STUDIA MATHEMATICA, T. LXXXVIII. (1988)

On relations between additive and multiplicative clustering operators

by

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Abstract. It is proved that a family $\mathcal{T}_t$, $t \geqslant 0$, of selfadjoint operators defines a one-parameter semigroup of multiplicative clustering operators iff its generator is an additive clustering operator. We show that 0 is not an eigenvalue of a multiplicative clustering operator.

1. Introduction. In the recent years there has been isolated a class of operators, called clustering operators [1, 6]. This notion appears in mathematical physics and corresponds to the fact that in a multi-particle system the particles try to group into clusters which do not interact with other clusters at large distances. This leads to the special structure of those operators.

The simplest models where the clustering operators appear are the $(\nu+1)$-dimensional Ising models with discrete and continuous time. The Transfer-matrix $\mathcal{T}_t$ of these models has a multiplicative clustering structure. This was conjectured by Minlos and Sinai in [9] and proved by Abdulla-Zade, Minlos and Pogosyan in [1] (for $\nu = 1$) and by Malyshev in [4] (for $\nu \geqslant 2$). The generator H of the semigroup $\mathcal{T}_t$ (in the continuous time model) turns out to be an additive clustering operator [7, 8]. Other examples of clustering operators and their basic properties are given in [8].

An additive clustering operator H defines the Hamiltonian of an infinite-particle system on the integer lattice and a multiplicative clustering operator $\mathcal{T}$ corresponds to the Transfer-matrix of such a system. Therefore it is natural to expect that: (a) e^{-tH} is a multiplicative clustering operator and (b) $\ln \mathcal{T}$ (if it exists) is an additive clustering operator. In the present paper we prove (a) and give one result concerning (b).

The author thanks V. A. Malyshev for valuable discussions.

2. Results. Let $C_{\mathbf{Z}^\nu}$ be the set of all finite subsets of the ν-dimensional integer lattice $\mathbf{Z}^\nu$. We consider the Hilbert space

$$\mathcal{H} = l_2(C_{\mathbf{Z}^\nu})$$

with orthogonal basis (e_T), $T \in C_{\mathbf{Z}^\nu}$,

$$e_T(T') = \delta_{TT'}, \qquad T, T' \in C_{\mathbf{Z}^\nu}.$$

DEFINITION 2.1. A selfadjoint operator $\mathcal{T}$ in $\mathcal{H}$ is called a *multiplicative clustering operator* with (clustering) parameters λ and β if its matrix elements have the following expansion:

$$(2.1)\qquad (e_\emptyset, \mathcal{T}e_\emptyset) = 1, \quad (e_T, \mathcal{T}e_{T'}) = \sum_{s\geqslant 1}\sum_{(L_i, L_i')}\prod_{i=1}^{s}\omega(L_i, L_i'), \qquad |T|+|T'|>0,$$

where the summation is over all (unordered) partitions $\{(L_1, L_1'), \ldots, (L_s, L_s')\}$, $s = 1, 2, \ldots$, of the pair (T, T') (i.e. $\bigcup L_i = T$, $\bigcup L_i' = T'$, $L_i\cap L_j = L_i'\cap L_j' = \emptyset$ for $i\neq j$) such that

$$(2.2)\qquad L_i\neq\emptyset, \quad L_i'\neq\emptyset, \quad i = 1, \ldots, s,$$

and the *clustering functions* $\omega(L, L') = \omega^{\mathcal{T}}(L, L')$ satisfy:

(a) $\omega(L+x, L'+x) = \omega(L, L')$, $x\in \boldsymbol{Z}^\nu$, where $\{x_1, \ldots, x_n\}+x = \{x_1+x, \ldots, x_n+x\}$.

(b) $\omega(L, L') = \overline{\omega(L', L)}$.

