by Lemma 2.1.6,

$$\operatorname{Ord} \tilde{M}_{L(X)|F} \leqslant \operatorname{Ord} M_{L(F)} = \dim F . \ ■$$

The space $X = \bigoplus_{n=1}^{\infty} I^n$ is not S-w.i.d. (apply Theorem 1.2.6). Then, by Theorem 3.1.3, $\dim X = \infty$. If ωX is the one point compactification of X then $\dim \omega X = \omega_0$. Consequently, Theorem 3.1.6 is not valid for arbitrary subspaces.

3.2. Dim and Ind. We study the relation between dim and the large transfinite dimension Ind. We will prove the inequality $\dim X \leqslant \operatorname{Ind} X$.

3.2.1. PROPOSITION. *Let X be a space, let $\tau = \{(A_i, B_i)\}_{i=1}^n \in \operatorname{Fin} L(X)$ and for $i = 1, \ldots, n$ let O_i be an open set in X such that*

$$A_i \subset O_i \subset \bar{O}_i \subset X - B_i .$$

Then for $F = \bigcap_{i=1}^{n} \operatorname{Fr} O_i$ we have

(1) $M^{\tau}_{L(X)} \subset \tilde{M}_{L(X)|F}$ *and hence*
(2) $\operatorname{Ord} M^{\tau}_{L(X)} \leqslant \dim F$.

Proof. Let $\sigma = \{(A_i, B_i)\}_{i=n+1}^m \in M^{\tau}_{L(X)}$. Then $\sigma \cup \tau \in M_{L(X)}$. Hence for open sets O_i in X, $i = n+1, \ldots, m$, such that $A_i \subset O_i \subset \bar{O}_i \subset X - B_i$ we have

$$\emptyset \neq \bigcap_{i=n+1}^{m} O_i \cap \bigcap_{i=1}^{n} O_i = \bigcap_{i=n+1}^{m} O_i \cap F .$$

Consequently, $\sigma \in \tilde{M}_{L(X)|F}$.

Assertion (2) holds since by Theorem 3.1.6 and (1) we must have

$$\operatorname{Ord} M^{\tau}_{L(X)} \leqslant \operatorname{Ord} \tilde{M}_{L(X)|F} = \dim F . \ ■$$

3.2.2. COROLLARY. *Let X be a space. If for $(A, B) \in L(X)$ we can find an open set O in X such that*

$$A \subset O \subset \bar{O} \subset X - B \quad \textit{and} \quad \dim \operatorname{Fr} O \leqslant \alpha ,$$

then $\operatorname{Ord} M^{(A,B)}_{L(X)} \leqslant \alpha$. ■

3.2.3. COROLLARY. *Let X be a space such that for every pair (A, B) of disjoint closed sets in X we can find an open set O in X such that*

$$A \subset O \subset \bar{O} \subset X - B \quad \textit{and} \quad \dim \operatorname{Fr} O < \alpha .$$

Then $\dim X \leqslant \alpha$. ■

3.2.4. THEOREM. *Let X be a space such that* $\operatorname{Ind} X$ *exists. Then*

$$\dim X \leqslant \operatorname{Ind} X .$$

Proof. By transfinite induction on $\operatorname{Ind} X = \alpha$.

The case $\alpha = -1$ is clear.

Assume the theorem holds for spaces with large transfinite dimension less than α. Let X be a space with $\operatorname{Ind} X = \alpha$. Then for every pair (A, B) of disjoint closed sets in X we can find an open set O in X such that

$$A \subset O \subset \bar{O} \subset X - B \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} O < \alpha .$$

Since our inductive hypothesis $\dim \operatorname{Fr} O \leqslant \operatorname{Ind} \operatorname{Fr} O < \alpha$ for every such open set O our theorem follows from Corollary 3.2.3. ■

In Section 5.1 we will exhibit a compact space X satisfying $\dim X = \omega_0$ and $\operatorname{Ind} X = \omega_0 + 1$. Consequently, Theorem 3.2.4 is best possible.

3.3. The relation with R. Pol's classification of w. i. d. **compact metric spaces.** In [P2], R. Pol introduced a transfinite classification of w.i.d. compact spaces by means of a topological invariant called *index*. We shall prove in this section that for a w.i.d. compact space X such that $\dim X \geqslant 1$ we have

$$\operatorname{index} X = \omega_0^{\dim X} \quad \text{(ordinal exponentiation)}.$$

We may therefore conclude that both classifications are equivalent and that the properties of index shown in [P2] are also valid for dim. Let us introduce R. Pol's classification.

