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On duality and interpolation for spaces of polyharmonic functions

by

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Abstract. We extend the results from our previous works on duality and interpolation to the case of m-polyharmonic functions, i.e. functions u for which $\Delta^m u = 0$. We give an explicit characterization of Sobolev and Besov spaces of such functions for $1 \leqslant p \leqslant \infty$, $-\infty < s < \infty$ and study the duality and interpolation relations between them. We also apply the results obtained to the study of the $\bar{\partial}$-Neumann problem and the Bergman projection.

I. Introduction. The present paper is the final part of a long series of papers ([18]–[24]) devoted to various aspects of duality and interpolation in spaces of harmonic functions. We generalize the results proved in [18]–[24] to the case of spaces of polyharmonic functions of finite order m, i.e. functions h for which $\Delta^m h = 0$. We call these functions *m-polyharmonic*. (For information about polyharmonic functions see [2]).

We make extended use of the fact that Sobolev and Besov norms restricted to the space of all m-polyharmonic functions on a smooth bounded domain are equivalent. In particular, we use it to get new estimates and duality results in the limit cases $p = 1$ and $p = \infty$ which will be proved in Appendix 2. This also permits us to get an explicit description of the Sobolev spaces $\mathrm{Harm}_p^s(m)(D)$ of m-polyharmonic functions for all $-\infty < s < \infty$, $1 \leqslant p < \infty$. The equality $\mathrm{Harm}_p^s(m)(D) = B_{p,p}^{s} \mathrm{Harm}(m)(D)$, $-\infty < s < \infty$, $1 \leqslant p < \infty$, also explains the slightly astonishing interpolation results, which were proved for the spaces of harmonic functions in [22] and [24] and are here generalized to the spaces of m-polyharmonic functions. We hope that the estimates concerning the polyharmonic functions can be useful in the study of the $\bar{\partial}$-Neumann problem. In fact, this was the main motivation to do the present work. We also discuss the possible generalizations of the above results.

Before we shall state our results more precisely we need to recall some definitions, notation and facts. We always denote by D a bounded domain in $\boldsymbol{R}^n$ with boundary of class C^∞, and by ϱ its defining function, i.e. $\varrho \in C^\infty(\boldsymbol{R}^n)$, $D = \{x \in \boldsymbol{R}^n\colon \varrho(x) < 0\}$, $\operatorname{grad} \varrho \neq 0$ on ∂D. By $W_p^s(D)$, $1 < p < \infty$, $-\infty < s < \infty$, we denote the Sobolev spaces on D, defined as follows. If $s \geqslant 0$ is an integer then $W_p^s(D)$ is the usual space of functions whose sth derivatives belong to $L^p(D)$, with

$$\|f\|_p^s = \|f\|_{L^p(D)} + \sum_{|\alpha|=s} \|\partial^\alpha f/\partial x^\alpha\|_{L^p(D)}.$$

We then denote by $\mathring{W}_p^s(D)$ the closure of $C_0^\infty(D)$ in $W_p^s(D)$. By $W_p^{-s}(D)$, $1 < p < \infty$, we denote the dual space to $\mathring{W}_q^s(D)$ with respect to the usual L^2 scalar product $\langle\cdot,\cdot\rangle_0$ on D, $q = p/(p-1)$. It consists of distributions g on D such that

$$g = \sum_{|\alpha|=s} D^\alpha g_\alpha + g_0,$$

where $g_0, g_\alpha \in L^p(D)$ (see [25]). We shall need the following important properties of the above-defined spaces:

(a) The mapping $f \to f/|\varrho|^s$ maps continuously $\mathring{W}_p^s(D)$ into $L^p(D)$.

(b) The mapping $f \to \Delta^s f$ is an isomorphism between $\mathring{W}_p^s(D)$ and $\mathring{W}_p^{-s}(D)$.

Property (a) follows from Muckenhoupt's inequalities [27] (see [25], Theorem 1.3.1/2]). Property (b) follows from [26, Theorem 5.4] (it could also be proved by using the Hölder estimates from [1] and interpolation). For $p = 2$, property (b) is elementary (see [18]).

If s is *not* an integer we define

$$W_p^s(D) = [W_p^{[s]}(D), W_p^{[s]+1}(D)]_{[s-[s]]}$$

—the value of the complex interpolation functor at $\theta = s-[s]$ (for the definitions of the complex interpolation functor $[\cdot,\cdot]_{[\theta]}$ and the real interpolation functor $(\cdot,\cdot)_{\theta,p}$ see [7], [14] and [31]). If $s > 0$ then the space $W_p^{-s}(D)$ is the dual of

$$\mathring{W}_q^s(D) \stackrel{\mathrm{df}}{=} [\mathring{W}_q^{[s]}(D), \mathring{W}_q^{[s]+1}(D)]_{[s-[s]]}, \qquad q = \frac{p}{p-1}.$$

It follows from [31], 3.4.3, that $\mathring{W}_q^s(D)$ is equal to the closure of $C_0^\infty(D)$ in $W_q^s(D)$ for $s \neq k+1/q$, $k = 0, 1, \ldots$ If $s = k+1/q$ there is a continuous inclusion $\mathring{W}_q^s(D) \subset \overline{C_0^\infty(D)} \subset W_q^s(D)$. A simple duality and interpolation argument shows that the mapping $f \to f/|\varrho|^s$ maps continuously $\mathring{W}_p^s(D)$ into $L^p(D)$. The results from [31] also imply that the spaces $W_p^s(D)$ defined as above are the spaces of the restrictions of functions (distributions) from $W_p^s(\mathbf{R}^n)$, the Sobolev spaces on $\mathbf{R}^n$. (If s is not an integer $W_p^s(\mathbf{R}^n)$ is often called "the space of Bessel potentials". Note that $W_p^s(\mathbf{R}^n) = F_{p,2}^s(\mathbf{R}^n)$ in the notation of [31].) We can thus extend the above definition to the case $p \leq 1$, by putting $W_p^s(D) =$ the restriction of $W_p^s(\mathbf{R}^n)$ to D, $-\infty < s < \infty$.

If $s > 0$ is not an integer then the Besov space can be defined for $1 \leq p < \infty$ as follows:

$$B_{pp}^s(D) = \Big\{f \in L^p(D)\colon \|f\|_{pp}^s = \|f\|_{L^p(D)} + \sum_{|\alpha|=[s]} \Big(\int_{D\times D} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+(s-[s])p}}\Big)^{1/p} < \infty\Big\}.$$

If $s > 0$ is an integer then we put

$$B_{pp}^s(D) = [B_{pp}^{s-1/2}, B_{pp}^{s+1/2}]_{[1/2]}.$$

For $s \neq 1/p+k$ and $1 < p < \infty$ we define $\mathring{B}_{pp}^s(D)$ as the closure of $C_0^\infty(D)$ in $B_{pp}^s(D)$, and $B_{qq}^{-s}(D)$, $q = p/(p-1)$, as the dual of $\mathring{B}_{pp}^s(D)$ as in the case of Sobolev spaces. If $s = 1/p+k$ we put

$$\mathring{B}_{pp}^s(D) = [\mathring{B}_{pp}^{s-\varepsilon}(D), \mathring{B}_{pp}^{s+\varepsilon}(D)]_{[1/2]}, \qquad B_{qq}^{-s}(D) = (\mathring{B}_{pp}^s(D))^*.$$

Again the results from [31] imply that the spaces defined above are the spaces of the restrictions of functions (distributions) from $B_{pp}^s(\mathbf{R}^n)$.

$\Lambda_\alpha(D)$ denotes the usual Hölder space and $L^p(D, |\varrho|^r)$ the L^p space with respect to the measure $|\varrho|^r dV$ on D if $p < \infty$. $L^\infty(D, |\varrho|^r)$ is the space of functions f such that $f|\varrho|^r \in L^\infty(D)$, with norm $\|f\|_{L^\infty(D,|\varrho|^r)} = \|f|\varrho|^r\|_{L^\infty(D)}$.

$\mathrm{Harm}(m)(D)$ will denote the space of all m-polyharmonic functions on D. Then

$$L^p\,\mathrm{Harm}(m)(D, |\varrho|^r) = L^p(D, |\varrho|^r) \cap \mathrm{Harm}(m)(D),$$

$$\Lambda_\alpha\,\mathrm{Harm}(m)(D) = \Lambda_\alpha(D) \cap \mathrm{Harm}(m)(D),$$

$$B_{pp}^s\,\mathrm{Harm}(m)(D) = B_{pp}^s(D) \cap \mathrm{Harm}(m)(D),$$

$$\mathrm{Harm}_p^s(m)(D) = W_p^s(D) \cap \mathrm{Harm}(m)(D).$$

We shall also need the space of Bloch m-polyharmonic functions $\mathrm{Bl\,Harm}(m)(D)$, defined as the space of m-polyharmonic functions h such that

$$\|h\|_{\mathrm{Bl}(D)} = \sup_{x\in D}(|\varrho(x)\,h(x)| + |\varrho(x)\,\mathrm{grad}\,h(x)|) < \infty.$$

Now, let T be a differential operator of order $2m$ with C^∞ coefficients on $\mathbf{R}^n$ such that σ_T, the principal symbol of T, does not vanish on $\bar{D}\times(\mathbf{R}^n\setminus\{0\})$. We also assume that the Dirichlet problem $Tu = v$, u vanishes on D up to order $m-1$, is uniquely solvable. Denote the operator solving this problem by G_T. Let T^* denote the formal adjoint of T. Note that T^* and TT^* must have the same properties as T.

Let P_T denote the orthogonal projection from $L^2(D)$ onto the space $L^2(D) \cap \mathrm{Ker}\,T$. We have

$$P_T u = u - T^* G_{TT^*} Tu = T^*(G_{T^*}u - G_{TT^*} Tu).$$

If $T = \Delta^m$ then Ker T is the space of m-polyharmonic functions. In this case we write $P_T = P_m$ and $G_T = G_m$.

The estimates from [1] and [26] imply that for every strongly elliptic T, P_T maps continuously $W_p^s(D)$ into $W_p^s(D)$, $s \geq 0$, and $\Lambda_\alpha(D)$ into $\Lambda_\alpha(D)$. By using the real interpolation functor one can easily check that P_T also maps $B_{pp}^s(D)$ into $B_{pp}^s(D)$, $s > 0$. Let us now define the family of Bell's operators L_T^r in the following manner:

$$L_T^r u = u - T^*\Big(\sum_{k=0}^{r-1} \theta_k \varrho^{k+2m}\Big),$$

$$\theta_t = \frac{\varphi}{(t+2m)!}(\sigma_{T^*}(\nabla\varrho))^{-1}\left(\frac{\partial}{\partial\eta}\right)^t L_T^t u, \qquad \eta = \frac{\sum_i \frac{\partial\varrho}{\partial x_i}\frac{\partial}{\partial x_i}}{|\nabla\varrho|^2},$$

$$L_T^1 u = u - T^*(\theta_0 \varrho^{2m}), \qquad \theta_0 = \frac{1}{(2m)!}\frac{\varphi u}{\sigma_{T^*}(\nabla\varrho)},$$

where φ is an arbitrarily chosen C^∞ function equal to 1 in a neighborhood of ∂D and to 0 in a neighborhood of the set $\{\nabla\varrho = 0\}$.

It follows directly from the definition of L_T^r that if $u \in C^\infty(\bar{D})$ then $P_T u = P_T L_T^r u$ and $L_T^r u$ vanishes on ∂D up to order $r-1$. The operators L_T^r were defined for $T = \Delta$ by S. Bell in [5]. Earlier S. Bell defined similar operators for $\bar{\partial}$ and the Bergman projection B onto the L^2 space of holomorphic functions in [4]. These operators were used to study the duality relations between spaces of holomorphic functions in [4], [6], [13] and [3], and between spaces of harmonic and pluriharmonic functions in [5], [18], [19], [20], [21], [22] and [24]. E. Straube in [28] and [29] constructed a single operator L, which maps continuously $W_2^s(D)$ into $\mathring{W}_2^s(D)$, $0 \leq s < \infty$, such that $P_1 u = P_1 Lu$. However, the construction of L is more complicated than the explicit construction of Bell's operators.