(c) $0\leqslant \omega(\{0\}, \{0\}) = \lambda\leqslant 1$ and

$$(2.3)\qquad |\omega(L, L')|\leqslant M\lambda(1-\lambda)\beta^{d_{L\cup L'}} \quad \text{if } |L\cup L'|\geqslant 2,$$

where $M>0$, $0\leqslant\beta<1$ are constants, $|B|$ denotes the cardinality of the set B and d_B, $B\subset \boldsymbol{Z}^\nu$, is the minimum length of a tree graph connecting points of B (the metric in $\boldsymbol{Z}^\nu$ is given by

$$\operatorname{dist}(x_1, x_2) = |x_1-x_2| = \sum |x_1^{(i)}-x_2^{(i)}|, \qquad x_j = (x_j^{(1)}, \ldots, x_j^{(\nu)}),\ j = 1, 2).$$

We say that a strongly continuous family $\mathcal{T}_t$, $t\in \boldsymbol{C}^1$, of multiplicative clustering operators in $\mathcal{H}$ defines a *holomorphic group of multiplicative clustering operators* with parameters a and β if their clustering functions $\omega_t = \omega^{\mathcal{T}_t}$ satisfy:

$$\omega_t(\{0\}, \{0\}) = e^{-at}, \quad a\geqslant 0,$$

$$(2.4)\qquad |\omega_t(L, L')|\leqslant M(t)|t|\, e^{-a\operatorname{Re} t}\beta^{d_{L\cup L'}} \quad \text{if } |L\cup L'|\geqslant 2,$$

where $0<M(t)<Me^{\delta t}$ for t from a neighbourhood of the set $[0, \infty)\subset \boldsymbol{C}^1$; here $M>0$, and $\delta\geqslant 0$ is a small constant ($\delta<a/3$).

DEFINITION 2.2. A selfadjoint operator H in $\mathcal{H}$ is called an *additive clustering operator* with parameters a and β if its matrix elements have the following expansion:

$$(2.5)\qquad (e_T, He_{T'}) = \sum_{\substack{\emptyset\neq L\subset T\\ \emptyset\neq L'\subset T'}}\omega(L, L'), \qquad T, T'\in C_{\boldsymbol{Z}^\nu},$$

where the (clustering) functions $\omega(L, L') = \omega^H(L, L')$ satisfy:

(a) $\omega(L, L') = \omega(L+x, L'+x)$, $x\in \boldsymbol{Z}^\nu$.

(b) $\omega(L, L') = \overline{\omega(L', L)}$.

(c) $\omega(\{0\}, \{0\}) = a$ and

$$(2.6)\qquad |\omega(L, L')|\leqslant M\beta^{d_{L\cup L'}} \quad \text{if } |L\cup L'|\geqslant 2,$$

where $M>0$, $a\geqslant 0$ and $0\leqslant\beta<1$ are constants.

Remark 2.1. The estimate (2.3) in the definition of a multiplicative clustering operator is slightly weaker than the estimate

$$|\omega(L, L')|\leqslant M\beta^{d_{L\cup L'(t)}}$$

used in earlier papers [1, 6]. (Here $L\cong L\times\{0\}\subset \boldsymbol{Z}^\nu\times\boldsymbol{R}^1$ and $L'(t) = L'\times\{t\}\subset \boldsymbol{Z}^\nu\times \boldsymbol{R}^1$.) Our estimate is more natural for the purposes of this paper. It covers the cases of small t as well as those of large t.

The main result of this work is the following:

THEOREM 1. (a) *Let H be an additive clustering operator with parameters a and β. Then there exists $\beta_0 = \beta_0(M, a)>0$ such that for $0\leqslant\beta<\beta_0$ the family $\mathcal{T}_t = e^{-tH}$, $t\in \boldsymbol{C}^1$, defines a holomorphic group of multiplicative clustering operators with parameters a and $\beta^{1-\varepsilon}$, where ε is a small constant depending on ν and a.*

(b) *Let $\mathcal{T}_t$, $t\in \boldsymbol{C}^1$, be a holomorphic group of multiplicative clustering operators with parameters a and β. Then the generator H of the semigroup $(\mathcal{T}_t)_{t\geqslant 0}$ is an additive clustering operator with parameters a and β.*

Remark 2.2. The following natural problem arises: can a multiplicative clustering operator $\mathcal{T}$ be represented as e^{-tH} for some additive clustering operator H? The necessary conditions for this are:

(i) 0 is not an eigenvalue of $\mathcal{T}$.

