Consequently $Q\eta$ is in the linear subspace generated by $T_1\xi_0, \ldots, T_{n-1}\xi_0$. So Q has a finite rank $\leqslant n-1$. If moreover the T_i commute, then Q and R commute, so $Q^2\eta = -QR\xi_0 = -RQ\xi_0 = 0$. Hence $Q^2 = 0$. ∎

Remark. Let P and Q be two different projections having the same range of dimension 1, defined on a complex vector space X. For every $\xi \in X$ the vectors $P\xi$ and $Q\xi$ are dependent and obviously there are linear combinations of P and Q having rank one. But $\alpha P+\beta Q \neq 0$ for any $\alpha, \beta \in \boldsymbol{C}$. So in general it is impossible to have $Q = 0$ in Theorem 2.

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Received December 5, 1986 (2253)

STUDIA MATHEMATICA, T. LXXXVIII. (1988)

Extension of C^∞ functions from sets with polynomial cusps

by

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Abstract. We give a simple construction of a continuous linear operator extending C^∞ functions from compact subsets of $\boldsymbol{R}^n$ with polynomial cusps including fat subanalytic sets.

1. Introduction. Whitney's extension theorem [15] yields a continuous linear operator extending C^k functions (k finite) defined on closed subsets X of $\boldsymbol{R}^n$. For C^∞ functions such an operator does not in general exist (see e.g. [12, p. 79]). However, Mityagin [4] and Seeley [7] proved the existence of an extension operator if X is a half-space of $\boldsymbol{R}^n$. Stein [9] showed that such an operator exists if X is the closure of a Lipschitz domain in $\boldsymbol{R}^n$ of class Lip 1. Stein's result was then extended by Bierstone [1] to the case of a domain with boundary which is Lipschitz of any order. By the main result of Bierstone [1] involving Hironaka's desingularization theorem, an extension operator exists if X is a *fat* (i.e. $\overline{\operatorname{int} X} \supset X$) closed subanalytic subset of $\boldsymbol{R}^n$. If X is Nash subanalytic (not necessarily fat) the existence problem was solved by Bierstone and Schwarz [3]. Recently Wachta [14] has constructed an extension operator for fat closed subanalytic sets in $\boldsymbol{R}^n$ without making use of the Hironaka desingularization theorem. For closed subsets of $\boldsymbol{R}^n$ admitting some polynomial cusps, the existence of an extension operator was shown by Tidten [10].

In this paper we construct an extension operator for the family of compact *uniformly polynomially cuspidal* (briefly, UPC) subsets of $\boldsymbol{R}^n$ (see Theorem 4.1). The UPC sets were introduced in [6] as follows.

Definition 1.1. A subset X of $\boldsymbol{R}^n$ is said to be UPC if there exist positive constants M and m, and a positive integer d such that for each point x in $\bar{X}$, one may choose a polynomial map $h_x\colon \boldsymbol{R} \to \boldsymbol{R}^n$ of degree at most d satisfying the following conditions:

(i) $h_x((0, 1]) \subset X$ and $h_x(0) = x$;

(ii) $\operatorname{dist}(h_x(t), \boldsymbol{R}^n - X) \geqslant Mt^m$ for all x in X and $t \in (0, 1]$.

Every bounded convex domain in $\boldsymbol{R}^n$ and every bounded Lipschitz domain are UPC. More generally, every subset of $\boldsymbol{R}^n$ with a parallelepiped property is UPC (see [6]). Using Hironaka's rectilinearization theorem and Łojasiewicz's inequality we proved in [6, Corollary 6.6] that every bounded fat subanalytic subset of $\mathbf{R}^n$ is UPC. Further examples of UPC sets are provided by the following

PROPOSITION 1.2. *Let K be a compact set in $\mathbf{R}^n$ and let f be a C^∞ mapping defined in $\mathbf{R}^n$ with values in $\mathbf{R}^n$ such that for each $x \in K$, $D(x) \neq 0$, D being the Jacobi determinant of f. If then K is UPC, so is the set $f(K)$.*