3.3.1. DEFINITION. Let X be a compact metric space. We say that a sequence $S = \{(A_i, B_i)\colon i \in N\}$ of pairs of disjoint closed sets in X is *separating* if for each pair (A, B) of disjoint closed sets in X the inclusions $A \subset A_i$ and $B \subset B_i$ hold for infinitely many i. Moreover, we define for such a separating sequence S,

$$M_S = \{\sigma \in \operatorname{Fin} N\colon \{(A_i, B_i)\}_{i\in\sigma} \text{ is essential in } X\}$$

(this motivated our definition of M_L).

We quote the following two results from [P2].

3.3.2. THEOREM. *Let S be a separating sequence in a compact metric space X. Then X is* w.i.d. *if and only if M_S is well-ordered w.r.t. the Brouwer- Kleene order.* ■

3.3.3. PROPOSITION. *Let S and S' be two separating sequences in a* w.i.d. *compact metric space X. Then*

$$\operatorname{type} M_S = \operatorname{type} M_{S'}. \ ■$$

The reader should compare the first statement with Theorem 3.1.3. It follows that for a w.i.d. compact metric space we may define $\operatorname{index} X$ to be type M_S for some separating sequence S.

3.3.4. PROPOSITION. *Let X be a space and let L and L' be collections of pairs of disjoint closed sets in X such that for every pair (A, B) in L there is a pair (A', B') in L' with $A \subset A'$ and $B \subset B'$. Then*

$$\operatorname{Ord} M_L \leqslant \operatorname{Ord} M_{L'}.$$

Proof. Define a function $\Phi\colon L \to L'$ by taking for every $(A, B) \in L$ an element $(A', B') = \Phi(A, B) \in L'$ as above. We will prove that Φ satisfies the conditions of Lemma 2.1.6.

Consider some $\sigma \in M_L$. Then

(1) $\Phi(\sigma)$ is essential, so that $\Phi(\sigma) \in M_{L'}$,

and by Lemma 3.1.4,

(2) $|\sigma| = |\Phi(\sigma)|$.

So we conclude that the conditions of Lemma 2.1.6 are satisfied. Consequently, since $\operatorname{Ord} M_{L'}$ exists, we have $\operatorname{Ord} M_L \leqslant \operatorname{Ord} M_{L'}$. ■

Let us state the following corollary to Proposition 3.3.4:

3.3.5. COROLLARY. *Let X be a space and let L be a collection of pairs of disjoint closed sets in X such that for every pair (F, G) of disjoint closed sets in X we have some $(A, B) \in L$ such that $F \subset A$ and $G \subset B$. Then*

$$\operatorname{Ord} M_L = \operatorname{Ord} M_{L(X)}. \ ■$$

3.3.6. LEMMA. *Let L be a collection of pairs of disjoint closed sets in a space X. Moreover, let $\sigma = \{(A_i, B_i)\}_{i=1}^n$ and $\sigma' = \{(C_i, D_i)\}_{i=1}^n \in \operatorname{Fin} L$ be such that $A_i \subset C_i$ and $B_i \subset D_i$ for every $i \in \{1, \dots, n\}$. Then*

$$M_L^\sigma \subset M_L^{\sigma'}.$$

Proof. Let $\gamma = \{(A_i, B_i)\}_{i=n+1}^m \in M_L^\sigma$. Then $\gamma \cup \sigma \in M_L$ and $\gamma \cap \sigma = \varnothing$. Putting $(C_i, D_i) = (A_i, B_i)$ for $i = n+1, \dots, m$, we see that $\gamma \cup \sigma = \{(A_i, B_i)\}_{i=1}^m$ and $\gamma \cup \sigma' = \{(C_i, D_i)\}_{i=1}^m$ satisfy $A_i \subset C_i$ and $B_i \subset D_i$ for $i = 1, \dots, m$. Then it is easily seen that

$$\gamma \cup \sigma' \in M_L.$$

Consequently, by Lemma 3.1.4, $\gamma \cap \sigma' = \varnothing$ and therefore $\gamma \in M_L^{\sigma'}$. ■

3.3.7. LEMMA. *Let $S = \{(A_i, B_i)\colon i \in N\}$ be a separating sequence in a space X. Then M_S satisfies property* $(*)$ *(Definition 2.3.2).*

Proof. (a) Follows fromt he fact that every subset of an essential family is itself essential.