(ii) $\mathcal{T}^t$ is a multiplicative clustering operator for $0<t<1$ ($t = 1/2$ suffices).

A partial answer to the question whether (i) is satisfied is given in Theorem 2 below. But the second problem, the multiplicativeness of $\sqrt{\mathcal{T}}$, remains open. Notice that for the Transfer-matrix of the Ising model in $\boldsymbol{Z}^{\nu+1}$ the first problem is unsolved.

THEOREM 2. *Let $\mathcal{T}$ be a multiplicative clustering operator with parameters λ and β. Then if β/λ is sufficiently small then 0 is not an eigenvalue of $\mathcal{T}$.*

Remark 2.3. V. A. Malyshev and I. A. Kashapov have announced this result, but their proof has not appeared in the literature (see note in [6.II] and also [3]). The invertibility of the Transfer-matrix in the gauge lattice model has been proved by K. Fredenhagen in [2].

3. Proof of Theorem 1(a). Let $\gamma = ((L_1, L_1'), \ldots, (L_k, L_k'))$, $k = k(\gamma)$, be a finite ordered system of pairs of nonempty subsets of $\boldsymbol{Z}^\nu$. We call such a system a *bond* if the system $\{L_1 \cup L_1', \ldots, L_k \cup L_k'\}$ of sets is connected (see [4] or [5]) and put

$$\tilde{\gamma} = \bigcup L_i \cup L_i',$$

$$T(\gamma) = L_1 \cup (L_2 \setminus L_1') \cup (L_3 \setminus L_2' \setminus L_1') \cup \ldots, \tag{3.1}$$

$$T'(\gamma) = L_k' \cup (L_{k-1}' \setminus L_k) \cup (L_{k-2}' \setminus L_{k-1} \setminus L_k) \cup \ldots$$

We also consider finite (unordered) systems $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, $m = m(\Gamma)$, of bonds. Call such a system *regular* if

$$T(\gamma_i) \cap T(\gamma_j) = \emptyset, \quad T'(\gamma_i) \cap T'(\gamma_j) = \emptyset, \quad i \neq j, \tag{3.2}$$

and *completely regular* if

$$\tilde{\gamma}_i \cap \tilde{\gamma}_j = \emptyset, \quad i \neq j. \tag{3.3}$$

Γ is called *connected* if the system $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m\}$ is connected (see [5]). If H is an additive clustering operator then for a given $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ we put

$$\tilde{\omega}_t(\Gamma) = \prod_{i=1}^{m} \frac{(-t)^{k(\gamma_i)}}{k(\gamma_i)!} \prod_{(L,L') \in \gamma_i} \omega^H(L, L'), \tag{3.4}$$

where $t \geqslant 0$.

LEMMA 3.1. *Let H be an additive clustering operator with clustering functions ω^H. Then the matrix elements of the operator $\mathcal{T}_t = e^{-tH}$ have the form*

$$(e_T, \mathcal{T}_t e_{T'}) = \sum_{\Gamma} \omega_t(\Gamma) + \delta_{|T \cup T'|,1}, \tag{3.5}$$

where the summation is over all completely regular systems Γ such that

$$T(\Gamma) = \bigcup T(\gamma_i) \subset T, \quad T'(\Gamma) = \bigcup T'(\gamma_i) \subset T', \quad T \setminus T(\Gamma) = T' \setminus T'(\Gamma). \tag{3.6}$$

Proof. From (2.5) it is easy to show that

$$(e_T, \mathcal{T}_t e_{T'}) = \sum_{n \geqslant 0} \frac{(-t)^n}{n!} (e_T, H^n e_{T'}) = \sum_{n \geqslant 0} \frac{(-t)^n}{n!} \prod_{(L_i, L_i')} \prod_{i=1}^{n} \omega^H(L_i, L_i'), \tag{3.7}$$

where the summation is over (ordered) systems of pairs $((L_i, L_i'))_{i=1}^n$ such that

$$\begin{gathered} L_i \neq \emptyset, \quad L_i' \neq \emptyset, \\ L_1 \subset T_1 = T, \quad L_1' \cap (T_1 \setminus L_1) = \emptyset, \\ L_2 \subset T_2 = L_1' \cup (T_1 \setminus L_1), \quad L_2' \cap (T_2 \setminus L_2) = \emptyset, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_n \subset T_n = L_{n-1}' \cup (T_{n-1} \setminus L_{n-1}), \quad L_n' \subset T', \\ T_n \setminus L_n = T' \setminus L_n'. \end{gathered} \tag{3.8}$$