Proof. By Definition 1.1, there exists a mapping $h\colon K \times [0, 1] \ni (x, t) \to h(x, t) \in K$ such that for each $x \in K$, $h(x, \cdot)$ is a polynomial of degree at most d, and

$$\text{(1.1)} \qquad \operatorname{dist}\big(h(x, t), \mathbf{R}^n - K\big) \geqslant Mt^m, \quad \text{for } x \in K \text{ and } t \in [0, 1],$$

the constants d, M and m being independent of x and t. Choose $c > 0$ so that $|D(x)| \geqslant c$, for each $x \in K$. By the implicit function theorem, there exist positive constants L and L_1 such that for each $x \in K$ and each $r \in (0, c]$,

$$\text{(1.2)} \qquad f\big(B(x, Lr)\big) \supset B\big(f(x), L_1 r\big),$$

$B(a, s)$ denoting the ball centered at a of radius s (see e.g. [12, p. 105–106]). Fix $y \in f(K)$ and choose $x \in K$ such that $y = f(x)$. Define

$$H(y, t) := f\big(h(x, t)\big), \quad \text{for } t \in [0, 1].$$

Then by (1.1) and (1.2),

$$\text{(1.3)} \qquad \operatorname{dist}\big(H(y, t), \mathbf{R}^n - f(K)\big) \geqslant Nt^m, \quad \text{for } t \in [0, 1],$$

where $N = L_1 \min(M, Lc)/L$. Take a positive integer $s \geqslant m$. Let $T(y, \cdot)$ be the Taylor polynomial at 0 of degree s of the function $H(y, \cdot)$. We have

$$\text{(1.4)} \qquad H(y, t) = T(y, t) + t^{s+1} R(y, t), \quad \text{for } (y, t) \in f(K) \times [0, 1],$$

where $R(y, t) = [1/(s+1)!](\partial^{s+1} H/\partial t^{s+1})(y, \Theta t)$ with $0 < \Theta < 1$ (Θ depending on t). Since the coefficients of each polynomial $h(x, \cdot)$ are uniformly bounded on K and since $\deg h(x, \cdot) \leqslant d$, for each $x \in K$, the remainder R is uniformly bounded in $f(K) \times [0, 1]$. Choose $\delta \in (0, 1]$ such that $|tR(y, t)| \leqslant N/2$ as $y \in f(K)$ and $t \in [0, \delta]$. Then by (1.3) and (1.4),

$$\text{(1.5)} \qquad \operatorname{dist}\big(T(y, t), \mathbf{R}^n - f(K)\big) \geqslant (N/2)\, t^m, \quad \text{for } (y, t) \in f(K) \times [0, \delta].$$

The proposition follows if we replace in (1.5) t by δt. Actually, it is sufficient to assume f is of class C^{s+1}.

From Proposition 1.2 we derive in particular that the class of (compact) sets with cusps considered by Tidten [10] is contained in the class of UPC sets. On the other hand, the set $X = \{(x, y) \in \mathbf{R}^2\colon\ 0 \leqslant x \leqslant 1,\ x^{2/3} \leqslant y \leqslant 2x^{2/3}\}$ is UPC (since it is semialgebraic) but it does not belong to Tidten's class.

In order to construct the extension operator we use the Lagrange interpolation polynomials corresponding to systems of Fekete–Leja's extremal points of X. In the proof of the extension property a crucial role is played by both Markov's inequality and Bernstein's theorem for UPC sets proved in [6]. For the convenience of the reader we restate them below.

THEOREM 1.3 (Markov's inequality). *Let X be a UPC subset of $\mathbf{R}^n$. Then there exists a constant $r > 0$ such that for each polynomial $p\colon \mathbf{R}^n \to \mathbf{R}$ of degree at most k, and each multiindex $\alpha \in \mathbf{Z}^n_+$, we have*

$$\|D^\alpha p\|_X \leqslant Ck^{r|\alpha|} \|p\|_X,$$

where C is a positive constant depending only on X. (Here $\|h\|_X$ stands for the supremum norm of a function h defined on X.)