(b) If $\sigma \in M_S$ then $\{(A_i, B_i)\}_{i\in\sigma}$ is essential.

Since S is separating, we can find for each $i \in \sigma$ infinitely many $j \in N$ such that $A_i \subset A_j$ and $B_i \subset B_j$. Select such $j(i)$ so that $j(i) \notin \sigma$ for every $i \in \sigma$. Let $\sigma' = \{j(i)\}_{i\in\sigma}$. Then (b1) is satisfied by our choice of $j(i)$. That $\sigma' \in M_S$ and that (b2) and (b3) are true for σ' follows directly from Lemmas 3.1.4 and 3.3.6.

Since there are clearly infinitely many pairwise disjoint such σ', we are done. ■

Now we can use Theorem 2.3.9 to give the relationship between dim and index.

3.3.8. THEOREM. *Let X be a compact space. Then the following statements are equivalent:*

(1) X *is* w.i.d.,

(2) $\operatorname{index} X$ *exists, and*

(3) $\dim X$ *exists.*

Moreover, the following equalities hold between $\operatorname{index} X$ *and* $\dim X$.

(a) $\dim X = -1$ *or* 0 *iff* $\operatorname{index} X = 0$,

(b) $\operatorname{index} X = \omega_0^{\dim X}$ *otherwise.*

Proof. (1) $\leftrightarrow$ (2): see [P2; Lemma 3.2].

(1) $\leftrightarrow$ (3): Theorem 3.1.3.

We have $\operatorname{index} X = 0$ iff $M_S = \emptyset$ for some separating sequence S. This is true iff $\dim X \leqslant 0$.

In the other case, if $M_S \neq \emptyset$ then M_S has property (*) by Lemma 3.3.7 and $\operatorname{Ord} M_S = \operatorname{Ord} M_{L(X)}$ by Corollary 3.3.5. Applying Theorem 2.3.9 we then get

$$\operatorname{index} X = \operatorname{type} M_S = \omega_0^{\operatorname{Ord} M_S} = \omega_0^{\operatorname{Ord} M_{L(X)}} = \omega_0^{\dim X}. \ \blacksquare$$

By Theorem 3.3.8 we can translate every result about index into a result about dim (and vice versa). For example, we can restate [P2; Theorem 5.1] as follows:

3.3.9. THEOREM. *Let $\mathscr{K}$ be a family of* w.i.d. *compact spaces. Then there exists a* w.i.d. *compact space X containing each member of $\mathscr{K}$ topologically if and only if*

$$\sup\{\dim K \colon K \in \mathscr{K}\} < \omega_1. \ \blacksquare$$

3.4. Sum theorems. We shall prove a few sum theorems needed in later sections. As usual $X \oplus Y$ denotes the topological sum of the spaces X and Y.

3.4.1. PROPOSITION. *Let $X = X_1 \oplus X_2$. Then for every $\tau \in \{\emptyset\} \cup \operatorname{Fin} L(X)$ we have*

$$\operatorname{Ord} M^{\tau}_{L(X)} \leqslant \max(\operatorname{Ord} \tilde{M}^{\tau}_{L(X)|X_1}, \operatorname{Ord} \tilde{M}^{\tau}_{L(X)|X_2}).$$

Proof. We prove that $M^{\tau}_{L(X)} \subset \tilde{M}^{\tau}_{L(X)|X_1} \cup \tilde{M}^{\tau}_{L(X)|X_2}$ for every $\tau \in \{\emptyset\} \cup \operatorname{Fin} L(X)$. Then the proposition will follow from Lemma 2.2.1.

Let $\sigma \in M^{\tau}_{L(X)}$. Then $\gamma = \sigma \cup \tau \in M_{L(X)}$, i.e., $\tau = \{(A_i, B_i)\}_{i=1}^n$ is essential in X. But because $X = X_1 \oplus X_2$ it now follows easily that either

$$\{(X_1 \cap A_i, X_1 \cap B_i)\}_{i=1}^n \quad \text{or} \quad \{(X_2 \cap A_i, X_2 \cap B_i)\}_{i=1}^n$$

is essential so that

$$\text{either } \gamma \in \tilde{M}_{L(X)|X_1} \quad \text{or} \quad \gamma \in \tilde{M}_{L(X)|X_2},$$

and hence

$$\sigma \in \tilde{M}^{\tau}_{L(X)|X_1} \cup \tilde{M}^{\tau}_{L(X)|X_2}. \ \blacksquare$$

3.4.2. COROLLARY. *Let $X = X_1 \oplus X_2$. Then* $\dim X = \max(\dim X_1, \dim X_2)$.

Proof. "$\geqslant$" Theorem 3.1.6.