(Here the vectors e_T belong to the domains of all H^n.) We divide each collection $((L_i, L_i'))$ satisfying (3.8) into connected (ordered) subsystems — bonds γ_j, $j = 1, \ldots, m$, and assign to it the system $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$. From (3.8) it follows that Γ is completely regular and satisfies (3.6). Conversely, to any completely regular system Γ of bonds satisfying (3.6) we can assign $n!/(k(\gamma_1)! \ldots k(\gamma_m)!)$ systems $((L_i, L_i'))_{i=1}^n$ satisfying (3.8); here $\sum k(\gamma_i) = n$. Different systems $((L_i, L_i'))$ correspond to different permutations of the set $\{1, \ldots, n\}$ leaving the natural orderings of the γ_i invariant. From this remark and (3.7) the lemma follows.

DEFINITION. Let $T, T' \subset \mathbf{Z}^\nu$. Then we define

$$\omega_t(T, T') = \delta_{|T \cup T'|,1} + \sum_{\Gamma} D(\Gamma) \tilde{\omega}_t(\Gamma), \tag{3.9}$$

where the summation is over all connected regular systems Γ of bonds such that

$$T(\Gamma) = T, \quad T'(\Gamma) = T'. \tag{3.10}$$

To define $D(\Gamma)$ for any $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, consider the graph $G = G_\Gamma$ with vertices $\{1, \ldots, m\}$, in which there is a unique edge between vertices i and j iff $\tilde{\gamma}_i \cap \tilde{\gamma}_j \neq \emptyset$. Then we put

$$D(\Gamma) = \mu_{\mathfrak{A}_G}(\mathbf{0}, \mathbf{1}),$$

(see [4, 55]), i.e. $D(\Gamma)$ is the Möbius function for the lattice $\mathfrak{A}_G$ of partitions of the graph G.

LEMMA 3.2. *We have*

$$(e_T, \mathcal{T}_t e_{T'}) = \sum_{(T_i, T_i')} \prod_i \omega_t(T_i, T_i'),$$

where the sum is over all partitions $\{(T_1, T_1'), \ldots, (T_s, T_s')\}$, $s = 1, 2, \ldots$, such that $T_i \neq \emptyset$, $T_i' \neq \emptyset$, $\bigcup T_i = T$, $\bigcup T_i' = T'$, $T_i \cap T_j = T_i' \cap T_j' = \emptyset$ for $i \neq j$.

Proof. From (3.9) and (3.4) we get

$$\sum_{(T_i, T_i')} \prod_{i=1}^{s} \omega_t(T_i, T_i') = \sum_{(T_i, T_i')} \sum_I \prod_{i \notin I} \delta_{|T_i \cup T_i'|,1} \sum' \prod_{i \in I} D(\Gamma_i) \tilde{\omega}_t(\Gamma_i)$$
$$= \sum'' \tilde{\omega}_t(\Gamma) \sum''' \prod_{i=1}^{r} D(\Gamma_i),$$

where $I \subset \{1, \ldots, s\}$, the sum $\sum'$ is over all ordered systems $(\Gamma_i, i \in I)$ with $T(\Gamma_i) = T_i$, $T'(\Gamma_i) = T_i'$ for $i \in I$, the sum $\sum''$ is over all systems Γ such that $T(\Gamma) \subset T$, $T'(\Gamma) \subset T'$, $T \setminus T(\Gamma) = T' \setminus T'(\Gamma)$, and the sum $\sum'''$ is over connected regular $\Gamma_1, \ldots, \Gamma_r$ such that $\bigcup \Gamma_i = \Gamma$. Therefore Lemma 3.2 follows from the fact that

$$\sum \prod_i D(\Gamma_i) = \begin{cases} 1 & \text{if } \Gamma \text{ is completely regular,} \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all $\Gamma_1, \ldots, \Gamma_r$ with $\bigcup \Gamma_i = \Gamma$ (see [4, 5]).