We denote by $\langle\cdot,\cdot\rangle_{r,T}$ the sesquilinear pairing defined as follows: $\langle u, v\rangle_{r,T} = \langle u, L_T^r v\rangle_0$, where $\langle\cdot,\cdot\rangle_0$ is the usual $L^2(D)$ scalar product.

If $T = \Delta^m$ then we denote L_T^r by L_m^r and $\langle\cdot,\cdot\rangle_{t,T}$ by $\langle\cdot,\cdot\rangle_{r,m}$. We shall show that L_m^r maps $L^2\,\mathrm{Harm}(m)(D)$ into $L^2(D)$, and thus if $u, v \in L^2\,\mathrm{Harm}(m)(D)$ then $\langle u, v\rangle_{r,m} = \langle u, v\rangle_0$.

II. Statement of the results.

PROPOSITION 1. *For every integer $m > 0$ the projection P_m maps continuously:*

(a) $L^\infty(D, |\varrho|^{-\alpha})$ *into* $\Lambda_\alpha\,\mathrm{Harm}(m)(D)$, $\alpha > 0$;

(b) $L^\infty(D, |\varrho|^\beta)$ *into* $L^\infty\,\mathrm{Harm}(m)(D, |\varrho|^\beta)$, $0 < \beta < 1$;

(c) $L^\infty(D)$ *into* $\mathrm{Bl\,Harm}(m)(D)$;

(d) $L^p(D, |\varrho|^{-ps})$ *into* $\mathrm{Harm}_p^s(m)(D)$, $-1+1/p < s < \infty$, $1 < p < \infty$.

PROPOSITION 2. (a) *For any integers $m \geq 1$ and $k \geq 0$ the mapping $T_k u = \varrho^k u$ maps continuously:*

1) $\mathrm{Harm}_p^s(m)(D)$ *into* $W_p^{s+k}(D)$, $-\infty < s < \infty$, $1 < p < \infty$;

2) $\Lambda_\alpha\,\mathrm{Harm}(m)(D)$ *into* $\Lambda_{\alpha+k}(D)$;

3) $L^\infty\,\mathrm{Harm}(m)(D, |\varrho|^\beta)$ *into* $\Lambda_{k-\beta}(D)$, $0 < \beta < 1$;

4) $\mathrm{Bl\,Harm}(m)(D)$ *into* $\Lambda_k(D)$.

(b) *For any integers $m \geq 1$ and $r \geq 1$ the operator L_m^r maps continuously:*

1) $\mathrm{Harm}_p^s(m)(D)$ *into* $W_p^s(D)$ *and* $\mathrm{Harm}_p^s(m)(D)$ *into* $\mathring{W}_p^s(D) \subset L^p(D, |\varrho|^{-ps})$ *if $s \geq 0$ and $r \geq s$, $1 < p < \infty$;*

2) $\Lambda_\alpha\,\mathrm{Harm}(m)(D)$ *into* $L^\infty(D, |\varrho|^{-\alpha})$ *if $r > \alpha$;*

3) $\mathrm{Bl\,Harm}(m)(D)$ *into* $L^\infty(D)$.

Propositions 1 and 2 imply the following duality theorem.

THEOREM 1. (a) $\mathrm{Harm}_p^s(m)(D)$ *and* $\mathrm{Harm}_q^{-s}(m)(D)$ *are mutually dual via the pairing $\langle\cdot,\cdot\rangle_{r,m}$ if $0 \leq s \leq r$ and $1/p+1/q = 1$, $1 < p < \infty$.*

$\Lambda_\alpha\,\mathrm{Harm}(m)(D)$ *represents the dual of the space* $\mathring{L}^1\,\mathrm{Harm}(m)(D, |\varrho|^\alpha)$, *which is the closure of* $L^2\,\mathrm{Harm}(m)(D, |\varrho|^\alpha)$ *in* $L^1(D, |\varrho|^\alpha)$, *via the pairing $\langle\cdot,\cdot\rangle_{r,m}$ if $r > \alpha$.*

$\mathrm{Bl\,Harm}(m)(D)$ *represents the dual of* $L^1\,\mathrm{Harm}(m)(D)$ *via the pairing $\langle\cdot,\cdot\rangle_{r,m}$ if $r \geq 1$.*

(b) *If $s < 1/p$, then* $\mathrm{Harm}_p^s(m)(D)$ *is equal to* $L^p\,\mathrm{Harm}(m)(D, |\varrho|^{-ps})$, *with equivalent norms.*

Propositions 1 and 2 and duality arguments permit us to get the following.

THEOREM 2. *For any $-\infty < s < \infty$, $1 < p < \infty$ and any integer $m \geq 1$*

$$B_{pp}^s\,\mathrm{Harm}(m)(D) = \mathrm{Harm}_p^s(m)(D)$$

with equivalent norms.

Remark 1. Theorem 2 is of some interest only for small s, namely for

$$s \leq \max\big((m-1)(1-2/p),\, m(2/p-1)\big).$$

Note that if $s > m-1+1/p$, then the spaces $B_{pp}^s(D)$ and $W_p^s(D)$ have the same traces on ∂D equal to $\prod_{j=0}^{m-1} B_{pp}^{s-j-1/p}(\partial D)$ (see [31]). Thus, roughly speaking, the m-polyharmonic extension of these traces must be the same. Interpolation with $L^2\,\mathrm{Harm}(m)(D)$ will prove our theorem for $s > \max((m-1)(1-2/p), m(2/p-1))$. (See also the proof of Remark 3 below.)

Remark 2. Theorem 2 together with part (b) of Theorem 1 gives us an explicit description of all Sobolev spaces of m-polyharmonic functions. If $s \geq 0$ is an integer, $\mathrm{Harm}_p^s(m)(D)$ is the classical Sobolev space. If $s > 0$ is not an integer then the equivalent Besov norm is given by an explicit formula (see Introduction). If $s < 1/p$ then the norm in $\mathrm{Harm}^s_p(m)(D)$ is the $L^p(D, |\varrho|^{-ps})$ norm. Note that if $0 < s < 1/p$ then the Besov and $L^p(D, |\varrho|^{-ps})$ norms are equivalent on m-polyharmonic functions for every m. Thus we get the following

COROLLARY 1. *Let f be a polyharmonic function of finite order on D. For any $0 < \beta < 1$ and $1 \leqslant p < \infty$ the following conditions are equivalent*:

(a) $$\int_D |f|^p |\varrho|^\beta \, dV < \infty.$$

(b) $$\int_{D\times D} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\beta}} dV < \infty.$$

We now state the following

THEOREM 3. *Let A^s_p be equal to $\mathrm{Harm}^s_p(m)(D)$ for $1 < p < \infty$, $-\infty < s < \infty$, to $\Lambda_s \mathrm{Harm}(m)(D)$ for $s > 0$, $p = \infty$, to $\mathrm{Bl}\,\mathrm{Harm}(m)(D)$ for $s = 0$, $p = \infty$, and to $L^\infty \mathrm{Harm}(D, |\varrho|^s)$ for $0 > s > -\infty$, $p = \infty$. Then*

$$[A^{s_1}_{p_1}, A^{s_2}_{p_2}]_{[\theta]} = A^s_p \quad \text{if } \min(p_1, p_2) < \infty, \qquad [A^{s_1}_\infty, A^{s_2}_\infty]^{\sim}_{[\theta]} = A^s_\infty,$$
$$(A^{s_1}_{p_1}, A^{s_2}_{p_2})_{\theta,p} = A^s_p,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad s = (1-\theta)s_1 + \theta s_2, \qquad 0 < \theta < 1$$

and $[A, B]^{\sim}_{[\theta]}$ denotes the completion of $[A, B]_{[\theta]}$ with respect to $A+B$.

Theorem 3 extends the results of [22] and [24].

Remark 3. Theorem 2 and some interpolation results similar to Theorem 3 are valid in a more general case. Let T be a differential operator of order $2m$ with $C^\infty(\bar{D})$ coefficients such that

$$|\langle T\varphi, \varphi\rangle| \geqslant c(\|\varphi\|^{(m)}_2)^2 \quad \text{on } \mathring{W}^m_2(D).$$

Then

(a) $(\mathrm{Harm}_T)^s_p(D) = B^s_{pp}\mathrm{Harm}_T(D)$,

where

$$(\mathrm{Harm}_T)^s_p(D) = W^s_p(D) \cap \mathrm{Ker}\, T,$$
$$B^s_{pp}\mathrm{Harm}_T(D) = B^s_{pp}(D) \cap \mathrm{Ker}\, T, \qquad -\infty < s < \infty, \ 1 < p < \infty.$$

(b) *If T is in addition selfadjoint then the spaces $A^s_p = B^s_{pp}\mathrm{Harm}_T(D)$, $-\infty < s < \infty$, $1 < p \leqslant \infty$, have the same interpolation properties as in Theorem 3.*

Remark 3 will be proved later.

Since $B^s_{\infty\infty}\mathrm{Harm}_T(D) = \Lambda_s(D) \cap \mathrm{Ker}\, T$, and $B^0_{pp}\mathrm{Harm}_T(D) = L^p\mathrm{Harm}_T(D)$ by (a), we see that (b) gives the possibility of interpolation between Hölder and L^p spaces of functions from $\mathrm{Ker}\, T$ as in the case of m-polyharmonic functions. However, we have no explicit characterization of the spaces $B^s_{pp}\mathrm{Harm}_T(D)$ for $s < 0$ (see Problem 4 at the end of this paper).

We can also consider the more general scale of spaces $F^s_{pq}(D)$ described in [31]. Note that $F^s_{p2}(D) = W^s_p(D)$ and $F^s_{pp}(D) = B^s_{pp}(D)$. It turns out that if T is such as in Remark 3 then $F^s_{pq}\mathrm{Harm}_T(D) = (\mathrm{Harm}_T)^s_p(D)$ for all $-\infty < s < \infty$, $0 < p < \infty$, $0 < q < \infty$.

It is also possible to use Theorem 1 to get an explicit characterization of the spaces $B^s_{pq}\mathrm{Harm}(m)(D)$ for $s < 0$. We deal with those general Triebel–Lizorkin spaces and Besov spaces in Appendix 1 in the final part of this paper.

The duality theorem cannot be extended to the case $p < 1$ since Sobolev and Besov spaces are only quasi-Banach for $p < 1$. However, the recent results of Franke [10], [11] (also mentioned in the Russian edition of [31]) permit us to show that P_T maps continuously B^s_{pp} into B^s_{pp} for $p \leqslant 1$, $s > n(1/p-1)$, and that one can adjoin the spaces $B^s_{pp}\mathrm{Harm}_T(D) = (\mathrm{Harm}_T)^s_p(D)$, $-\infty < s < \infty$, $p \leqslant 1$, to the interpolation scale described in Remark 3. We deal with this case in Appendix 2. In particular, we show there that $B^{-s}_{11}\mathrm{Harm}(m)(D) = \mathring{L}^1\mathrm{Harm}(m)(D, |\varrho|^s)$ and $B^{-s}_{\infty\infty}\mathrm{Harm}(m)(D) = L^\infty\mathrm{Harm}(m)(D, |\varrho|^s)$ for $s > 0$.

Putting aside these generalizations, we return to our polyharmonic functions.