THEOREM 1.4 (Bernstein's theorem). *Let X be a UPC compact set in $\mathbf{R}^n$. A real-valued function f defined on X is the restriction to X of a C^∞ function g on $\mathbf{R}^n$ if and only if for each $r > 0$,*

$$\lim_{k \to \infty} k^r \operatorname{dist}_X(f, \mathscr{P}_k) = 0,$$

where $\mathscr{P}_k$ is the linear space of (the restrictions to X of) all polynomials from $\mathbf{R}^n$ to $\mathbf{R}$ of degree at most k, and

$$\operatorname{dist}_X(f, \mathscr{P}_k) = \inf\{\|f - p\|_X\colon\ p \in \mathscr{P}_k\}.$$

In the subanalytic case Theorems 1.3 and 1.4 were proved in [5]. Both Markov's inequality and Bernstein's theorem hold true if X is a compact subset of $\mathbf{R}^n$ such that *Siciak's extremal function* of X, introduced in [8] by the formula

$$\Phi_X(x) = \sup_{k \geqslant 1} \{\sup\{|p(x)|^{1/k}\colon\ p \in \mathscr{P}_k,\ \|p\|_X \leqslant 1\}\}, \quad \text{for } x \in \mathbf{C}^n,$$

has the following *Hölder continuity property* (cf. [6, Remark 3.2]):

$$\text{(HCP)} \qquad \Phi_X(x) \leqslant 1 + C_1 \delta^\mu \quad \text{if } \operatorname{dist}(x, X) \leqslant \delta \leqslant 1,$$

with some positive constants C_1 and μ independent of δ. Consequently, in order that our extension operator exist it is sufficient that X have HCP. In [6, Theorem 4.1] we proved

THEOREM 1.5. *If X is a UPC compact subset of $\mathbf{R}^n$ then Φ_X has HCP with the constants C_1 and μ depending on the constants M, m and d of Definition 1.1.*

So far, we do not know any other examples of (fat) compact subsets X of $\mathbf{R}^n$ with Φ_X having HCP.

2. C^∞ functions on compact sets in $\boldsymbol{R}^n$. Let X be a fat compact subset of the space $\boldsymbol{R}^n$ (i.e. $X = \overline{\operatorname{int} X}$). A C^∞ *function* on X is a function $f\colon X \to \boldsymbol{R}$ such that there exists a C^∞ function g on $\boldsymbol{R}^n$ with $g|X = f$. Let $C^\infty(X)$ be the space of all such functions. For a compact set E in $\boldsymbol{R}^n$ and $k = 0, 1, \ldots$, we define

$$q_{E,k}(f) := \inf\{|g|_E^k\colon\ g \in C^\infty(\boldsymbol{R}^n),\ g|X = f\},$$

where

$$|g|_E^k := \max_{|\alpha| \leqslant k} \|D^\alpha g\|_E.$$

Let τ_1 be the topology in $C^\infty(X)$ defined by the seminorms $q_{E,k}$. Then τ_1 is exactly the quotient topology of the space $C^\infty(\boldsymbol{R}^n)/I(X)$, where $C^\infty(\boldsymbol{R}^n)$ is endowed with the natural topology τ_0 defined by the seminorms $|\cdot|_E^k$, and $I(X) = \{g \in C^\infty(\boldsymbol{R}^n)\colon\ g|X = 0\}$. Since the space $(C^\infty(\boldsymbol{R}^n), \tau_0)$ is complete and $I(X)$ is a closed subspace of $C^\infty(\boldsymbol{R}^n)$, the quotient space $C^\infty(\boldsymbol{R}^n)/I(X)$ is also complete, whence $(C^\infty(X), \tau_1)$ is a Fréchet space.