To finish the proof of Theorem 1(a) we shall prove the estimate (2.4). In order to do this we transform (3.9). Notice that $D(\Gamma)$ does not change if we replace one bond $\gamma \in \Gamma$, $\gamma = ((L_1, L_1'), \ldots, (L_s, L_s'))$, with

$$\gamma' = ((L_1, L_1'), \ldots, (L_j, L_j'), (\{x\}, \{x\}), (L_{j+1}, L_{j+1}'), \ldots, (L_s, L_s')),$$

where $x \in (L_{j+1} \cup T(\gamma)) \cap (L_j' \cup T'(\gamma))$; only $k(\gamma)$ and $\prod \omega^H(L, L')$ change in (3.4). Any bond γ with $|\tilde{\gamma}| > 1$ can be obtained with the help of a sequence of the above operations from a unique bond γ' consisting of pairs (L, L') such that $|L \cup L'| > 1$. If $\tilde{\gamma} = \{x\}$ then $\gamma = ((\{x\}, \{x\}), \ldots, (\{x\}, \{x\}))$ and we say that γ is obtained with the help of the sequence of the above operations from a unique *distinguished bond* γ' with $k(\gamma') = 0$ and $\tilde{\gamma}' = \{x\}$. For γ' of the above two types we define the numbers $r_1, \ldots, r_{k(\gamma')+1}$, where $r_1 = 1$ for distinguished γ', and for $\gamma' = ((L_1, L_1'), \ldots, (L_k, L_k'))$, $k = k(\gamma')$,

$$r_1 = |T(\gamma')|, \quad r_j = |(L_j \cup T(\gamma')) \cap (L_{j-1}' \cup T'(\gamma'))|,$$
$$j = 2, \ldots, k, \quad r_{k+1} = |T'(\gamma')|.$$

Notice that since γ' is a bond

(3.11) $$r_j \geqslant \max\{|T(\gamma') \cap T'(\gamma')|, 1\}, \quad j = 1, \ldots, k+1.$$

Using the above arguments, from (3.4) and (3.9) we get

(3.12) $$\omega_t(T, T') = \delta_{|T \cup T'|,1} + \sum_\Gamma{}' D(\Gamma) \prod_{\gamma' \in \Gamma} (-t)^{k(\gamma')} \prod_{(L,L') \in \gamma'} \omega^H(L, L')$$
$$\times \sum_{l_1 = 0, \ldots, l_{k+1} = 0}^{\infty} \frac{(-tar_1)^{l_1} \ldots (-tar_{k+1})^{l_{k+1}}}{(k + l_1 + \ldots + l_{k+1})!},$$

where the sum $\sum'$ extends over all connected systems $\Gamma = \{\gamma_1', \ldots, \gamma_m'\}$ of distinguished bonds and bonds consisting of pairs (L, L') with $|L \cup L'| > 1$ such that $T(\Gamma) = T$, $T'(\Gamma) = T'$; here $k = k(\gamma')$.

If $T = T' = \{x\}$ then the above sum is simply

$$\omega_t(\{x\}, \{x\}) = e^{-at}.$$

For other pairs (T, T') we use the following lemma.

Lemma 3.3. (a) *The following formula is true*:

(3.13) $$\sum_{l_1 = 0, \ldots, l_{k+1} = 0}^{\infty} \frac{z_1^{l_1} \ldots z_{k+1}^{l_{k+1}}}{(k + l_1 + \ldots + l_{k+1})!} = \sum_{i=1}^{k+1} \frac{e^{z_i}}{\prod_{j \neq i} (z_i - z_j)}.$$

(b) *We have the estimate*

(3.14) $$|f(t)| = \left| t^k \sum_{i=1}^{k+1} \frac{e^{-atr_i}}{\prod_{j \neq i} at(r_i - r_j)} \right|$$
$$\leqslant \begin{cases} (C|t|)^k & \textit{for } |at| < 1 \ (t \in \mathbf{C}^1), \\ (C/a)^k e^{-(\min r_i - 1/3) a \operatorname{Re} t} & \textit{for } \operatorname{Re}(at) \geqslant 1, \end{cases}$$

where the r_i are positive integers and $C > 0$ is an absolute constant independent of a, t, k, r_i.