THEOREM 4. *Let $k \geqslant m > 0$ be integers. The projection P_m restricted to $\mathrm{Harm}(k)(D)$ is bounded in every $L^p(D, |\varrho|^t)$ norm, $0 \leqslant t < \infty$, $1 \leqslant p < \infty$, $\min(1/t, p) < \infty$.*

Let now D be a strictly pseudoconvex domain in C^n. We denote by B the orthogonal projection from $L^2(D)$ onto the space of square-integrable holomorphic functions (the *Bergman projection*). We proved in [21] and [22] that B restricted to the space of harmonic functions is bounded in $L^p(D, |\varrho|^t)$ norms if $1 \leqslant p < \infty$, $t \geqslant 0$. Thus Theorem 4 yields immediately

THEOREM 5. *For every integer $m > 0$, the projection B restricted to $\mathrm{Harm}(m)(D)$ is bounded in every $L^p(D, |\varrho|^t)$ norm for $0 \leqslant t < \infty$, $1 \leqslant p < \infty$.*

Much information about Sobolev spaces of holomorphic functions on the unit ball can be found in [3] (see also [8]). It should also be mentioned that Theorem 5 remains valid if we replace the Bergman projection B by the orthogonal projection Q onto the space of pluriharmonic functions or the projection S_r onto the space of the real parts of holomorphic functions (see [20]).

Let D be a strictly pseudoconvex domain, and let ϱ be a defining function of D which is strictly plurisubharmonic in a neighborhood of ∂D. Let

$$L = \sum_{i,j} \frac{\partial \varrho}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$$

be the Kähler metric on D induced by the potential ϱ. Denote by $\langle \cdot, \cdot \rangle_L$ the scalar product induced by L.

Let $\Box_L = \bar\partial \bar\partial_L^* + \bar\partial_L^* \bar\partial$ (the adjoint is taken with respect to $\langle \cdot, \cdot \rangle_L$). Let N_L be the operator solving the $\bar\partial$-Neumann problem $\Box_L \omega = \xi$, ω, ξ $(0, q)$-differential forms, $\omega \in \operatorname{Dom} \Box_L$ (see [9] for details). Lieb and Range in [15]–[17] constructed an integral representation for the operator $\bar\partial_L^* N_L$ and obtained the Hölder estimates for the operators $\bar\partial_L^* N_L$ and N_L. They proved that $\bar\partial_L^* N_L$ maps $\Lambda_\alpha(0, q+1)(D)$ into $\Lambda_{\alpha+1/2}(0, q)(D)$ and N maps $\Lambda_\alpha(0, q)(D)$ into $\Lambda_{\alpha+1}(0, q)(D)$ ($\Lambda_\alpha(0, q)(D)$ denotes here the space of $(0, q)$-forms with coefficients from $\Lambda_\alpha(D)$).

Remark 3 permits us to show that the Lieb–Range results imply the following

THEOREM 6. *The operator N_L maps continuously $W_p^s(0, q)(D)$ into $W_p^{s+1}(0, q)(D)$ for $0 \leqslant s < \infty$, $1 < p < \infty$, and the operator $\bar\partial_L^* N_L$ maps continuously $W_p^s(0, q+1)(D)$ into $W_p^{s+1/2}(0, q)(D)$, $0 \leqslant s < \infty$, $1 < p < \infty$.*

$W_p^s(0, q)(D)$ denotes here the space of $(0, q)$-forms with coefficients from $W_p^s(D)$.

If D is equal to the unit ball then the usual Euclidean metric on D is equal to L since $\varrho = |z|^2 - 1$. Thus we have $\Box_L = -\frac{1}{2}\Delta$ and we get the following application of Theorem 1:

COROLLARY 2. *If $D = B(0, 1)$ then N extends to a continuous mapping from the space of $(0, q)$-forms with coefficients in $L^p \operatorname{Harm}(m)(D, |\varrho|^t)$ into the space of $(0, q)$-forms with coefficients in $L^p \operatorname{Harm}(m+1)(D, |\varrho|^{t-p})$ if $t > p-1$ and in $\operatorname{Harm}_p^{-t/p+1}(m+1)(D)$ if $t \leqslant p-1$, $1 < p < \infty$.*

The operator $\bar\partial^* N$ extends to a mapping from the space of $(0, q+1)$-forms with coefficients in $L^p \operatorname{Harm}(m)(D, |\varrho|^t)$ into the space of $(0, q)$-forms with coefficients in $L^p \operatorname{Harm}(m+1)(D, |\varrho|^{t-p/2})$ if $t > p/2 - 1$ and in $\operatorname{Harm}_p^{-t/p+1/2}(m+1)(D)$ if $t \leqslant p/2-1$. Here $1 \leqslant p < \infty$. [For $p = 1$ we take $\mathring{L}^1 \operatorname{Harm}(m)(D, |\varrho|^t)$, which is the closure of $L^2 \operatorname{Harm}(m)(D)$ in $\mathring{L}^1(D, |\varrho|^t)$.]

The last corollary justifies our conjecture from [22] that the operators N and $\partial^* N$ should "behave better" on forms with harmonic coefficients. For $(0, 1)$-forms the Hölder and W_p^s estimates were first proved by Greiner and Stein [12].

We end the present paper with a list of open problems.

III. Proofs.

1. Proof of Proposition 1. We begin with two lemmas:

LEMMA 1. (a) *For $h \in \Lambda_\alpha \operatorname{Harm}(m)(D)$ we have*

$$|D^k h(x)| \leqslant c \frac{\|h\|_\alpha}{|\varrho(x)|^{k-\alpha}} \quad \text{if } k > [\alpha].$$

(b) *For $h \in \operatorname{Bl} \operatorname{Harm}(m)(D)$ we have*

$$|D^k h(x)| \leqslant c \frac{\|h\|_{\mathrm{Bl}(D)}}{|\varrho(x)|^k}.$$

(c) *For $h \in L^\infty \operatorname{Harm}(m)(D, |\varrho|^t)$ we have*

$$|D^k h(x)| \leqslant c \frac{\|h\|_{L^\infty(D, |\varrho|^t)}}{|\varrho(x)|^{k+t}}.$$

Proof. Let $x \in D$, $\delta = \operatorname{dist}(x, \partial D)$. We can assume that $x = 0$. Let $u \in \Lambda_\alpha \operatorname{Harm}(m)(D)$. We have for $m > k > 0$

$$\int_{B(0,\delta/2)} \Delta^{m-k} (|x|^2 - \tfrac{1}{4}\delta^2)^{2(m-k)} u(x)\, dV_x = \int_{B(0,\delta/2)} (|x|^2 - \tfrac{1}{4}\delta^2)^{2m-2k} \Delta^{m-k} u(x)\, dV_x$$
$$= \|(|x|^2 - \tfrac{1}{4}\delta^2)^{2m-2k}\|_{L^1(D)} \Delta^{m-k} u(0)$$
$$+ \int_0^{\delta/2} r^{n-1} (r^2 - \tfrac{1}{2}\delta^2)^{2m-2k} \int_{|y|<r} \frac{\Delta^{m-k+1} u}{|y|^{n-2}}\, dV_y\, dr.$$

The above formula permits us to show after elementary calculations that $|\Delta^{m-1} u(x)| \leqslant c \|u\|_{\Lambda_\alpha} / \delta(x)^{2m-2-\alpha}$ and next, by induction on k, that $|\Delta^{m-k} u(x)| \leqslant c \|u\|_{\Lambda_\alpha} / \delta(x)^{2m-2k-\alpha}$. In particular, $|\Delta u(x)| \leqslant c \|u\|_{\Lambda_\alpha} / \delta(x)^{2-\alpha}$.

We now proceed as in the lemma in the proof of Theorem 2 in [19] and write $u|_{B(0,\delta/2)} = u_1 + h$, where h is harmonic, $u_1 = 0$ on $\partial B(0, \delta/2)$. We use the mean value theorem for h and the fact that

$$u_1(x) = \int_{B(0,\delta/2)} G(x, y) \Delta u(y)\, dV_y,$$

where $G(x, y)$ is the Green function of $B(0, \delta/2)$ to get the desired estimates for all derivatives of u. The same procedure permits us to prove parts (b) and (c) of the lemma.

Similar estimates of the derivatives of a polyharmonic function can also be found in [2].

LEMMA 2. *Assume that Proposition 2(a1) holds for $(m-1)$-polyharmonic functions, i.e. for each $k > 0$, $T_k f = \varrho^k f$ maps continuously $\operatorname{Harm}_p^s(m-1)(D)$ into $W_p^{s+k}(D)$* for $-\infty < s < \infty$, $1 < p < \infty$. *Then:*

(i) *For all* $\varphi \in C^\infty(\bar{D})$ *and* $-\infty < s < \infty$, $u \to \Delta^k(\varrho^k u\varphi)$ *maps continuously* $\operatorname{Harm}_p^s(m)(D)$ *into* $W_p^{s-k}(D)$.

(ii) T_k *maps continuously* $\operatorname{Harm}_p^s(m)(D)$ *into* $W_p^{s+k}(D)$ *for* $s \geqslant 0$.

Proof. (i) Let $u \in \operatorname{Harm}_p^s(m)(D)$. For $k = 1$ we have

$$\Delta(\varrho u\varphi) = \Delta(\varrho\varphi)\cdot u + \varrho\varphi\Delta u + 2\sum_i \frac{\partial(\varrho\varphi)}{\partial x_i}\frac{\partial u}{\partial x_i}.$$

Now, $\varrho\Delta u \in W_p^{s-1}(D)$ since Δu is $(m-1)$-polyharmonic. Thus $\Delta(\varrho u\varphi) \in W_p^{s-1}(D)$ and $\|\Delta(\varrho u\varphi)\|_p^{s-1} \leqslant c\|u\|_p^s$.

Assume now that (i) holds for all $l \leqslant k$. Then

$$\Delta^{k+1}(\varrho^{k+1}u\varphi) = \Delta^k(\varrho^{k+1}\varphi\Delta u)\,\Delta^k(\varrho^{k+1}\Delta\varphi\cdot u) + (k+1)\Delta^k(\varrho^k\Delta\varrho\cdot\varphi u)$$
$$+2\sum_i\Bigg[\Delta^k\left(\varrho^{k+1}\frac{\partial\varphi}{\partial x_i}\frac{\partial u}{\partial x_i}\right) + (k+1)\Bigg\{\Delta^k\left(\varrho^k\varphi\frac{\partial\varrho}{\partial x_i}\frac{\partial u}{\partial x_i}\right)$$
$$+\tfrac{1}{2}k\Delta\left(\Delta^{k-1}\left(\varrho^{k-1}\left(\frac{\partial\varrho}{\partial x_i}\right)^2\varphi u\right)\right)$$
$$+\Delta^k\left(\varrho^k\frac{\partial\varrho}{\partial x_i}\frac{\partial\varphi}{\partial x_i}u\right)\Bigg\}\Bigg] \in W_p^{s-k-1}(D)$$

by the assumptions of the lemma and the inductive assumption on k. We have again

$$\|\Delta^{k+1}(\varrho^{k+1}u\varphi)\|_p^{s-k-1} \leqslant c\|u\|_p^s.$$

(ii) If $s \geqslant 0$ then $\varrho^k u = G_k(\Delta^k\varrho^k u)$. The estimates from [26] (Theorem 5.4) yield that

$$\|\varrho^k u\|_p^{s+k} \leqslant c\|\Delta^k\varrho^k u\|_p^{s-k} \quad \text{(cf. [18], [22]).}$$

We now prove parts (a) and (b) of Proposition 1. We proceed in exactly the same manner as in [19] and [22]. We have

$$P_m u = u - \Delta^m G_{2m}\Delta^m u = \Delta^m(G_m u - G_{2m}\Delta^{2m}G_m u).$$

Note that the function in parentheses on the right is the $2m$-polyharmonic extension of $\operatorname{Tr} G_m u$ from ∂D to D. ($\operatorname{Tr} f$ is equal here to $(f\,|\,\partial D, \ldots, \partial^k f/\partial n^k|_{\partial D}, \ldots, \partial^{2m-1}f/\partial n^{2m-1}|_{\partial D})$). We have

$$G_m u(x) = \int_D G_m(x, y)u(y)\,dV_y = \int_D V_m(x, y)u(y)\,dV_y - \int_D G_m(x, y)\,dV_y,$$

where $G_m(x, y)$ is a Green function for the Dirichlet problem $\Delta^m f = g$, f vanishes on ∂D up to order $m-1$, $V_m(x, y)$ is a fundamental solution of the equation $\Delta^m f = 0$ and $G_m(x, y)$ is m-polyharmonic with respect to x ($G_m(x, y) = G_m(y, x)$) and such that for every $y \in D$, $G_m(x, y)$ vanishes on ∂D up to order $m-1$.