Now consider the mapping $J\colon C^\infty(X) \ni f \to J(f) = (D^\alpha g|X)_{\alpha \in \boldsymbol{Z}_+^n}$, where $g \in C^\infty(\boldsymbol{R}^n)$ and $g|X = f$. By Whitney's extension theorem [15], J is a linear bijection of the space $C^\infty(X)$ onto the space $\mathscr{E}(X)$ of C^∞ Whitney fields $F = (F^\alpha)_{\alpha \in \boldsymbol{Z}_+^n}$, where each F^α is a continuous function on X. $\mathscr{E}(X)$ is a Fréchet space with the topology τ_2 defined by the seminorms

$$\|F\|_X^k = |F|_X^k + \sup\{|(R_x^k F)^\alpha(y)|/|x-y|^{k-|\alpha|}\colon\ x, y \in X,\ x \neq y,\ |\alpha| \leqslant k\}$$

($k = 0, 1, \ldots$), where

$$|F|_X^k = \sup\{|F^\alpha(x)|\colon\ x \in X,\ |\alpha| \leqslant k\},$$

$$(R_x^k F)^\alpha(y) = F^\alpha(y) - \sum_{|\beta| \leqslant k-|\alpha|} (1/\beta!)\, F^{\alpha+\beta}(x)(y-x)^\beta.$$

The linear bijection J is a continuous mapping from $(C^\infty(X), \tau_1)$ to $(\mathscr{E}(X), \tau_2)$. This follows from the fact that if the geodesic distance on a compact set E in $\boldsymbol{R}^n$ is equivalent to the Euclidean distance, then the seminorms $\|\ \|_E^k$ and $|\ |_E^k$ are equivalent too (see [15]). Hence in particular, if E is a cube (containing X), the seminorms $\|\ \|_E^k$ and $|\ |_E^k$ are equivalent. Consequently, by Banach's theorem, the spaces $(C^\infty(X), \tau_1)$ and $(\mathscr{E}(X), \tau_2)$ are isomorphic.

Now we equip $C^\infty(X)$ with another topology. Following an idea of Zerner [16], we set $d_{-1}(f) := \|f\|_X$, $d_0(f) := \operatorname{dist}_X(f, \mathscr{P}_0)$, and for $k = 1, 2, \ldots$,

$$d_k(f) := \sup_{l \geqslant 1} l^k \operatorname{dist}_X(f, \mathscr{P}_l).$$

Then by Jackson's theorem (see e.g. [11]), each d_k is a seminorm on $C^\infty(X)$.

Denote by τ_3 the topology in $C^\infty(X)$ defined by the system of the seminorms d_k ($k = -1, 0, \ldots$). We have

PROPOSITION 2.1. *If X is a UPC compact set in $\boldsymbol{R}^n$ then $(C^\infty(X), \tau_3)$ is a Fréchet space.*

Proof. Let (f_s) be a Cauchy sequence in $(C^\infty(X), \tau_3)$. Then (f_s) is uniformly convergent on X to a continuous function f. Hence, since the function $h \to \operatorname{dist}_X(h, \mathscr{P}_l)$ is continuous in the Banach space $C(X)$ of all continuous functions on X with the uniform norm $\|\ \|_X$, for each $\varepsilon > 0$, $d_k(f_s - f_t) \leqslant \varepsilon$ implies $d_k(f - f_s) \leqslant \varepsilon$. Therefore $d_k(f) \leqslant d_k(f-f_s) + d_k(f_s) < \infty$, and by Theorem 1.4, $f \in C^\infty(X)$.