"$\leqslant$" Proposition 3.4.1 for the case $\tau = \emptyset$. $\blacksquare$

3.4.3. LEMMA. *Let X be a space, and let X_1 and X_2 be closed subsets of X such that $X = X_1 \cup X_2$. Let $\sigma = \{(A_i, B_i)\}_{i=1}^n \in \operatorname{Fin} L(X)$ be such that*

$$\sigma \notin \tilde{M}_{L(X)|X_1} \cup \tilde{M}_{L(X)|X_2}.$$

Then we can find open sets O_i in X for $i = 1, \ldots, n$ such that

$$A_i \subset O_i \subset \bar{O}_i \subset X - B_i \quad \text{and} \quad \bigcap_{i=1}^n \operatorname{Fr} O_i \subset X_1 \cap X_2.$$

Proof. Because $\sigma \notin \tilde{M}_{L(X)|X_K}$ for $K = 1, 2$, we can find open sets O_i^K in X such that

$$A_i \subset O_i^K \subset \bar{O}_i^K \subset X - B_i \quad \text{for } i = 1, \ldots, n, \quad \text{and}$$

$$\bigcap_{i=1}^n \operatorname{Fr} O_i^K \cap X_K = \emptyset \quad \text{for } K = 1, 2.$$

For $i = 1, \ldots, n$ we let U_i be an open set in X such that

$$A_i \subset U_i \subset \bar{U}_i \subset X - B_i \quad \text{and} \quad U_i \subset O_i^1 \cap O_i^2.$$

If we put $O_i = (O_i^1 - X_2) \cup U_i \cup (O_i^2 - X_1)$ for $i = 1, \ldots, n$ we obtain

$$\operatorname{Fr} O_i \subset (\operatorname{Fr} O_i^1 \cap X_1) \cup (X_1 \cap X_2) \cup (\operatorname{Fr} O_i^2 \cap X_2).$$

Then the O_i, $i = 1, \ldots, n$, are as required. $\blacksquare$

We now prove a sum theorem for dim which is similar to a result obtained by A. R. Pears [Pe] for Ind. In this theorem the lower sum is used as defined in Definition 2.2.2.

3.4.4. THEOREM. *Let X be a space and let X_1 and X_2 be closed subsets of X such that $X = X_1 \cup X_2$. Then* $\dim X \leqslant \max(\dim X_1, \dim X_2) \oplus (\dim(X_1 \cap X_2) + 1)$.

Proof. The case $X_1 \cap X_2 = \emptyset$ was already delt with in Corollary 3.4.2 so that we may assume that $X_1 \cap X_2 \neq \emptyset$. Put $M = M_{L(X)}$, $M_1 = \tilde{M}_{L(X)|X_1} \cup \tilde{M}_{L(X)|X_2}$ and $M_2 = \tilde{M}_{L(X)|X_1 \cap X_2}$, respectively. By Lemma 2.2.1 and Theorem 3.1.6 we have

$$\alpha = \operatorname{Ord} M_1 = \max(\operatorname{Ord} \tilde{M}_{L(X)|X_1}, \operatorname{Ord} \tilde{M}_{L(X)|X_2}) = \max(\dim X_1, \dim X_2).$$

Put $\beta = \operatorname{Ord} M_2 = \dim(X_1 \cap X_2)$. Note that $\beta \leqslant \alpha$ by the subspace Theorem 3.1.6. We shall prove that M, M_1 and M_2 are as in Lemma 2.2.4. Let $\sigma = \{(A_j, B_j)\}_{j=1}^n \in M$. Consider $i = \min(\{k \colon \{(A_j, B_j)\}_{j=1}^k \notin M_1, 1 \leqslant k \leqslant n\} \cup \{n\})$. Clearly,

$$\{(A_j, B_j)\}_{j=1}^{i-1} \in M_1 \cup \{\emptyset\}.$$

CLAIM. $\sigma_2 = \{(A_j, B_j)\}_{j=i+1}^n \in M_2 \cup \{\emptyset\}$.