I have not found the formula (3.13) in the literature. The proof of this lemma is left to the Appendix.

We return to the estimate of $\omega_t(T, T')$. We use the following estimate of $D(\Gamma)$ proved in [5]:

(3.15) $$|D(\Gamma)| \leqslant \prod_{\gamma' \in \Gamma} C_1^{d_{\tilde{\gamma}'}} \leqslant \prod_{\gamma'} \prod_{(L,L') \in \gamma'} C_1^{d_{L \cup L'}},$$

where $C_1 = C_1(\nu)$ is an absolute constant. Notice also that

(3.16) $$\sum_{\gamma' \in \Gamma} \sum_{(L,L') \in \gamma'} d_{L \cup L'} \geqslant d_{T \cup T'}, \quad \sum k(\gamma') \geqslant 1.$$

From (3.11), (3.12) and (3.14)–(3.16) we get

$$|\omega_t(T, T')| \leqslant \begin{cases} \sum_\Gamma{}' \prod_{\gamma'} (C|t|)^k \prod_{(L,L')} (C_2 \beta)^{d_{L \cup L'}} & \text{if } |at| < 1, \\ \sum_\Gamma{}' \prod_{\gamma'} e^{-(\min r_i - 1/3) \operatorname{Re}(at)} (C/a)^k \prod_{(L,L')} (C_2 \beta)^{d_{L \cup L'}} & \text{if } \operatorname{Re}(at) \geqslant 1. \end{cases}$$

As in [6.II, p. 218] one can show that for fixed $|\Gamma| = m(\Gamma)$ and $k(\gamma_i')$, $i = 1, \ldots, |\Gamma|$,

$$\sum_\Gamma{}' \prod_{\gamma'} \prod_{(L,L')} (C_2 \beta)^{d_{L \cup L'}} \leqslant (C_3 \beta^{1-\varepsilon})^{d_{T \cup T'}} \beta^{\varepsilon \sum k(\gamma')}$$

for some small $\varepsilon > 0$. Hence we get the final estimate

$$|\omega_t(T, T')| \leqslant \begin{cases} \tilde{M} |t| \beta_1^{d_{T \cup T'}} & \text{if } |at| < 1, \\ \tilde{M} e^{-2\operatorname{Re}(at)/3} \beta_1^{d_{T \cup T'}} & \text{if } \operatorname{Re}(at) \geqslant 1, \end{cases}$$

where $\beta_1 = \beta^{1-\varepsilon}$. This finishes the proof of Theorem 1(a).

4. Proof of Theorem 1(b). Let $\mathcal{T}_t$ be a one-parameter holomorphic group of multiplicative clustering operators with clustering functions obeying the estimate (2.4).

Lemma 4.1. *We have*

$$\omega_0(T, T') = \delta_{|T \cup T'|,1}.$$

Proof. This follows from the fact that the clustering functions of multiplicative clustering operators are uniquely determined by their matrix elements [8].

Differentiating the formula (2.1) for $\mathcal{T}_t$ at $t=0$ and using Lemma 4.1 we obtain the formula (2.5). Hence one only needs to prove the estimate (2.6). But this follows from the Cauchy formula

$$\omega^H(T, T') = \frac{\partial}{\partial t}\omega_t(T, T')\Big|_{t=0} = \frac{1}{2\pi i}\oint_{|t|=r}\frac{\omega_t(T, T')}{t^2}\,dt$$

and the estimate (2.4). Theorem 1(b) is proved.