PROPOSITION 2.2. *If X is a UPC compact set in $\boldsymbol{R}^n$ then the topologies τ_1 and τ_3 coincide.*

Proof. Put $\varepsilon_0 = 1$ and for each $k \geqslant 1$, set $\varepsilon_k = (1/(C_1 k))^{1/\mu}$, where the constants C_1 and μ are so chosen that the extremal function Φ_X satisfies HCP (see Theorem 1.5), and $C_1 \geqslant 1$. For $k = 0, 1, \ldots$, there exist C^∞ functions u_k on $\boldsymbol{R}^n$ such that $0 \leqslant u_k \leqslant 1$, $u_k = 1$ in a neighborhood of X, $u_k(x) = 0$ if $\operatorname{dist}(x, X) \geqslant \varepsilon_k$, and for all $x \in \boldsymbol{R}^n$ and $\alpha \in \boldsymbol{Z}_+^n$, $|D^\alpha u_k(x)| \leqslant C_\alpha \varepsilon_k^{-|\alpha|}$ with some constants C_α depending only on α (see e.g. [12, p. 77]). Fix $f \in C^\infty(X)$ and define

$$g = \sum_{k=0}^{\infty} u_k \cdot (p_k - p_{k-1}),$$

where $p_{-1} := 0$ and for $k = 0, 1, \ldots$, p_k is a polynomial of degree at most k such that $\|f - p_k\|_X = \operatorname{dist}_X(f, \mathscr{P}_k)$. Note that if E is a fixed compact set in $\boldsymbol{R}^n$ such that $X \subset \operatorname{int} E$, then the topology τ_1 coincides with that given by the family of the seminorms $q_{E,k}$ ($k = 0, 1, \ldots$). Set $X_k = \{x \in \boldsymbol{R}^n\colon \operatorname{dist}(x, X) \leqslant \varepsilon_k\}$ for $k = 0, 1, \ldots$, and choose E to be the set X_0. Then $E \supset X_k$, for each k. By Theorems 1.5 and 1.3, for each $l = 0, 1, \ldots$ and each $\alpha \in \boldsymbol{Z}_+^n$ with $|\alpha| \leqslant l$, we get

$$\begin{aligned}
\|D^\alpha g\|_E &\leqslant \|D^\alpha u_0 p_0\|_{X_0} + \sum_{k=1}^{\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \|D^\beta u_k\|_{X_k} \|D^{\alpha-\beta}(p_k - p_{k-1})\|_{X_k} \\
&\leqslant C_\alpha \|f\|_X + \sum_{k=1}^{\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} C_\beta \varepsilon_k^{-|\beta|} (1 + C_1 \varepsilon_k^{\mu})^k \|D^{\alpha-\beta}(p_k - p_{k-1})\|_X \\
&\leqslant C_\alpha \|f\|_X + \sum_{k=1}^{\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} C_\beta (C_1 k)^{|\beta|/\mu} (1 + 1/k)^k C k^{r|\alpha-\beta|} \|p_k - p_{k-1}\|_X \\
&\leqslant C_\alpha d_{-1}(f) + C_2 d_0(f) + C_2 \sum_{k=2}^{\infty} k^{|\beta|/\mu + r|\alpha-\beta|} \operatorname{dist}_X(f, \mathscr{P}_{k-1}) \\
&\leqslant C_\alpha d_{-1}(f) + C_2 d_0(f) + C_3 d_{sl+2}(f),
\end{aligned}$$

where s is an integer such that $s \geqslant \max(1/\mu, r)$, and the constants C_2 and C_3 depend only on n, X and l. The proposition now follows from Banach's theorem.

Finally, let τ_4 denote the topology for $C^\infty(X)$ defined by the seminorms

$$|f|_X^k = \sup\{|D^\alpha f(x)|:\ x \in \operatorname{int} X,\ |\alpha| \leqslant k\}.$$

Assume X has the following *Whitney extension property* (WEP): For every C^∞ function f on $\operatorname{int} X$ whose partial derivatives of all orders are uniformly continuous on $\operatorname{int} X$, there exists a C^∞ function g in $\boldsymbol{R}^n$ such that $g = f$ on $\operatorname{int} X$. Then the space $(C^\infty(X), \tau_4)$ is complete and again by Banach's theorem, the topologies τ_1, τ_2 and τ_4 coincide.