Proof of the claim. Since $i = n$ implies $\sigma_2 = \emptyset$ we may assume that $i < n$. Let $\sigma_1 = \{(A_j, B_j)\}_{j=1}^i$. Observe that $\sigma_1 \notin M_1 = \tilde{M}_{L(X)|X_1} \cup \tilde{M}_{L(X)|X_2}$ so that by Lemma 3.4.3 there is an open set O_j in X for each $j = 1, \ldots, i$ such that

$$A_j \subset O_j \subset \bar{O}_j \subset X - B_j, \quad F = \bigcap_{j=1}^i \operatorname{Fr} O_j \subset X_1 \cap X_2.$$

Then, by virtue of Proposition 3.2.1(1), we see that

$$M^{\sigma_1} \subset \tilde{M}_{L(X)|F} \subset \tilde{M}_{L(X)|X_1 \cap X_2} = M_2.$$

Since $\sigma_1\cup\sigma_2=\sigma\in M$ and $\sigma_1\cap\sigma_2=\emptyset$ we have $\sigma_2\in M^{\sigma_1}\subset M_2$. This proves our claim.

Now we can apply Lemma 2.2.4 and obtain

$$\dim X=\operatorname{Ord} M_{L(X)}\leqslant\Phi(\alpha,\beta)=\max(\dim X_1,\dim X_2)\oplus(\dim(X_1\cap X_2)+1).\ \blacksquare$$

The following sum theorem will be used when we deal with s.c.d. spaces in Section 4.4.

3.4.5. Proposition. *Let X be a space and let $\tau\in\{\emptyset\}\cup\operatorname{Fin}L(X)$. If $X=U\cup V$, with U and V open in X then*

$$\operatorname{Ord} M^\tau_{L(X)}\leqslant\Phi(\max(\operatorname{Ord}\tilde{M}^\tau_{L(X)|\bar U},\ \operatorname{Ord}\tilde{M}^\tau_{L(X)|\bar V}),\ \operatorname{Ord} M^{(A,B)}_{L(X)}),$$

where $(A,B)=(X-U,X-V)\in L(X)$.

Proof. We put $M=M^\tau_{L(X)}$, $M_1=\tilde{M}^\tau_{L(X)|\bar U}\cup\tilde{M}^\tau_{L(X)|\bar V}$ and $M_2=M^{(A,B)}_{L(X)}$ and prove that they are as in Lemma 2.2.4. Let $\sigma=\{(A_j,B_j)\}_{j=1}^n\in M$. Consider

$$i=\max(\{k:\ \{(A_j,B_j)\}_{j=k}^n\notin M_2,\ 1\leqslant k\leqslant n\}\cup\{1\}).$$

Clearly $\{(A_j,B_j)\}_{j=i+1}^n\in M_2\cup\{\emptyset\}$. We prove $\sigma_1=\{(A_j,B_j)\}_{j=1}^{i-1}\in M_1\cup\{\emptyset\}$. Assume that $\sigma_1\neq\emptyset$. To begin, consider $\sigma_2=\{(A_j,B_j)\}_{j=i}^n$. Observe that $\sigma_2\notin M_2=M^{(A,B)}_{L(X)}$ so that

(i) $\sigma_2\cup\{(A,B)\}\notin M_{L(X)}$ or

(ii) $(A,B)\in\sigma_2$.

In both cases we find open sets O_j for $j=i,\ldots,n$ and an open set O in X such that

$$A_j\subset O_j\subset\bar O_j\subset X-B_j\quad\text{for } j=i,\ldots,n,\qquad A\subset O\subset\bar O\subset X-B$$

and

$$\operatorname{Fr} O\cup\bigcap_{j=i}^n\operatorname{Fr} O_j=\emptyset.$$

For $C=\bar O$ and $D=X-O$ we have $C\subset\bar V$ and $D\subset\bar U$. Then also

$$\tilde{M}^\tau_{L(X)|C}\subset\tilde{M}^\tau_{L(X)|\bar V}\quad\text{and}\quad\tilde{M}^\tau_{L(X)|D}\subset\tilde{M}^\tau_{L(X)|\bar U}.$$

Now assume

$$\sigma_1\notin\tilde{M}^\tau_{L(X)|C}\quad\text{and}\quad\sigma_1\notin\tilde{M}^\tau_{L(X)|D}.$$

Then, since $\sigma_1\cap\tau=\emptyset$, for $\gamma=\sigma_1\cup\tau$ we have

$$\gamma\notin\tilde{M}_{L(X)|C}\quad\text{and}\quad\gamma\notin\tilde{M}_{L(X)|D}.$$