The fundamental solution $V_m(x, y)$ is

(1) $V_m(x, y) = c(n)|x-y|^{2m-n}$ if n is odd or if n is even and $n > 2m$,

(2) $V_m(x, y) = c(n)|x-y|^{2m-n}\ln|x-y|$ if $2m \geqslant n$ and n is even.

Let

$$w(x) = \int_D V_m(x, y)u(y)\,dV_y.$$

Then

$$P_m u = \Delta^m(w(x) - G_{2m}\Delta^{2m}w(x))$$

is a $2m$-polyharmonic extension of $\operatorname{Tr} w(x)$ on ∂D in view of the estimates from [1] (Th. 12.10 and what follows). It remains to prove:

(a) If $u = |\varrho|^\alpha m$, $m \in L^\infty(D)$, $0 < \alpha < \infty$, then

$$w(x)|_{\partial D} \in \Lambda_{2m+\alpha}(\partial D), \qquad \frac{\partial^j w}{\partial n^j}(x)\Big|_{\partial D} \in \Lambda_{2m-j+\alpha}(\partial D), \qquad 0 \leqslant j \leqslant 2m-1.$$

(b) If $u = m/|\varrho|^\beta$, $0 < \beta < 1$, $m \in L^\infty(D)$, then

$$w(x)|_{\partial D} \in \Lambda_{2m-\beta}(\partial D), \qquad \frac{\partial^j w}{\partial n^j}(x)\Big|_{\partial D} \in \Lambda_{2m-j-\beta}(\partial D), \qquad 0 \leqslant j \leqslant 2m-1.$$

If $x \in \partial D$ then

$$w(x) = c(n)\int_D (|x-y|^2 + \varrho(x)\varrho(y))^{m-n/2}u(y)\,dV_y = w_0(x)$$

or

$$w(x) = c(n)\int_D (|x-y|^2 + \varrho(x)\varrho(y))^{m-n/2}u(y)\ln(|x-y|^2 + \varrho(x)\varrho(y))\,dV_y = w_0(x).$$

If $u = |\varrho|^\alpha m$ we differentiate $w_0(x)$ $k = 2m + [\alpha] + 1$ times in order to get gradient estimates and use the Hardy–Littlewood lemma. The standard calculations show that $|D^\gamma w_0(x)| \leqslant c\|m\|_{L^\infty}/|\varrho(x)|^{1-\alpha}$ if $|\gamma| = k$. Thus $w_0(x) \in \Lambda_{2m+\alpha}(D)$ and $w(x)|_{\partial D} \in \Lambda_{2m+\alpha}(\partial D)$.

If $u = m/|\varrho|^\beta$ we proceed as in [24], assuming first that $\operatorname{supp} m \Subset D$ and differentiating $w_0(x)$ $2m$ times. We then have

$$|D^\gamma w_0(x)| \leqslant c\|m\|_{L^\infty}\int_D \frac{1}{(|x-y|^2 + \varrho(x)\varrho(y))^{n/2}}\cdot\frac{1}{|\varrho(y)|^\beta}\,dV_y$$

if $|\gamma| = 2m$. The last integral can be written as

$$\int_D \frac{\sum_i (\partial\varrho/\partial y_i)^2 \varphi(y)\,dV_y}{(|x-y|^2+\varrho(x)\varrho(y))^{n/2}|\varrho(y)|^\beta} + \int_D \frac{(1-\sum_i (\partial\varrho/\partial y_i)^2)\varphi(y)\,dV_y}{(|x-y|^2+\varrho(x)\varrho(y))^{n/2}|\varrho(y)|^\beta}$$

where $\varphi(y) = \psi(y)/\sum_i(\partial\varrho/\partial y_i)^2$, $\psi \in C_0^\infty(\boldsymbol{R}^n)$, $\psi = 0$ in a neighborhood of the set $\{\operatorname{grad}\varrho = 0\}$, and $\psi = 1$ in a neighborhood of ∂D. The second integral is a bounded function. We can now integrate by parts in the first integral and prove that it is bounded by $c/|\varrho(x)|^\beta$. Thus $|D^\gamma w_0(x)| \leqslant c\|m\|_{L^\infty}/|\varrho(x)^\beta$ and thus $w_0|_{\partial D} \in \Lambda_{2m-\beta}(\partial D)$.

For every $m \in L^\infty(D)$ there exists a sequence $m_n \to m$ in every $L^p(D)$, $p < \infty$, such that $\|m_n\|_{L^\infty} \leqslant \|m\|_{L^\infty}$. Thus we can extend the above estimates to arbitrary $m \in L^\infty(D)$. The same procedure can be applied to the functions $w_j = \partial^j w/\partial n^j$, which ends the proof of Proposition 1(a), (b).

Proposition 1(c) follows from interpolation. For all i and m the mapping $(\partial/\partial x_i)P_m$ maps continuously $L^\infty(D, |\varrho|^{-\alpha})$ into $L^\infty(D, |\varrho|^{1-\alpha})$ and $L^\infty(D, |\varrho|^\alpha)$ into $L^\infty(D, |\varrho|^{1+\alpha})$ by Lemma 1 $(0 < \alpha < 1)$. Thus $(\partial/\partial x_i)P_m$ maps continuously

$$L^\infty(D) = [L^\infty(D, |\varrho|^{-\alpha}),\ L^\infty(D, |\varrho|^\alpha)]_{[1/2]}$$

into

$$L^\infty(D, |\varrho|) = [L^\infty(D, |\varrho|^{1-\alpha}),\ L^\infty(D, |\varrho|^{1+\alpha})]_{[1/2]}$$

(see [24]). This means that P_m maps continuously $L^\infty(D)$ into $\mathrm{Bl}\,\mathrm{Harm}(m)(D)$.

We now prove part (d) of Proposition 1 under the assumptions of Lemma 3.

As was said in the introduction, the projection P_m maps continuously $L^p(D)$ into $L^p(D)$ for $1 < p < \infty$. It follows from Lemma 2 that for every β, $|\beta| = k$, the mapping $(\partial^{|\beta|}/\partial x^\beta)P_m$ maps continuously $L^p(D)$ into $L^p(D, |\varrho|^{pk})$ since $\varrho^l P_m u \in W_p^l(D)$ for each $l \leqslant k$ and $u \in L^p(D)$ and thus $\varrho^k(\partial^{|\beta|}/\partial x^\beta)P_m u \in L^p(D)$. Lemma 1 implies that if $k \geqslant [\alpha]+1$ then $(\partial^{|\beta|}/\partial x^\beta)P_m$ maps continuously $L^\infty(D, |\varrho|^{-\alpha})$ into $L^\infty(D, |\varrho|^{-k-\alpha})$. In [22, Proposition 2] we have proved that

$$[L^p(D, |\varrho|^s),\ L^\infty(D, |\varrho|^r)]_{[\theta]} = L^q(D, |\varrho|^t), \quad q = \frac{p}{1-\theta},\ t = s+\frac{p\theta r}{1-\theta}.$$

Putting $s = 0$, $r = -\alpha$ or $s = 0$, $r = k-\alpha$ we prove by interpolation that $(\partial^{|\beta|}/\partial x^\beta)P_m$ maps continuously

$$L^{p/(1-\theta)}(D, |\varrho|^{-\alpha\theta p/(1-\theta)}) \quad \text{into} \quad L^{p/(1-\theta)}(D, |\varrho|^{\theta(k-\alpha)p/(1-\theta)+pk}).$$

Let $s_1 = \theta\alpha$, $p_1 = p/(1-\theta)$. We have $L^{p_1}(D, |\varrho|^{p_1(k-s_1)}) \subset W_{p_1}^{-k+s_1}(D)$. Hence P_m maps $L^{p_1}(D, |\varrho|^{-p_1 s_1})$ into $W_{p_1}^{s_1}(D)$.

Thus we have proved Proposition 1(d) for m-polyharmonic functions under the condition that Proposition 2(a1) is valid for $(m-1)$-polyharmonic functions. Proposition 1(d) for $-1/p+1 < s < 0$ follows immediately from Proposition 1(c) and interpolation.

2. Proof of Proposition 2 and Theorem 1. Parts (a2)–(a4) of Proposition 2 follow immediately from Lemma 1. Parts (b2) and (b3) also follow from Lemma 1 and the construction of L_m^r, since $L_m^r u$ consists of terms of the type $\varrho^k D^\alpha u\cdot$(smooth function), where $k \geqslant r$ and $|\alpha| \leqslant k$. Thus the fact that $(\mathring{L}^1\,\mathrm{Harm}(m)(D, |\varrho|^\alpha))^* \cong \Lambda_\alpha\,\mathrm{Harm}(m)(D)$ can be proved in the same manner as in [19], namely for every $u \in \Lambda_\alpha\,\mathrm{Harm}(m)(D)$, $\langle\cdot, L_m^r u\rangle$ is a continuous functional on $\mathring{L}^1\,\mathrm{Harm}(m)(D, |\varrho|^\alpha)$, and every continuous functional φ on $\mathring{L}^1\,\mathrm{Harm}(m)(D, |\varrho|^\alpha)$ can be extended to a continuous functional on $L^1(D, |\varrho|^\alpha)$ and hence represented by a function $\mu \in L^\infty(D, |\varrho|^{-\alpha})$. The function $u = P_m\mu \in \Lambda_\alpha\,\mathrm{Harm}(m)(D)$ represents φ on $\mathring{L}^1\,\mathrm{Harm}(m)(D, |\varrho|^\alpha)$ via $\langle\cdot,\cdot\rangle_{r,m}$. Analogously we can prove that $\mathrm{Bl}\,\mathrm{Harm}(m)(D)$ represents the dual of $L^1\,\mathrm{Harm}(m)(D)$.

Hence it remains to prove Proposition 2(a1), (b1) and Theorem 1 for $1 < p < \infty$. We do it by induction on m.

In the case of $m = 1$, i.e. in the case of harmonic functions the needed facts were proved in [22]. Let us assume that they are valid for all $k \leqslant m-1$. Then Proposition 2(a1) yields that Proposition 1(d) is valid for $k = m$. Proposition 1(d) and the construction of L_m^r imply that L_m^r maps $L^p\,\mathrm{Harm}(m)(D)$ into $L^p(D)$ and $\mathrm{Harm}_p^r(m)(D)$ into $\mathring{W}_p^r \subset L^p(D, |\varrho|^{-pr})$. Thus complex interpolation implies that Proposition 2(b1) holds for $k = m$.

We can now repeat the proof of the theorem from [22] in order to prove that $\mathrm{Harm}_q^{-s}(m)(D)$ and $\mathrm{Harm}_p^s(m)(D)$, $1/q+1/p = 1$, are mutually dual via the pairing $\langle\cdot,\cdot\rangle_{r,m}$, $r \geqslant s$. Briefly, Proposition 2(b1) yields that every $u \in \mathrm{Harm}_p^s(m)(D)$ represents a continuous functional on $\mathrm{Harm}_q^{-s}(m)(D)$ via $\langle\cdot,\cdot\rangle_{r,m}$. If φ is a continuous functional on $\mathrm{Harm}_q^{-s}(m)(D)$ then it can be extended to a continuous functional on $W_q^{-s}(D)$ and then represented by $\mu \in \mathring{W}_p^s(D)$. The function $u = P_m\mu \in \mathrm{Harm}_p^s(m)(D)$ represents φ on $\mathrm{Harm}_q^{-s}(m)(D)$. If ψ is a continuous functional on $\mathrm{Harm}_p^s(m)(D)$ then it can be extended to $\tilde\psi \in (W_p^s(D))^*$. Since P_m maps $\mathring{W}_p^s(D)$ into $W_p^s(D)$ and is selfadjoint, P_m extends to a continuous mapping from $(W_p^s(D))^*$ into $W_q^{-s}(D)$. Thus $P_m(\tilde\psi)$ represents ψ via $\langle\cdot,\cdot\rangle_{r,m}$. (The above proof is based on the ideas from [4], [5] and [8].)