We note that there exist UPC compact sets in $\boldsymbol{R}^n$ which do not have WEP (take e.g.

$$X = [0, 1]\times[-1, 1] - \{(x, y) \in \boldsymbol{R}^2:\ 0 < x \leqslant 1,\ 0 < y < \exp(-1/x)\}).$$

A sufficient (but not necessary) condition in order that X have WEP is that it satisfy the following *strong regularity condition* (see [2, Proposition 2.16]):

(SRC) There exist a positive integer k and a positive constant C such that any two points $x, y \in X$ can be joined by a rectifiable arc σ which lies in $\operatorname{int} X$ except perhaps for finitely many points, and satisfies $C|\sigma|^k \leqslant |x-y|$.

(That SRC is not necessary in order that X have WEP follows from an example given by Wachta [13].) In particular, if X is a fat subanalytic compact set in $\boldsymbol{R}^n$, then by Bierstone [2, Theorem 6.17] it satisfies SRC, and in this case the topologies τ_1, τ_2, τ_3, and τ_4 all coincide.

3. Lagrange interpolation polynomials. Let now $\mathscr{P}_k$ denote the vector space of all polynomials from $\boldsymbol{K}^n$ into $\boldsymbol{K}$ of degree at most k, where $\boldsymbol{K} = \boldsymbol{C}$ or $\boldsymbol{K} = \boldsymbol{R}$. Let

$$\varkappa:\ \{1, 2, \ldots\} \ni j \to \varkappa(j) = (\varkappa_1(j), \ldots, \varkappa_n(j)) \in \boldsymbol{Z}_+^n$$

be a one-to-one mapping such that for each j, $|\varkappa(j)| \leqslant |\varkappa(j+1)|$. Let m_k denote the number of monomials $x^\alpha := x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ of degree at most k. One can easily verify that $m_k = \binom{n+k}{k}$. Let $e_j(x) := x^{\varkappa(j)}$, for $j = 1, 2, \ldots$ The set of monomials $e_1, \ldots, e_{m_k}$ is a basis of the space $\mathscr{P}_k$.

Let $t^{(k)} = \{t_1, \ldots, t_k\}$ be a system of k points of $\boldsymbol{K}^n$. Consider the Vandermonde determinant

$$V(t^{(k)}) = V(t_1, \ldots, t_k) := \det[e_j(t_i)],$$

where $i, j \in \{1, \ldots, k\}$. If $V(t^{(k)}) \neq 0$, we define

$$L^{(j)}(x, t^{(k)}) := V(t_1, \ldots, t_{j-1}, x, t_{j+1}, \ldots, t_k)/V(t^{(k)}).$$

Since

$$L^{(j)}(t_i, t^{(k)}) = \delta_{ij} \quad \text{(Kronecker's symbol)},$$

we get the following *Lagrange interpolation formula* (cf. [8, Lemma 2.1]).

(LIF) *If $p \in \mathscr{P}_k$ and $t^{(m_k)}$ is a system of m_k points of $\boldsymbol{K}^n$ such that $V(t^{(m_k)}) \neq 0$, then*

$$p(x) = \sum_{j=1}^{m_k} p(t_j) L^{(j)}(x, t^{(m_k)}), \quad \textit{for } x \in \boldsymbol{K}^n.$$

Let X be a compact subset of $\boldsymbol{K}^n$. A system $t^{(k)}$ of k points $t_1, \ldots, t_k$ of X is called a *Fekete–Leja system of extremal points* of X of order k if $V(t^{(k)}) \geqslant V(s^{(k)})$ for all systems $s^{(k)} = \{s_1, \ldots, s_k\} \subset X$. Observe that if $t^{(k)}$ is a system of extremal points of X such that $V(t^{(k)}) \neq 0$, then

$$|L^{(j)}(x, t^{(k)})| \leqslant 1 \quad \text{on } X, \text{ for } j = 1, \ldots, k. \tag{3.1}$$