According to Lemma 3.4.3 for $\gamma=\{(F_j,G_j)\}_{j=1}^m$ we can find open sets V_j, $j=1,\ldots,m$ such that

$$F_j\subset V_j\subset\bar V_j\subset X-G_j\quad\text{and}\quad\bigcap_{j=1}^m\operatorname{Fr} V_j\subset C\cap D=\operatorname{Fr} O.$$

Consequently, $\bigcap_{j=1}^m\operatorname{Fr} V_j\cap\bigcap_{j=i}^n\operatorname{Fr} O_j=\emptyset$ and therefore

$$\sigma_2\cup\gamma=\sigma_2\cup\sigma_1\cup\tau=\sigma\cup\tau\notin M_{L(X)}$$

so that $\sigma\notin M^\tau_{L(X)}$. Contradiction.

It follows that $\sigma_1\in\tilde{M}^\tau_{L(X)|C}\cup\tilde{M}^\tau_{L(X)|D}\subset M_1$. We conclude that indeed M, M_1 and M_2 are as in Lemma 2.2.4. Thus we find

$$\operatorname{Ord} M\leqslant\Phi(\operatorname{Ord} M_1,\operatorname{Ord} M_2).$$

Also, by Lemma 2.2.1, $\operatorname{Ord} M_1=\max(\operatorname{Ord}\tilde{M}^\tau_{L(X)|\bar U},\operatorname{Ord}\tilde{M}^\tau_{L(X)|\bar V})$, so we are done. $\blacksquare$

3.5. The product with the Cantor set. We prove in this section that for a locally compact space X we have

$$\dim X=\dim(X\times C),$$

here C denotes the Cantor set.

The proof of the following lemma will be left to the reader.

3.5.1. Lemma. *Let X be a space and let $\mathscr{U}$ be a finite collection of clopen sets in X.*

Then there exist a disjoint finite collection $\mathscr{D}$ of clopen sets in X such that for every $U\in\mathscr{U}$ there is a subcollection $\mathscr{D}'\subset\mathscr{D}$ with

$$U=\bigcup\{D:\ D\in\mathscr{D}'\}.\ \blacksquare$$

3.5.2. Lemma. *Let X be a compact space and let (A,B) be a pair of disjoint closed sets in $X\times C$.*

Then we can find a disjoint clopen cover $\{D_1,\ldots,D_p\}$ of C and for each $j=1,\ldots,p$ a pair (F_j,G_j) of disjoint closed sets in X such that:

$$A\subset\bigcup_{j=1}^p(F_j\times D_j)\quad\text{and}\quad B\subset\bigcup_{j=1}^p(G_j\times D_j).$$

Proof. By a standard argument, [E1; 7.4.10], and by Lemma 3.5.1 we can find a finite open cover $\mathscr{U}$ of X and a disjoint clopen cover $\mathscr{D}$ of C such that for every $U\in\mathscr{U}$ and $D\in\mathscr{D}$

$$(\bar U\times D)\cap A=\emptyset\quad\text{or}\quad(\bar U\times D)\cap B=\emptyset.\tag{1}$$

Index $\mathscr{D}$ as $\{D_1,\ldots,D_p\}$ and for each $j=1,\ldots,p$ set

$$F_i=\Pi_X(A\cap(D_i\times X))\quad\text{and}\quad G_i=\Pi_X(B\cap(D_i\times X))$$

(here Π_X: $X\times C\to X$ is the projection onto the first factor of the product $X\times C$).

By compactness of X and C and (1), F_i and G_i are closed and disjoint. Clearly,

$$A\subset\bigcup_{j=1}^p(F_j\times D_j)\quad\text{and}\quad B\subset\bigcup_{j=1}^p(G_j\times D_j).\ \blacksquare$$

3.5.3. PROPOSITION. *Let X be a compact space. Let*

$$\tau = \{(A_i, B_i)\}_{i=1}^n \in \{\emptyset\} \cup \operatorname{Fin} L(X)$$

and put $\tau' = \{(A_i \times C, B_i \times C)\}_{i=1}^n$.

Then $\tau' \in L(X \times C)$ *and* $\operatorname{Ord} M^{\tau}_{L(X)} = \operatorname{Ord} M^{\tau'}_{L(X\times C)}$.

Proof. For notational simplicity denote $L(X\times C)$ by L'. We shall prove the proposition by transfinite induction on $\alpha = \operatorname{Ord} M^{\tau'}_{L'}$.