On the other hand, $\mathrm{Harm}_p^s(m)(D)$ represents the dual of $L^q\,\mathrm{Harm}(m)(D, |\varrho|^{qs})$ via the same pairing. This implies that $\mathrm{Harm}_q^{-s}(m)(D) = L^q\,\mathrm{Harm}(m)(D, |\varrho|^{qs})$. Thus Proposition 2(a1) holds for $k = m$.

In order to end the proof of Theorem 1 we must show that $\mathrm{Harm}_p^s(m)(D) = L^p\,\mathrm{Harm}(m)(D, |\varrho|^{-ps})$ for $0 < s < 1/p$. By Proposition 1(b), P_m maps $L^\infty(D, |\varrho|^\alpha)$ into itself if $0 < \alpha < 1$. Since P_m maps $L^p(D)$ into itself,

$1<p<\infty$, we can use complex interpolation to show that P_m maps $L^q(D, |\varrho|^{-qs})$ into itself for $-1+1/q<s<0$. Since P_m is selfadjoint and $L^p(D, |\varrho|^{ps})$, $1/p+1/q=1$, represents the dual of $L^q(D, |\varrho|^{qs})$ via $\langle\cdot,\cdot\rangle_0$, the operator P_m must map $L^p(D, |\varrho|^{ps})$ into itself. Thus $L^p\,\mathrm{Harm}(m)(D, |\varrho|^{ps}) = \mathrm{Harm}_p^{-s}(m)(D)$, $-1/p<s<0$, $1<p<\infty$, being the ranges of the same set under the projection P_m. The above duality relations show that the norms must also be equivalent.

3. Proof of Theorem 2. Theorem 2 is obviously a special case of Remark 3(a), but we wish to give here another proof since it can be useful in extending our results to the case of domains with boundary of class C^k (see [19], [22]).

First, assume that $s>1/p-1$. Since P_m maps $W_p^s(D)$ into itself (for $1/p-1<s<0$ this follows from $W_q^{-s}(D) = \mathring{W}_q^{-s}(D)$ and from the fact that P_m is selfadjoint) and

$$(W_p^{s_1}(D), W_p^{s_2}(D))_{\theta,p} = B_{pp}^s(D), \quad s=(1-\theta)s_1+\theta s_2,\ s_1\neq s_2,$$

P_m maps $B_{pp}^s(D)$ into itself if $s>1/p-1$. In particular, we have

$$(\mathrm{Harm}_p^{s_1}(m)(D), \mathrm{Harm}_p^{s_2}(m)(D))_{\theta,p} = B_{pp}^s\,\mathrm{Harm}(m)(D)$$
$$\text{if } \min(s_1, s_2)>1/p-1,\ s=(1-\theta)s_1+\theta s_2.$$

Thus L_m^r, $r\geqslant \max(s_1, s_2)$, maps $B_{pp}^s\,\mathrm{Harm}(m)(D)$ into

$$(L^p(D, |\varrho|^{-ps_1}), L^p(D, |\varrho|^{-ps_2}))_{\theta,p} = L^p(D, |\varrho|^{-ps})$$

by Theorem 5.5.1 of [7]. Thus

$$B_{pp}^s\,\mathrm{Harm}(m)(D) = P_m\big(L^p(D, |\varrho|^{-ps})\big) = \mathrm{Harm}_p^s(m)(D) \quad \text{for } s>1/p-1.$$

The Sobolev and Besov norms are both equivalent to $\|L_m^r u\|_{L^p(D,|\varrho|^{-ps})}$.

Now we can use the real interpolation functor $(\cdot,\cdot)_{\theta,p}$ to prove that L_m^r, $r\geqslant s$, maps $B_{pp}^s\,\mathrm{Harm}(m)(D)$ into $\mathring{B}_{pp}^s(D)$ if $s\neq k+1/p$, and complex interpolation to prove that L_m^r maps $B_{pp}^s\,\mathrm{Harm}(m)(D)$ into $\mathring{B}_{pp}^s(D)$ for $s=k+1/p$. Then we use the same considerations as in the proof of Theorem 1 to show that $B_{qq}^{-s}\,\mathrm{Harm}(m)(D)$ and $B_{pp}^s\,\mathrm{Harm}(m)(D)$ are mutually dual via $\langle\cdot,\cdot\rangle_{r,m}$, $r\geqslant s$, $1/q+1/p=1$. Finally, we obtain

$$B_{qq}^{-s}\,\mathrm{Harm}(m)(D) = L^q\,\mathrm{Harm}(m)(D, |\varrho|^{qs}) = \mathrm{Harm}_q^{-s}(m)(D).$$

4. Proof of Remark 3 and Theorem 3. We begin with a proof of part (a) of Remark 3. We can assume that T is selfadjoint (since we can always replace T by T^*T). Thus for each integer k, the Dirichlet problem is uniquely solvable for T^k. Since T^k is of order $2mk$, the estimates from [26] (Th. 5.7.2) show that G_{T^k} maps continuously $W_p^{-km}(D)$ onto $\mathring{W}_p^{km}(D)$, $1<p<\infty$. It is well known that G_{T^k} maps $W_p^s(D)$ into $W_p^{s+2km}(D)$ for $s\geqslant 0$. Then complex interpolation shows that G_{T^k} maps $W_p^s(D)$ into $W_p^{s+2mk}(D)$ for each $s\geqslant -mk$. If we use the real interpolation functor $(\cdot,\cdot)_{\theta,p}$ we can prove that G_{T^k} maps $B_{pp}^s(D)$ into $B_{pp}^{s+2mk}(D)$ for $s>-mk$.

We now prove that

$$(\mathrm{Harm}_{T^k})_p^s(D) = B_{pp}^s\,\mathrm{Harm}_{T^k}(D) \quad \text{for } s>mk.$$

Let $u\in(\mathrm{Harm}_{T^k})_p^s(D)$. The trace $\mathrm{Tr}\,u$ on the boundary belongs to $\prod_{i=1}^{mk-1} B_{pp}^{s-i-1/p}(\partial D) = V(\partial D)$. It follows from [31, 2.7.2] that there exists an extension operator $S\colon V(\partial D)\to W_p^s(D)\cap B_{pp}^s(D)$. We have

$$u = S(\mathrm{Tr}\,u) - G_{T^k}T^k S(\mathrm{Tr}\,u)\in B_{pp}^s\,\mathrm{Harm}_{T^k}(D).$$

Thus $(\mathrm{Harm}_{T^k})_p^s(D)\subset B_{pp}^s\,\mathrm{Harm}_{T^k}(D)$. The opposite inclusion can be proved in the same way.

Let now s be arbitrary, and choose k so large that $s>-(k-1)m$. The operator G_{T^k} maps $(\mathrm{Harm}_T)_p^s(D)$ into $(\mathrm{Harm}_{T^{k+1}})_p^{s+2mk}(D) = B_{pp}^{s+2mk}\,\mathrm{Harm}_{T^{k+1}}(D)$ since $s+2mk>m(k+1)$. Thus

$$(\mathrm{Harm}_T)_p^s(D) = B_{pp}^s\,\mathrm{Harm}_T(D).$$

Let us prove (b). For each k the projector P_{T^k} maps continuously $B_{pp}^s(D)$ onto $B_{pp}^s\,\mathrm{Harm}_{T^k}(D)$ for $1<p\leqslant\infty$, $s>0$ ($B_{\infty\infty}^s(D) = \Lambda_s(D)$). Thus the scale of spaces $B_{pp}^s\,\mathrm{Harm}_{T^k}(D)$ has the same interpolation properties as the scale $B_{pp}^s(D)$, $1<p\leqslant\infty$, $s\geqslant 0$.

By Theorem 12.10 of [1] the operator G_{T^k} maps continuously $B_{\infty\infty}^s(D)$ into $B_{\infty\infty}^{s+2mk}(D)$ for $s>-mk-1$. If k is so large that $\min(s_1, s_2)>-km$ then

$$G_{T^k}([B_{p_1p_1}^{s_1}\,\mathrm{Harm}_T(D), B_{p_2p_2}^{s_2}\,\mathrm{Harm}_T(D)]_{[\theta]})$$
$$\subset [B_{p_1p_1}^{s_1}\,\mathrm{Harm}_{T^{k+1}}(D), B_{p_2p_2}^{s_2}\,\mathrm{Harm}_{T^{k+1}}(D)]_{[\theta]} = B_{pp}^s\,\mathrm{Harm}_{T^{k+1}}(D),$$
$$s=(1-\theta)s_1+\theta s_2,\ \frac{1-\theta}{p_1}+\frac{\theta}{p_2}=\frac{1}{p}.$$

Thus

$$[B_{p_1p_1}^{s_1}\,\mathrm{Harm}_T(D), B_{p_2p_2}^{s_2}\,\mathrm{Harm}_T(D)]_{[\theta]}\subset B_{pp}^s\,\mathrm{Harm}_T(D),$$

s, p as above. On the other hand, T^k maps $B_{pp}^{s+2mk}\,\mathrm{Harm}_{T^{k+1}}(D)$ into $[B_{p_1p_1}^{s_1}\,\mathrm{Harm}_T(D), B_{p_2p_2}^{s_2}\,\mathrm{Harm}_T(D)]_{[\theta]}$ and *onto* $B_{pp}^s\,\mathrm{Harm}_T(D)$. Hence

$$B_{pp}^s\,\mathrm{Harm}_T(D) = [B_{p_1p_1}^{s_1}\,\mathrm{Harm}_T(D), B_{p_2p_2}^{s_2}\,\mathrm{Harm}_T(D)]_{[\theta]},$$

s, p as above. In the same manner we can prove that

$$(B_{p_1p_1}^{s_1}\,\mathrm{Harm}_T(D), B_{p_2p_2}^{s_2}\,\mathrm{Harm}_T(D))_{\theta,p} = B_{pp}^s\,\mathrm{Harm}_T(D)$$

where s, p are as above.

Theorem 3 is now an immediate consequence of Remark 3. The only thing that remains to be proved is that $B^0_{\infty\infty}\operatorname{Harm}(m)(D) = \operatorname{Bl}\operatorname{Harm}(m)(D)$ and $B^{-s}_{\infty\infty}\operatorname{Harm}(m)(D) = L^\infty\operatorname{Harm}(m)(D, |\varrho|^s)$, $s > 0$. These facts will be proved in Appendix 2.

5. Proof of Theorem 4. We prove Theorem 4 for $1 \leqslant p < \infty$. The case $p = \infty$ will be considered separately in Appendix 2.

Let $1 < p < \infty$. We have by Theorem 1

$$L^p\operatorname{Harm}(k)(D, |\varrho|^t) = \operatorname{Harm}_p^{-t/p}(k)(D).$$

Thus for every $u \in L^2\operatorname{Harm}(k)(D)$ and $q = p/(p-1)$

$$\|P_m u\|_{L^p(D,|\varrho|^t)} = \sup_{\substack{v\in \operatorname{Harm}_q^{t/p}(k)(D)\\ \|v\|\leqslant 1}} |\langle P_m u, v\rangle|.$$

Since P_m is selfadjoint the last expression is bounded by

$$\|u\|_p^{-t/p}\|P_m v\|_q^{t/p} \leqslant c\,\|u\|_{L^p(D,|\varrho|^t)}.$$

Since the smooth functions are dense in $L^p\operatorname{Harm}(k)(D, |\varrho|^t)$, P_m extends to a continuous mapping of $L^p\operatorname{Harm}(k)(D, |\varrho|^t)$ into itself.