5. Proof of Theorem 2. Denote by D the diagonal part and by V the off diagonal part of $\mathcal{T}$:

$$D\psi(T) = \lambda^{|T|}\psi(T), \qquad V = \mathcal{T} - D.$$

Assume that $\psi \in \mathcal{H}$ is an eigenfunction of $\mathcal{T}$ with eigenvalue 0. Then

$$\mathcal{T}\psi = D(\mathrm{id} + D^{-1}V)\psi = 0.$$

LEMMA 5.1. *We have*

$$\|D^{-1}V\| \leqslant \mathrm{const}\,\beta(1-\lambda)$$

with const *independent of β and λ provided β/λ is sufficiently small.*

From this lemma it immediately follows that $\psi = 0$, which completes the proof of Theorem 2.

Proof of Lemma 5.1. We use the results of [6.I]. It is easy to see that the operator $D^{-1}V$ is clustering (i.e. is of the form (2.1) with $\prod \omega(L_i, L_i')$ replaced with $\tilde{\omega}((L_1, L_1'), \ldots, (L_s, L_s'))$, where $\tilde{\omega} = \tilde{\omega}^{D^{-1}V}$ is some translation invariant function of s arguments, [6, 8]). The clustering functions of $D^{-1}V$ are

$$\tilde{\omega}((\{x_1\}, \{x_1\}), \ldots, (\{x_s\}, \{x_s\})) = 0,$$

$$\tilde{\omega}((L_1, L_1'), \ldots, (L_s, L_s')) = \prod \lambda^{-|L_i|}\omega^{\mathcal{T}}(L_i, L_i') \quad \text{if } \prod |L_i \cup L_i'| > 1,$$

and satisfy the estimate

$$|\tilde{\omega}((L_1, L_1'), \ldots, (L_s, L_s'))| \leqslant \prod \mathrm{const}\,\lambda(1-\lambda)(\beta/\lambda)^{d_{L_i \cup L_i'}}.$$

Denote by P_n the orthogonal projection of $\mathcal{H}$ onto $\{f\colon f(T) = 0$ if $|T| \neq n\}$. Then from the results of [6.I] (Corollary 1 from Lemma 2.2) we have

$$\|P_n D^{-1} V P_m\| \leqslant \mathrm{const}\,\lambda(1-\lambda)\begin{cases}(C\beta/\lambda)^{\max(m,n)} & \text{if } m>0,\ n>0,\\ 0 & \text{if } m=0 \text{ or } n=0.\end{cases}$$

From this we obtain

$$\|D^{-1}V\| \leqslant \tfrac{1}{2}\sup_n \sum_m \|P_n D^{-1} V P_m\| + \tfrac{1}{2}\sup_m \sum_n \|P_n D^{-1} V P_m\|$$
$$\leqslant \mathrm{const}\,\lambda(1-\lambda)\beta/\lambda.$$

Lemma 5.1 is proved.

Appendix. Proof of Lemma 3.3. (a) We first prove the formula (3.13). It is easy to see that it is enough to show the identity

$$\text{(A.1)}\qquad \sum_{i=1}^{k+1}(-1)^i x_i^m \prod_{\substack{1\leqslant r<s\leqslant k+1\\ r,s\neq i}}(x_r - x_s) = \begin{cases} 0 & \text{if } m<k,\\ \prod\limits_{1\leqslant i<j\leqslant k+1}(x_i - x_j)\ \left[\sum x_1^{\alpha_1}\ldots x_{k+1}^{\alpha_{k+1}}\right] & \text{if } m\geqslant k,\end{cases}$$

where the sum is over all $\alpha_1 \geqslant 0, \ldots, \alpha_{k+1} \geqslant 0$ with $\sum \alpha_i = m-k$.

From the Vandermonde formula it follows that the left side of (A.1) is

$$\text{(A.2)}\qquad \det\begin{bmatrix} x_1^m & x_1^{k-1} & \ldots & x_1 & 1\\ x_2^m & x_2^{k-1} & \ldots & x_2 & 1\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ x_{k+1}^m & x_{k+1}^{k-1} & \ldots & x_{k+1} & 1\end{bmatrix}$$

and that

$$\text{(A.3)}\qquad \prod_{1\leqslant i<j\leqslant k+1}(x_i - x_j) = \det\begin{bmatrix} x_1^k & x_1^{k-1} & \ldots & x_1 & 1\\ \vdots & \ldots & \ldots & \ldots & \ldots\\ x_{k+1}^k & x_{k+1}^{k-1} & \ldots & x_{k+1} & 1\end{bmatrix}.$$

If $m<k$ then (A.1) follows from the fact that in (A.2) two columns are equal. The case $m=k$ is also obvious.