For $i = 1, \ldots, n$ write $C_i = A_i \times C$ and $D_i = B_i \times C$.

$\alpha = 0$: Since always $\operatorname{Ord} M^{\tau}_{L(X)} \geqslant 0$ we are done.

Assume that the proposition is true for all $\beta < \alpha$. Assume first that α is a successor, say $\alpha = \beta + 1$.

Let $b = (A_b, B_b) \in L' - \tau'$ be such that $\operatorname{Ord} M^{\tau' \cup \{b\}}_{L'} = \beta$. Then, by Lemma 3.5.2, for the pair (A_b, B_b) of disjoint closed sets in $X \times C$ we can find a clopen cover $\{D_1, \ldots, D_p\}$ of C and a collection $\{(F_1, G_1), \ldots, (F_p, G_p)\}$ of pairs of disjoint closed sets in X such that

$$A_b \subset \bigcup_{j=1}^{p} (F_j \times D_j) \quad \text{and} \quad B_b \subset \bigcup_{j=1}^{p} (G_j \times D_j).$$

For $j = 1, \ldots, p$ define the following sets:

$$Y_j = X \times D_j \quad \text{and} \quad M_j = \tilde{M}_{L'|Y_j}.$$

Observe that $X \times C = \bigoplus_{j=1}^{p} Y_j$ so that by Proposition 3.4.1,

$$\operatorname{Ord} M^{\tau' \cup \{b\}}_{j_0} = \beta$$

for some $j_0 \in \{1, \ldots, p\}$. Observe also $A_b \cap Y_{j_0} \subset F_{j_0} \times D_{j_0}$ and $B_b \cap Y_{j_0} \subset G_{j_0} \times D_{j_0}$.

Now set $A_{n+1} = F_{j_0}$, $B_{n+1} = G_{j_0}$, $C_{n+1} = A_{n+1} \times C$, $D_{n+1} = B_{n+1} \times C$ and let

$$c = (A_{n+1}, B_{n+1}), \qquad d = (C_{n+1}, D_{n+1}).$$

Now $A_b \cap Y_{j_0} \subset C_{n+1} \cap Y_{j_0}$ and $B_b \cap Y_{j_0} \subset D_{n+1} \cap Y_{j_0}$ so that since $\tau' \cup \{b\} \in M_{j_0}$ we have $\tau' \cup \{d\} \in M_{j_0}$. Then $d \notin \tau'$ by Lemma 3.1.4 and so $c \notin \tau$. By Lemma 3.3.6

$$M^{\tau' \cup \{b\}}_{j_0} \subset M^{\tau' \cup \{d\}}_{j_0}$$

so that

$$\operatorname{Ord} M^{\tau' \cup \{d\}}_{L'} \geqslant \operatorname{Ord} M^{\tau' \cup \{d\}}_{j_0} \geqslant \operatorname{Ord} M^{\tau' \cup \{b\}}_{j_0} \geqslant \beta.$$

Consequently by our inductive assumption, $\operatorname{Ord} M^{\tau \cup \{c\}}_{L(X)} \geqslant \beta$, and hence

$$\operatorname{Ord} M^{\tau}_{L(X)} \geqslant \alpha$$

since $c \notin \tau$.

Finally assume that α is a limit ordinal. For every $\beta < \alpha$ we have $\operatorname{Ord} M^{\tau'}_{L'} \geqslant \beta$ and therefore $\operatorname{Ord} M^{\tau}_{L(X)} \geqslant \beta$. Consequently, $\operatorname{Ord} M^{\tau}_{L(X)} \geqslant \alpha$. Conversely: Identify X with $X \times \{0\}$.

Then $A_i \subset A_i \times C$ and $B_i \subset B_i \times C$ for each i.

Hence $M^{\tau}_{L'} \subset M^{\tau'}_{L'}$. But also $M^{\tau}_{L(X)} \subset M^{\tau}_{L'}$. It follows that

$$\operatorname{Ord} M^{\tau}_{L(X)} \leqslant \operatorname{Ord} M^{\tau'}_{L(X\times C)}. \ \blacksquare$$

3.5.4. THEOREM. *Let X be a compact space. Then*

$$\dim X = \dim (X \times C).$$

Proof. $\leqslant$: Theorem 3.1.6.