If $p = 1$ then we must use the fact that each Banach space imbeds isomorphically into its second dual. Thus we again have for $u \in L^2\operatorname{Harm}(k)(D)$

$$\|P_m u\|_{L^1(D,|\varrho|^t)} = \sup_{\substack{v\in \Lambda_t\operatorname{Harm}(k)(D)\\ \|v\|\leqslant 1}} |\langle P_m u, v\rangle| \leqslant c\,\|u\|_{L^1(D,|\varrho|^t)}.$$

Thus P_m extends to a continuous map of $L^1\operatorname{Harm}(k)(D, |\varrho|^t)$ into itself.

Remark. In the same manner we can prove that the operator G_m solving the Dirichlet problem extends to a continuous mapping from $L^p\operatorname{Harm}(k)(D, |\varrho|^t)$ into $\operatorname{Harm}_p^{2m-t/p}(k+m)(D)$ for every $1 < p < \infty$.

6. Proof of Theorem 6. The operator $T = \Box_L$ is selfadjoint with respect to the scalar product $\langle\cdot,\cdot\rangle_L$, and this scalar product is equivalent to the Euclidean scalar product

$$\langle\omega, \xi\rangle_0 = \sum_J \langle\omega_J, \xi_J\rangle_0, \qquad \omega = \sum_J \omega_J\, d\bar z_J,\ \xi = \sum_J \xi_J\, d\bar z_J.$$

We can treat $(0, q)$-forms as vector-valued functions and we have $\langle\omega, \xi\rangle_L = \langle\omega, A(\xi)\rangle_0$, where A is a matrix invertible on $\bar D$ with $C^\infty(\bar D)$ coefficients. The operator $\Box_L = T$ is a strongly elliptic operator of order 2 such that the Dirichlet problem is uniquely solvable for T. We shall consider the operator $N_L - G_T$, which maps $\operatorname{Harm}_T(D)$ into itself.

The operator T fulfills the assumptions of Remark 3 and thus the spaces $B^s_{pp}\operatorname{Harm}_T(D)$, $-\infty < s < \infty$, $1 < p \leqslant \infty$, form an interpolation scale and $B^s_{pp}\operatorname{Harm}_T(D) = (\operatorname{Harm}_T)^s_p(D)$ $(B^s_{\infty\infty} = \Lambda_s)$. Moreover, since $\langle\cdot,\cdot\rangle_0$ and $\langle\cdot,\cdot\rangle_L$ are equivalent in the above-explained sense, the spaces $W_r^{-s}(0, q)(D)$ and $\mathring{W}^s_p(0, q)(D)$, $r = p/(p-1)$, are mutually dual via the pairing $\langle\cdot,\cdot\rangle_L$.

Consider the map L^k_T constructed as in the introduction with $T^* = T$. Then L^k_T maps $(\operatorname{Harm}_T)^s_p(D)$ into $\mathring{W}^s_p(D)$ if $0 \leqslant s \leqslant k$. (Since T is of second order, the proof of this fact is the same as in the case of $T = \Delta$; see [18], [22].) We also have $L^k_T u - u \perp \operatorname{Harm}_T(D)$ with respect to $\langle\cdot,\cdot\rangle_L$. Hence we have the same situation as in Theorem 1 and obtain

(i) $(\operatorname{Harm}_T)^{-s}_r(D)$ *and* $(\operatorname{Harm}_T)^s_p(D)$ *are mutually dual via the pairing* $\langle u, L^k_T v\rangle_L$, $k \geqslant s$.

Now the estimates from [17] yield that $N_L - G_T$ maps $\Lambda_s\operatorname{Harm}_T(D)$ into $\Lambda_{s+1}\operatorname{Harm}_T(D)$. On the other hand, the classical Kohn estimate [9] shows that $N_L - G_T$ maps $L^2\operatorname{Harm}_T(D)$ into $(\operatorname{Harm}_T)^1_2(D)$. Then Remark 3 and (i) imply by the standard interpolation and duality argument that

(ii) $N_L - G_T$ *maps* $(\operatorname{Harm}_T)^s_p(D)$ *into* $(\operatorname{Harm}_T)^{s+1}_p(D)$ *for all* $-\infty < s < \infty$ *and* $1 < p < \infty$. (Recall that $N_L - G_T$ is selfadjoint with respect to $\langle\cdot,\cdot\rangle_L$.)

In order to prove the needed estimates for N_L we must observe that (ii) implies that N_L maps continuously $(\operatorname{Harm}_T)^s_p(D)$ into $W^{s+1}_p(0, q)(D)$ for $-1 \leqslant s < \infty$, $1 < p < \infty$.

Let now $\omega \in W^s_p(0, q)(D)$ and let $P_{T,L}$ be the orthogonal projection on $L^2\operatorname{Harm}_T$ (orthogonal with respect to $\langle\cdot,\cdot\rangle_L$). We have $\omega = P_{T,L}\,\omega + \Box_L G_{T^2}\Box_L\omega$. Since $G_{T^2}\Box_L\omega \in \operatorname{Dom}\Box_L$, we obtain

$$N_L\omega = N_L P_{T,L}\,\omega + G_{T^2}\Box_L\omega.$$

Hence N_L maps $W^s_p(0, q)(D)$ into $W^{s+1}_p(0, q)(D)$ for $s \geqslant 0$ (this can even be proved for $0 > s > -1+1/p$ but we shall not do it here).

The same methods applied to the operators $\bar\partial_L N_L$ and $\bar\partial^*_L N_L$ permit us to prove the rest of Theorem 6.

IV. Appendix 1. We now consider the general spaces $F^s_{pq}(D)$ and $B^s_{pq}(D)$. The definitions of these spaces for $0 < p < \infty$ and $0 < q < \infty$ can be found in [31]. Recall that for $1 \leqslant p < \infty$, we have $W^s_p(D) = F^s_{p2}(D)$ and $B^s_{pp}(D) = F^s_{pp}(D)$, $-\infty < s < \infty$.

We shall use the above notation also for $0 < p < 1$. The following interpolation formulas hold:

(1) $$[B^s_{p_1p_1}(D), W^s_{p_2}(D)]_{[\theta]} = F^s_{pq}(D), \qquad \frac{1-\theta}{p_1} + \frac{\theta}{p_2} = \frac{1}{p},\ q = \frac{2p_1}{2-2\theta+p_1\theta},$$

(2) $$(W^{s_1}_{p_1}(D), W^{s_2}_{p_2}(D))_{\theta,q} = B^s_{pq}(D), \qquad \frac{1-\theta}{p_1} + \frac{\theta}{p_2} = \frac{1}{p},\ s = (1-\theta)s_1 + \theta s_2.$$

Thus all spaces $B^s_{pq}(D)$, $0 < p, q < \infty$, and $F^s_{pq}(D)$, $0 < p < \infty$, $2p/(p+2) < q < \infty$, can be obtained via interpolation from the spaces $B^s_{pp}(D)$ and $W^s_p(D)$, $0 < p, q < \infty$. The formulas (1) and (2) together with the definitions of $B^s_{pp}(D)$ and $W^s_p(D)$ given above for $1 \leqslant p < \infty$ could serve as equivalent definitions of the spaces B^s_{pq}, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, and $F^s_{pq}(D)$, $1 \leqslant p < \infty$, $2p/(p+1) < q < 2p$.

We now prove the following

Proposition A. *Let T be a strongly elliptic operator as in Remark* 3(a). *Then*

$$F^s_{pq}\operatorname{Harm}_T(D) = B^s_{pp}\operatorname{Harm}_T(D)$$

for $-\infty < s < \infty$ *and* $1 < q < \infty$, $1 < p < \infty$.

Proof. The interpolation formula (1) yields that G_{Tk} maps $F^s_{pq}(D)$ into $F^{s+2mk}_{pq}(D)$ for $s > -mk$, $2p/(p+1) < q < 2p$. We also have $\operatorname{Tr} F^s_{pq}(D) = \operatorname{Tr} W^s_p(D)$ for $s > mk$, $1 < q < \infty$. Moreover, [31, 2.7.2] implies that the extension operator S maps $\operatorname{Tr} W^s_p(D) = V(\partial D)$ into $\bigcap_q F^s_{pq}(D)$.

Thus we can now repeat word by word the proof of Remark 3(a) and prove our proposition for $2p/(p+1) < q < 2p$. In order to prove Proposition A for all q, $1 < q < \infty$, we fix p and observe that the estimates from [31] yield that Proposition A is valid for all $1 < q < \infty$, $2m < s < \infty$ ($2m$ is the order of T). Thus we can use complex interpolation between the spaces $F^s_{pq}(D)$, $-\infty < s < \infty$, $2p/(p+1) < q < 2p$, and $F^s_{pq}(D)$, $2m < s < \infty$, $1 < q < \infty$, and prove our proposition for all $1 < q < \infty$.

Remark B. It will follow from Appendix 2 (see below) that Proposition A is in fact valid for $0 < q < \infty$ and $0 < p < \infty$.

Roughly speaking, if we deal with kernels of strongly elliptic differential operators, we have only one interpolation scale to consider—the scale of Besov spaces B^s_{pq}.

Let us now describe the interpolation scale $B^s_{pq}\operatorname{Harm}(m)(D)$, $1 < p < \infty$, $1 < q < \infty$.

If $s > 0$ and s is not an integer then for $u \in B^s_{pq}(D)$

$$\|u\|_{B^s_{pq}(D)} = \|u\|_{L^p(D)} + \sum_{|\alpha|=[s]} \Big(\int_{\boldsymbol{R}^n} |h|^{-n-q\{s\}} \Big(\int_{D_h} |D^\alpha f(x+h) - D^\alpha f(x)|^p \, dV_x\Big)^{q/p} dV_h\Big)$$

where $\{s\} = s - [s]$, $D_h = D \cap \{x \in \boldsymbol{R}^n \colon x+h \in D\}$. Thus in this case the Besov norm has an explicit form.

To describe the spaces $B^s_{pq}\operatorname{Harm}(m)(D)$ for other s we shall use Proposition 2, Theorem 1 and the interpolation formula (2). Let $s > -1+1/p$ and $s = (1-\theta)s_1 + \theta s_2$, $s_1 < s < s_2$, $s_1 > -1+1/p$. The formula (2) yields that L^r_m maps continuously $B^s_{pq}\operatorname{Harm}(m)(D)$ into $\big(L^p(D, |\varrho|^{-ps_1}), L^p(D, |\varrho|^{-ps_2})\big)_{\theta,q}$, $r \geqslant s_2$, and P_m maps the last space onto $B^s_{pq}\operatorname{Harm}(m)(D)$. Since in addition L^r_m maps continuously $B^s_{pq}\operatorname{Harm}(m)(D)$ into $\mathring{B}^s_{pq}\operatorname{Harm}(m)(D)$ (defined in the same way as for $\mathring{B}^s_{pq}$), we can repeat the proof of Theorem 1 to obtain for $1/p + 1/p_1 = 1$, $1/q + 1/q_1 = 1$

$$\begin{aligned}(B^s_{pq}\operatorname{Harm}(m)(D))^* &= B^{-s}_{p_1q_1}\operatorname{Harm}(m)(D) \\ &= \big(L^{p_1}(D, |\varrho|^{p_1s_1}), L^{p_1}(D, |\varrho|^{p_1s_2})\big)_{\theta,q_1} \cap \operatorname{Harm}(m)(D).\end{aligned}$$

The space on the right has the following norm (see [7, 5.7, Ex. 10]):

$$\|u\| = \Big(\int_0^\infty t^{\theta q_1/p_1} \Big(\int_{t<|\varrho|^{p_1(s_2-s_1)}} |u|^{p_1} |\varrho|^{p_1s_1} \, dV\Big)^{q_1/p_1} dt/t\Big)^{1/q_1}.$$

We can now change the notation and write $-s = s$, $p_1 = p$, $q_1 = p$. In addition we can choose s_1, s_2 in the above formula in such a way that $s_2 - s_1 = 1/p_1$ or if $s > 1/q_1 - 1/p_1$ we can take $s_2 - s_1 = 1$, $\theta = 1/q_1$ and put $t = \tau^{p_1}$. After those operations we get the following

Proposition C. *If $s < 1/p$, then the following norms are equivalent to the B^s_{pq} norm on $B^s_{pq}\operatorname{Harm}(m)(D)$*:

$$\|u\|_\theta = \Big(\int_0^\infty t^{\theta q/p} \Big(\int_{|\varrho|>t} |u|^p |\varrho|^{-(sp+\theta)} \, dV\Big)^{q/p} dt/t\Big)^{1/q}$$

for any $0 < \theta < \min(1, 1-ps)$.