To consider the case $m>k$ we compute the coefficient a_β of the monomial

$$x^\beta = x_1^{\beta_1}\ldots x_{k+1}^{\beta_{k+1}}, \qquad \sum \beta_i = m + k(k-1)/2,$$

in the expansion of the right-hand side of (A.1). The contribution to a_β from (A.3) comes only from monomials $x^\alpha = x_1^{\alpha_1}\ldots x_{k+1}^{\alpha_{k+1}}$ with $\alpha_i \leqslant \beta_i$. Each such monomial is multiplied by a unique monomial $x^{\beta-\alpha}$ with coefficient 1 coming from $\sum x^\alpha$ in (A.1). From this one can easily see that

$$\text{(A.4)}\qquad a_\beta = \det\begin{bmatrix} 0\ldots 0 & 1\ldots 1\\ \ldots & \ldots\\ 0\ldots 0 & 1\ldots 1\end{bmatrix}\begin{matrix}(\beta_1 \text{ zeros})\\ \\ (\beta_{k+1} \text{ zeros})\end{matrix}.$$

Now, $a_\beta = \operatorname{sign}\pi$ if there exists a permutation π of rows of the above matrix leading to

$$\begin{bmatrix} 1 & & 1 \\ & \ddots & \\ 0 & & 1 \end{bmatrix},$$

and 0 in other cases. But these coefficients coincide with those of (A.2), which completes the proof of the formula (3.13).

(b) If $r_i \neq r_j$ for $i \neq j$ then the estimate (3.14) for $\operatorname{Re}(at) \geqslant 1$ is obvious, and when $|at| \leqslant 1$ one can use the Cauchy formula

$$t^{-k} f(t) = \frac{1}{2\pi i} \oint_{|\zeta|=2} \frac{(\zeta/a)^{-k} f(\zeta/a)}{\zeta - at} d\zeta.$$

The difficulty appears when some r_i and r_j coincide. Consider first the case $\operatorname{Re}(at) \geqslant 1$. Assume that $r_1 = r_2 = \ldots = r_{l+1} = r$ and $r_i \neq r_j$ in other cases (other groups of coinciding r's are considered analogously). We use the Cauchy formula

$$\text{(A.5)} \quad f(t) = a^{-k}\left(\frac{1}{2\pi i}\right)^l \oint_{|\zeta_2 - r| = 1/(3l)} \oint_{|\zeta_3 - r| = 2/(3l)} \ldots \oint_{|\zeta_{l+1} - r| = 1/3} \frac{d\zeta_2 \ldots d\zeta_{l+1}}{\prod (\zeta_i - r)}$$
$$\times \left\{ \frac{e^{-atr_1}}{\prod_{i=2}^{l+1} (r-\zeta_i) \prod_{j>l+1} (r-r_j)} + \sum_{i=2}^{l+1} \frac{e^{-at\zeta_i}}{\prod (\zeta_i - \zeta_j) \prod (\zeta_i - r_j)} \right.$$
$$\left. + \sum_{i>l+1} \frac{e^{-atr_i}}{\prod (r_i - \zeta_j) \prod (r_i - r_j)} \right\}.$$

We have the following estimates:

$$\text{(A.6)} \quad \begin{gathered} |\zeta_i - \zeta_j| \geqslant |i-j|/(3l), \\ |r_i - \zeta_j| \geqslant \tfrac{2}{3} \quad \text{for } i > l+1, \\ C_1^l \leqslant \prod_{i=1}^{l+1} i/(3l) = (l+1)!\,(3l)^{-l-1} \leqslant C_2^l. \end{gathered}$$

From (A.5) and (A.6) the estimate (3.14) for $\operatorname{Re}(at) \geqslant 1$ can be easily obtained. The case $|at| < 1$ is treated as before. Lemma 3.3 is proved.

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Received March 3, 1985 (2040)
Revised version December 11, 1986