$\geqslant$: This follows from Proposition 3.5.3 by taking $\tau' = \tau = \emptyset$. ■

3.5.5. PROPOSITION. *Let Y be a space and let X be an open subspace of Y, such that $\omega_0 \leqslant \dim X < \infty$. Moreover, assume that $\dim(Y - X)$ is finite. Then*

$$\dim Y = \dim X.$$

Proof. The set $B = Y - X$ is closed in Y.

Let $P_n(X)$ for $n = 1, 2, \ldots$ be as in Definition 1.2.3. Note that $\dim P_n(X) = n$ by [E2; 3.1.14].

According to Theorem 1.2.6(2) we see that $A = X - \bigcup_{n=1}^{\infty} P_n(X)$ is compact, and hence A is closed in Y.

Using the normality of Y we can find closed sets F and G in Y such that

$$A \subset F, \quad B \subset G, \quad A \cap G = B \cap F = \emptyset$$

and

$$F \cup G = Y.$$

By Theorem 1.2.6(3) $G_1 = G \cap X \subset P_n(X)$ for some n so that $\dim G_1$ is finite. By [E2; 3.1.7] we obtain

from $X = F \cup G_1$ that $\dim F \geqslant \omega_0$, and

from $G = G_1 \cup B$ that $\dim G < \omega_0$.

Consequently, by Theorems 3.4.4 and 3.1.6.

$$\dim Y = \dim (F \cup G) = \dim F = \dim (F \cup G_1) = \dim X. \ \blacksquare$$

3.5.6. PROPOSITION. *A space X is* S-w.i.d. *iff $X \times C$ is* S-w.i.d.

Proof. $\Leftarrow$: Apply Theorems 3.1.3 and 3.1.6.

$\Rightarrow$: Let the subset $A_{\omega_0}(X)$ of X be as defined in 1.2.3. Then $A_{\omega_0}(X)$ satisfies (1)–(3) of Theorem 1.2.6.

From the product-theorem, [E2; 3.2.14] it follows that

$$P_n(X \times C) = P_n(X) \times C$$

so that

$$A = A_{\omega_0}(X \times C) = A_{\omega_0}(X) \times C.$$

It suffices to show that A satisfies (1)–(3) of Theorem 1.2.6. Condition (2) is satisfied since $A_{\omega_0}(X)$ is compact. Now an application of Theorems 3.1.3 and 3.5.4 yields (1).

Finally, for (3), the natural projection $\Pi\colon X\times C\to X$ will be used. By the Kuratowski Theorem, [E1; 1.3.16], Π is closed.

Let F be a closed set in $X\times C$ disjoint from A. Then $\Pi(F)$ is a closed set in X disjoint from $A_{\omega_0}(X)$, so that $\Pi(F)\subset P_n(X)$ for some n. Consequently,

$$F\subset\Pi(F)\times C\subset P_n(X)\times C=P_n(X\times C)$$

and we are done. ■

3.5.7. THEOREM. *Let X be a locally compact space. Then*

$$\dim X=\dim(X\times C).$$

Proof. Since the cases where $\dim X$ is finite and $\dim X=\infty$ are delt with in [E2; 3.2.14] and Proposition 3.5.6, respectively, we may restrict ourselves to the case where $\omega_0\leqslant\dim X<\infty$ and X is S-w.i.d.

Let Y be the one-point compactification of X. Then X and Y are as in Proposition 3.5.5. This is also true for $X\times C$ and $Y\times C$ by Proposition 3.5.6. Consequently by Theorem 3.5.4,

$$\dim X=\dim Y=\dim Y\times C=\dim X\times C. \ ■$$

3.5.8. Remark. It is easily seen that the results and proofs in this section remain valid when C is replaced by any compact zero-dimensional space. Several questions remain open. Naturally, we have

3.5.9. QUESTION. Can the condition "locally compact" in Theorem 3.5.7 be weakened?

Solving this question will also have consequences for the results in Section 4.2. Other problems connected with this one are

3.5.10. QUESTION. For compact spaces X and a finite dimensional space Y does the inequality

$$\dim(X\times Y)\leqslant\dim X+\dim Y$$

hold? In particular, do we have

$$\dim(X\times I^n)=\dim X\times n?$$

3.5.11. QUESTION. Is there some ordinal valued function Φ such that for any two compact spaces X and Y we have

$$\dim(X\times Y)\leqslant\Phi(\dim X,\dim Y)?$$

Finally, let us note that the following special case of Question 3.5.11 is also still open:

3.5.12. QUESTION. Is the product of two (compact) S-w.i.d. spaces again S-w.i.d.?

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