If in addition $s < 1/p - 1/q$ then the above norms are equivalent to

$$|||u||| = \Big(\int_0^\infty \Big(\int_{|\varrho|>\tau} |u|^p |\varrho|^{-sp-p/q} \, dV\Big)^{q/p} d\tau\Big)^{1/q}.$$

By considering the derivatives we can now get the explicit description of the spaces $B^s_{pq}\operatorname{Harm}(m)(D)$ for all s.

In the case $m = 1$, Proposition C implies the following characterization of the space $B^s_{pq}(\partial D)$, $1 < p, q < \infty$, $s > 0$:

Corollary D. *The following conditions are equivalent*:

1) $u \in B^s_{pq}(\partial D)$, $1 < p, q < \infty$, $s > 0$.

2) *$u \in L^p(\partial D)$ and for some integer $k > s + 1/q$*

$$(*) \qquad \int_0^\infty \Big(\int_{|\varrho|>t} |\tilde{u}|^p |\varrho|^{p(k-s-1/p-1/q)} \, dV\Big)^{q/p} dt < \infty$$

where $\tilde{u}$ denotes the harmonic extension of u over D.

3) *$u \in L^p(\partial D)$ and $(*)$ holds for every $k > s + 1/q$.*

V. Appendix 2. Let T be a strongly elliptic operator of order $2m$ for which the Dirichlet problem is uniquely solvable.

PROPOSITION A. *The operator G_T solving the Dirichlet problem for T maps continuously $B^s_{pq}(D) \to B^{s+2m}_{pq}(D)$ and $F^s_{pq}(D) \to F^{s+2m}_{pq}(D)$ for $-m+(1/p-1)n < s < \infty$, $0 < p \leqslant 1$, $0 < q < \infty$.*

COROLLARY B. (a) *The projection P_T maps continuously $B^s_{pq}(D)$ and $F^s_{pq}(D)$ into itself for $(1/p-1)n < s < \infty$, $0 < p \leqslant 1$, $0 < q < \infty$.*

(b) *$F^s_{pq}\operatorname{Harm}_T(D) = B^s_{pq}\operatorname{Harm}_T(D)$ for $0 < p \leqslant \infty$, $-\infty < s < \infty$, $0 < q < \infty$.*

(c) *The scale of spaces $B^s_{pp}\operatorname{Harm}_T(D)$, $0 < p \leqslant \infty$, $-\infty < s < \infty$, has the same interpolation properties as in Remark* 3(b) *provided that T is selfadjoint.*

Proof. In order to prove Proposition A we must use the results of Franke [11]. His estimates imply that G_T maps continuously $F^s_{pq}(D)$ into $F^{s+2m}_{pq}(D)$ and $B^s_{pq}(D)$ into $B^{s+2m}_{pq}(D)$ for $s > m(1/p-1)$, $0 < p \leqslant 1$. The estimates from [11] also yield that G_T maps continuously $B^s_{pq}(D)$ into $B^{s+2m}_{pq}(D)$ and $F^s_{pq}(D)$ into $F^{s+2m}_{pq}(D)$ if $-m < s < \infty$, $1 < p < \infty$, $1 < q < \infty$. This can be easily proved by using the estimates for $W^s_p(D)$ and the interpolation formulas from Appendix 1.

It follows from [31, 4.3.4, Remark 1] that G_T has the interpolation property with respect to the Calderón–Torchinsky construction (see [31] for details) and hence we can use it to interpolate between $F^{s_1}_{p_1 q_1}(D)$ and $F^{s_2}_{p_2 q_2}(D)$ for $s_1 > (1/p-1)n$, $0 < p_1 \leqslant 1$, $0 < q_1 < \infty$ and $s_2 > -m$, $1 < p_2 < \infty$, $1 < q_2 < \infty$ and get the required results for the spaces $F^s_{pq}(D)$.

In order to prove Proposition A for the spaces $B^s_{pq}(D)$ it suffices to use the real interpolation functor $(\cdot, \cdot)_{\theta,q}$.

Now, Corollary B(a) follows immediately from the definition of P_T. Corollary B(b) follows from the fact that $\operatorname{Tr} F^s_{pq}(D) = B^{s-1/p}_{pp}(\partial D)$ for $s > \max(1/p, (1/p-1)n)$ in the same way as in Remark 3 and in Proposition A from Appendix 1. The proof of (c) is the same as that of Remark 3(b).

Corollary B(a) yields in particular that for every m, P_m maps continuously $W^s_1(D)$ into itself and $B^s_{11}(D)$ into itself if only $s > 0$. Corollary B(b) implies that $\operatorname{Harm}^s_1(m)(D) = B^s_{11}\operatorname{Harm}(m)(D)$ for $-\infty < s < \infty$.

We now prove

PROPOSITION C.

(a) $\operatorname{Harm}^s_1(m)(D) = B^s_{11}\operatorname{Harm}(m)(D) = \mathring{L}^1\operatorname{Harm}(m)(D, |\varrho|^{-s})$ *for* $s < 1$.

If $0 \leqslant s < 1$ *then* $\mathring{L}^1\operatorname{Harm}(m)(D, |\varrho|^{-s}) = L^1\operatorname{Harm}(m)(D, |\varrho|^{-s})$.

(b) $B^s_{\infty\infty}\operatorname{Harm}(m)(D) = L^\infty\operatorname{Harm}(m)(D, |\varrho|^{-s})$ *for* $s < 0$,

$B^0_{\infty\infty}\operatorname{Harm}(m)(D) = \operatorname{Bl}\operatorname{Harm}(m)(D)$.

(c1) $B^{-s}_{\infty\infty}\operatorname{Harm}(m)(D) = L^\infty\operatorname{Harm}(m)(D, |\varrho|^s)$ *represents the dual of* $\operatorname{Harm}^s_1(m)(D)$ *via the pairing* $\langle L^r_m u, v\rangle_0$ *if* $r \geqslant [s]+1$, $s > 0$.

(c2) $B^{-s}_{11}\operatorname{Harm}(m)(D)$ *represents the dual of* $\Lambda^0_s\operatorname{Harm}(m)(D)$, *which is the closure of* $C^\infty(\bar{D}) \cap \operatorname{Harm}(m)(D)$ *in* $\Lambda_s(D)$ ($\Lambda_s = B^s_{\infty\infty}$ *for* $s > 0$), *via the same pairing.*

(c3) $L^1\operatorname{Harm}(m)(D)$ *represents the dual of* $\operatorname{Bl}^0\operatorname{Harm}(m)(D)$, *which is the closure of* $C^\infty(\bar{D}) \cap \operatorname{Harm}(m)(D)$ *in* $\operatorname{Bl}(D)$, *via the same pairing* ($s = 0$).

Proof. Let $u \in L^1\operatorname{Harm}(m)(D, |\varrho|^s) \cap C^\infty(\bar{D})$, $s > 0$, $s-[s] > 0$. We have

$$\|u\|_{L^1(D,|\varrho|^s)} = \sup_{\substack{v \in L^\infty(D,|\varrho|^{-s}) \\ \|v\| \leqslant 1}} |\langle u, v\rangle_0| = \sup_{\substack{v \in L^\infty(D,|\varrho|^{-s}) \\ \|v\| \leqslant 1}} |\langle u, L^r_m P_m v\rangle_0|.$$

Propositions 1 and 2 imply that $\varphi = L^r_m P_m v \in \Lambda_s(D)$, φ vanishes on ∂D up to order $[s]$ and $\|\varphi\|_{\Lambda_s(D)} \leqslant c\|v\|_{L^\infty(D,|\varrho|^{-s})}$. Thus the function $\tilde{\varphi}$ on $\boldsymbol{R}^n$ equal to φ on D and to 0 outside D belongs to $\Lambda_s(\boldsymbol{R}^n)$.

Let $\tilde{u} = Su$, where S is the extension operator from D to $\boldsymbol{R}^n$ described in [31, 3.3.4]. By [31, 2.11.2], $(B^{-s}_{11}(\boldsymbol{R}^n))^* = \Lambda_s(\boldsymbol{R})$. We also have $\|Su\|_{B^{-s}_{11}(\boldsymbol{R}^n)} \leqslant c\|u\|_{B^{-s}_{11}(D)}$ by the continuity of S. Hence

$$|\langle u, L^r_m P_m v\rangle| = |\langle \tilde{u}, \tilde{\varphi}\rangle_{\boldsymbol{R}^n}| \leqslant c\|\tilde{u}\|_{B^{-s}_{11}(\boldsymbol{R}^n)}\|\tilde{\varphi}\|_{\Lambda_s(\boldsymbol{R}^n)}$$

$$= c\|\tilde{u}\|_{B^{-s}_{11}(\boldsymbol{R}^n)}\|\varphi\|_{\Lambda_s(D)} \leqslant c\|u\|_{B^{-s}_{11}(D)}\|v\|_{L^\infty(D,|\varrho|^{-s})}.$$

Thus $\|u\|_{L^1(D,|\varrho|^s)} \leqslant c\|u\|_{B^{-s}_{11}(D)}$ and $B^{-s}_{11}\operatorname{Harm}(m)(D) \subset \mathring{L}^1\operatorname{Harm}(m)(D, |\varrho|^s)$.

In order to prove the opposite inclusion, we first prove that $B^{-1}_{11}(D)$ represents the dual of $\Lambda^0_s(D)$, which is the closure of $C^\infty_0(D)$ in $\Lambda_s(D)$. (Warning! $\Lambda^0_s(D)$ is *not equal* to the space $\mathring{\Lambda}_s(D)$ of functions which vanish on ∂D up to order s.)

If we extend the functions from $\Lambda^0_s(D)$ to $\boldsymbol{R}^n$ putting zero outside D then we get an isomorphic imbedding of $\Lambda^0_s(D)$ into the space $\Lambda^0_s(\boldsymbol{R}^n)$, which is the closure of the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ in $\Lambda_s(\boldsymbol{R}^n)$. By [31, 2.11.2, Remark 2], $B^{-s}_{11}(\boldsymbol{R}^n)$ is the dual of $\Lambda^0_s(\boldsymbol{R}^n)$. Thus the dual of $\Lambda^0_s(D)$ must be equal to the restriction of $B^{-s}_{11}(\boldsymbol{R}^n)$ to D, i.e. to $B^{-s}_{11}(D)$. Hence

$$\|u\|_{B^{-s}_{11}(D)} = \sup_{\substack{v \in \Lambda^0_s(D) \\ \|v\| \leqslant 1}} |\langle u, v\rangle|.$$

Since

$$|\langle u, v\rangle| = |\langle u, L^r_m P_m v\rangle| \leqslant \|u\|_{L^1(D,|\varrho|^s)}\|L^r_m P_m v\|_{L^\infty(D,|\varrho|^{-s})}$$

$$\leqslant c\|u\|_{L^1(D,|\varrho|^s)}\|P_m v\|_{\Lambda_s(D)} \leqslant c\|u\|_{L^1(D,|\varrho|^s)}\|v\|_{\Lambda_s(D)},$$

by Proposition 2, we have $\|u\|_{B^{-s}_{11}(D)} \leqslant c\|u\|_{L^1(D,|\varrho|^s)}$. The smooth functions on $\bar{D}$ are dense in $B^{-s}_{11}\operatorname{Harm}(m)(D)$ since Δ^k, $k > s$, is a retraction of $B^{2k-s}_{11}\operatorname{Harm}(m+k)(D)$ onto $B^{-s}_{11}\operatorname{Harm}(m)(D)$ (the coretraction is obviously G_k), and the smooth functions on $\bar{D}$ are dense in $B^{2k-s}_{11}\operatorname{Harm}(m+k)(D)$

because of the regularity of P_{k+m} (see Corollary B(a)). Thus part (a) of Proposition C is proved for $s<0$, $s\neq[s]$.