Following Siciak [8] we shall say that the set X is *unisolvent* if for each k and $p \in \mathscr{P}_k$, $p = 0$ on X implies $p = 0$ in $\boldsymbol{K}^n$. Let X be a unisolvent compact subset of $\boldsymbol{K}^n$ and let $f:\ X \to \boldsymbol{K}$. For each k, let $t^{(m_k)}$ be a system of extremal points of X of order m_k. Define

$$L_k f(x) = \sum_{j=1}^{m_k} f(t_j) L^{(j)}(x, t^{(m_k)}). \tag{3.2}$$

$L_k f$ is called the *Lagrange interpolation polynomial* of f of degree k.

Suppose f is continuous on X. Let p_k be any polynomial of degree k such that $\|f-p_k\|_X = \operatorname{dist}_X(f, \mathscr{P}_k)$. Then by LIF, (3.1) and (3.2), we have

$$\begin{aligned} \|f-L_k f\|_X &\leqslant \|f-p_k\|_X + \|L_k f - L_k p_k\|_X \\ &\leqslant (m_k+1)\|f-p_k\|_X \leqslant 4k^n \operatorname{dist}_X(f, \mathscr{P}_k). \end{aligned} \tag{3.3}$$

4. Extension operator. Let now X be a UPC compact subset of $\boldsymbol{R}^n$ and let u_k be the C^∞ functions defined in the proof of Proposition 2.2. Since $\operatorname{int} X \neq \emptyset$, the set X is unisolvent. Given $f \in C^\infty(X)$, let $L_k f$ be the Lagrange interpolation polynomial defined by (3.2). Then our extension operator is defined as follows:

$$Lf = u_1 L_1 f + \sum_{k=1}^{\infty} u_k(L_{k+1} f - L_k f). \tag{4.1}$$

The series (4.1) defines a C^∞ function on $\boldsymbol{R}^n$ the restriction of which to X is equal to f. For, if $\alpha \in \boldsymbol{Z}_+^n$, by Theorems 1.5 and 1.3, and by (3.3), we get (cf.

the proof of Proposition 2.2)

$$\sup_{\mathbf{R}^n}|D^\alpha Lf| \leqslant \sup_{X_1}|D^\alpha(u_1 L_1 f)| + \sum_{k=1}^{\infty}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\sup_{X_k}|D^\beta u_k D^{\alpha-\beta}(L_{k+1}f - L_k f)|$$
$$\leqslant M_1\|f\|_X + M_2 \sup_k k^{s|\alpha|+n+2}\operatorname{dist}_X(f, \mathscr{P}_k)$$
$$\leqslant M[d_{-1}(f) + d_{s|\alpha|+n+2}(f)],$$

where the constants M_1 and M_2 depend only on α, X and n, and $M = \max(M_1, M_2)$. Thus, by Proposition 2.2, we have proved

THEOREM 4.1. *If X is a UPC compact set in $\mathbf{R}^n$, then the operator* (4.1) *is a continuous linear operator from the space $(C^\infty(X), \tau_1)$ (or, what is the same, $(C^\infty(X), \tau_3)$, or else $(\mathscr{E}(X), \tau_1)$) to the space $C^\infty(\mathbf{R}^n)$ endowed with the natural topology τ_0.*

Remark 4.2. If X is not UPC, then in general there exists no continuous linear operator from $C^\infty(X)$ to $C^\infty(\mathbf{R}^n)$. By an example of Tidten [10], that is the case if $X = \{(x, y)\in\mathbf{R}^2\colon 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant\varphi(x)\}$, where φ is a C^∞ function infinitely flat at 0, and $\varphi(x) > 0$ as $x > 0$.

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Received November 28, 1986 (2249)

Added in proof (September 1987). If X is fat, it can also be shown that the following are equivalent: (i) Markov's inequality, (ii) Bernstein's theorem, (iii) the existence of a continuous linear operator $L\colon (C^\infty(X), \tau_3)\to(C^\infty(\mathbf{R}^n), \tau_0)$.