The second part of the above proof remains valid for $s=[s]$ and thus $\mathring{L}^1\,\mathrm{Harm}(m)(D,|\varrho|^s)\subset B^s_{11}\,\mathrm{Harm}(m)(D)$. The opposite inclusion follows from interpolation since

$$\begin{aligned}B^s_{11}\,\mathrm{Harm}(m)(D)&=[B^{s-\varepsilon}_{11}\,\mathrm{Harm}(m)(D),\,B^{s+\varepsilon}_{11}\,\mathrm{Harm}(m)(D)]_{[1/2]}\\&\subset[L^1(D,|\varrho|^{s-\varepsilon}),\,L^1(D,|\varrho|^{s+\varepsilon})]_{[1/2]}=L^1(D,|\varrho|^s).\end{aligned}$$

Thus (a) holds for every $s<0$.

We have $B^0_{11}\,\mathrm{Harm}(m)(D)\subset L^1(D)$ since $B^0_{11}\,\mathrm{Harm}(m)(D)=\mathrm{Harm}^0_1(m)(D)\subset W^0_1(D)\subset L^1(D)$. This implies that L^r_m maps continuously $B^s_{11}\,\mathrm{Harm}(m)(D)$ into $L^1(D,|\varrho|^{-s})$ just as for Hölder spaces, $r\geqslant[s]+1$, $s>0$.

Let us now prove (c1) and (b). Take $s>0$, $s-[s]>0$. The space $B^{-s}_{\infty\infty}(D)$ is the dual of $\mathring{B}^s_{11}(D)$. The mapping L^r_m maps continuously $B^s_{11}\,\mathrm{Harm}(m)(D)$ into $\mathring{B}^s_{11}(D)$. In view of Proposition A this fact can be proved in exactly the same way as for $p>1$, using the already proven part of (a).

Now, $B^{-s}_{\infty\infty}\,\mathrm{Harm}(m)(D)$ represents the dual of $B^s_{11}\,\mathrm{Harm}(m)(D)$ via $\langle L^r_m\cdot,\cdot\rangle_0$. Indeed, it is obvious that every element of $B^{-s}_{\infty\infty}\,\mathrm{Harm}(m)(D)$ determines a continuous functional on $B^s_{11}\,\mathrm{Harm}(m)(D)$. Let now φ be such a functional. We extend φ to an element $\tilde{\varphi}$ of $(B^s_{11}(D))^*$. Since P_m maps $\mathring{B}^s_{11}(D)$ into $B^s_{11}(D)$ and is selfadjoint, P_m maps continuously $(B^s_{11}(D))^*$ into $B^{-s}_{\infty\infty}(D)$. Thus $P_m\tilde{\varphi}$ is the element of $B^{-s}_{\infty\infty}\,\mathrm{Harm}(m)(D)$ representing φ.

The mapping L^r_m maps $B^s_{11}\,\mathrm{Harm}(m)(D)$ into $L^1(D,|\varrho|^{-s})$. Thus every function from $L^\infty\,\mathrm{Harm}(m)(D,|\varrho|^s)$ represents a functional on $B^s_{11}\,\mathrm{Harm}(m)(D)$. Thus $L^\infty\,\mathrm{Harm}(m)(D,|\varrho|^s)\subset B^{-s}_{\infty\infty}\,\mathrm{Harm}(m)(D)$. The opposite inclusion follows from Lemma 1 applied to the $(k+m)$-polyharmonic functions, $k>s$, since $B^{-s}_{\infty\infty}\,\mathrm{Harm}(m)(D)$ is the retract of $\Lambda_{2k-s}\,\mathrm{Harm}(m+k)(D)$ under Δ^k. Thus (b) is proved for noninteger s. Complex interpolation permits us to prove (b) for all s.

Let us now prove the rest of (a). Let $0<s<1$. Since L^1_m maps $B^s_{11}\,\mathrm{Harm}(m)(D)$ into $L^1(D,|\varrho|^{-s})$ and P_m maps $L^1(D,|\varrho|^{-s})$ into itself, we have $B^s_{11}\,\mathrm{Harm}(m)(D)\subset L^1(D,|\varrho|^{-s})$. For every m-polyharmonic u

$$\|u\|_{B^s_{11}(D)}\approx\sup_{\substack{v\in L^\infty\mathrm{Harm}(D,|\varrho|^s)\\ \|v\|\leqslant 1}}|\langle u,v\rangle|\leqslant c\,\|u\|_{L^1(D,|\varrho|^{-s})}.$$

Hence $L^1\,\mathrm{Harm}(D,|\varrho|^{-s})=B^s_{11}\,\mathrm{Harm}(m)(D)$.

Now $\mathrm{Bl}\,\mathrm{Harm}(m)(D)$ represents the dual of $L^1\,\mathrm{Harm}(m)(D)$ via the pairing $\langle u,L^1_m v\rangle$. We shall prove in the sequel that $B^0_{11}\,\mathrm{Harm}(m)(D)$ represents the dual of $\mathrm{Bl}^0\,\mathrm{Harm}(m)(D)$ via the pairing $\langle L^1_m u,v\rangle$. Thus $\|u\|_{B^0_{11}(D)}\leqslant c\,\|u\|_{L^1(D)}$ if u is m-polyharmonic. Hence $B^0_{11}\,\mathrm{Harm}(m)(D)=L^1\,\mathrm{Harm}(m)(D)$.

It now remains to prove (c2) and (c3).

Let us prove (c2). From the above-proved duality between $\Lambda^0_s(D)$ and $B^{-s}_{11}(D)$ it follows that every $v\in B^{-s}_{11}\,\mathrm{Harm}(m)(D)$ determines a continuous functional on $\Lambda^0_s\,\mathrm{Harm}(m)(D)$, since $u\in\Lambda^0_s\,\mathrm{Harm}(m)(D)$ iff $L^r_m u\in\Lambda^0_s(D)$. We can now proceed in the standard way, extending $\varphi\in(\Lambda^0_s\,\mathrm{Harm}(m)(D))^*$ to $\tilde{\varphi}\in(\Lambda_s(D))^*$ and taking $v=P_m\tilde{\varphi}\in(\Lambda^0_s(D))^*=B^{-s}_{11}(D)$.

In order to prove (c3) we use the fact that $B^0_{11}(D)$ is the dual of the space $B^{00}_{\infty\infty}(D)$, which is the closure of $C^\infty_0(D)$ in $B^0_{\infty\infty}(D)$ (see [31, 2.11.2, Remark 2]). The operator P_m maps continuously $B^0_{\infty\infty}(D)$ into itself (this follows from Th. 12.10 of [1] and interpolation). The mapping L^1_m maps $\mathrm{Bl}\,\mathrm{Harm}(m)(D)$ into $B^0_{\infty\infty}(D)$, since $L^\infty(D)\subset B^0_{\infty\infty}(D)$. Thus L^1_m maps $\mathrm{Bl}^0\,\mathrm{Harm}(m)(D)$ into $B^{00}_{\infty\infty}(D)$. Hence as before each $v\in B^0_{11}\,\mathrm{Harm}(m)(D)$ determines a continuous functional on $\mathrm{Bl}^0\,\mathrm{Harm}(m)(D)$ and $B^0_{11}\,\mathrm{Harm}(m)(D)=\big(P_m(B^{00}_{\infty\infty}(D))\big)^*$. Thus $B^0_{11}\,\mathrm{Harm}(m)(D)=L^1\,\mathrm{Harm}(m)(D)$ represents the dual of $\mathrm{Bl}^0\,\mathrm{Harm}(m)(D)$.

In the above proof we have used implicitly the fact that the smooth functions on $\bar{D}$ are dense in $L^1\,\mathrm{Harm}(m)(D)$. This fact needs a special proof, which is the same as the proof for $m=1$ given in [21].

Remark D. It can be easily proved using the fact that $C^\infty_0(D)$ is dense in $C_0(\bar{D})=\{f\in C(\bar{D}):\,f|_{\partial D}=0\}$ that if $m=1$ then

$$\mathrm{Bl}^0\,\mathrm{Harm}(D)=\{u\in\mathrm{Bl}\,\mathrm{Harm}(D):\ \varrho\nabla u\to 0\ \text{if}\ \varrho\to 0\},$$

$$\Lambda^0_s\,\mathrm{Harm}(D)=\{u\in\Lambda_s\,\mathrm{Harm}(D):\ |\varrho|^{1-s}\nabla u\to 0\ \text{if}\ \varrho\to 0\},\quad 0<s<1.$$

Remark E. Let us make a trivial but useful observation. Since for $D\subset C^n$ the holomorphic functions form a closed subspace of the space of harmonic functions with respect to any norm considered here, we have now an explicit description of Sobolev and Besov spaces of holomorphic functions for every smooth bounded domain in C^n. The additional assumptions on D are needed only to establish duality and interpolation relations between these spaces.

Proposition C (c2)–(c3) remains valid if we replace the spaces of harmonic functions with the corresponding spaces of holomorphic functions on strictly pseudoconvex domains (cf. [19], [21]). If D is the unit disc in C then $\mathrm{Bl}^0\,\mathrm{Hol}(D)$ is the classical Bloch class B_0.

Part (a) of Proposition C permits us to get Corollary 1 for $p=1$, and part (c1) proves Theorem 4 for $p=\infty$.

VI. Open problems. 1. It was proved in [24] that if $m=1$ then the mapping $u\to P(\varrho^k u)$ is an isomorphism between $\mathrm{Harm}^s_p(D)$ and $\mathrm{Harm}^{s+k}_p(D)$,

$-\infty < s < \infty$. Is this true for $m > 1$, i.e. is $u \to P_m(\varrho^k u)$ an isomorphism? What can be said about the mapping $u \to P_m(|\varrho|^s u)$, where $s > 0$ is noninteger?

2. Find an explicit characterization of the spaces $\mathrm{Harm}_p^s(m)(D)$ for $p < 1$. Is it in particular true that $\mathrm{Harm}_p^0(m)(D) = L^p \mathrm{Harm}(m)(D)$ for $p < 1$?

3. Are the smooth functions on $\bar{D}$ dense in $L^1 \mathrm{Harm}(m)(D, |\varrho|^s)$ if $s > 0$?

4. For which strongly elliptic second order operators T Lemma 1 and Proposition 1(a), (b) hold? If these facts are true for T then all results of the present paper remain valid if we replace the spaces $\mathrm{Harm}(m)(D)$ with the spaces $\mathrm{Harm}_{T^m}(D)$. It may also be interesting to try to extend these estimates to the case of pseudodifferential operators, since for every positive-definite elliptic operator T of order $2m$ the operator $T^{1/m}$ is a well-defined pseudodifferential operator (see [30]). We hope that this could be an interesting problem for specialists in PDE.

5. Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$. Is the Bergman projection B continuous from $W_p^s(D)$ into itself if $p \leqslant 1$ and $s > n(1/p-1)$?

6. Let D be a smooth bounded domain in $\mathbf{C}^n$. Is the operator N solving the $\bar{\partial}$-Neumann problem considered with respect to the Euclidean metric regular in Hölder norms? Note that in view of Corollary B in Appendix 2 the Hölder estimates for such an N yield automatically the estimates for N in every $W_p^s(D)$ norm, $0 \leqslant s < \infty$, $1 < p < \infty$, without any additional assumption on the domain D. The same is obviously true for the Bergman projection B: if, for a smooth bounded domain D, B is regular in Hölder norms then it must be regular in all norms $W_p^s(D)$, $1 < p < \infty$, $s \geqslant 0$. In particular, is B regular in Hölder norms if D is a pseudoconvex domain with real-analytic boundary?

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