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Logarithmically concave functions and sections of convex sets in $\boldsymbol{R}^n$

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Abstract. We prove that if $f\colon \boldsymbol{R}^k \to [0, \infty)$ is an even logarithmically concave function on $\boldsymbol{R}^k$ and $p \geqslant 1$, then $\|x\| = \left(\int_0^\infty f(rx)\, r^{p-1}\, dr\right)^{-1/p}$ defines a seminorm on $\boldsymbol{R}^k$ (provided it is finite everywhere). This result is used to provide estimates on the least constant γ_k ($k \geqslant 1$) with the property that for $n \in N$, $n > k$, every n-dimensional normed space E has a representation on $\boldsymbol{R}^n$, with unit ball C, say, such that if H and K are k-codimensional subspaces of $\boldsymbol{R}^n$ then $|H \cap C|/|K \cap C| \leqslant \gamma_k$.

1. Introduction. In this paper we prove a number of related results concerning logarithmically concave functions and convex sets in $\boldsymbol{R}^n$.

Throughout the paper we will denote volume in a Euclidean space by $|\cdot|$ as well as using this symbol for the usual Euclidean norm of a vector.

The first of the principal results is that every logarithmically concave function on $\boldsymbol{R}^n$ naturally generates a collection of norms on $\boldsymbol{R}^n$ which can be used to simplify the study of the original function. This fact is then applied to a problem concerning the volumes of sections of convex sets. The first result is essentially an inequality which bears some similarity to those of Prékopa [6] and Leindler [5]. The latter problem was considered by Hensley [4] who proved that for each $k \geqslant 1$, there is a constant γ_k so that for every $n > k$, every n-dimensional normed space E has a representation on $\boldsymbol{R}^n$, with unit ball C, say, such that for any two k-codimensional subspaces H and K of $\boldsymbol{R}^n$,

$$\frac{|H \cap C|}{|K \cap C|} \leqslant \gamma_k.$$

By applying our first result we shall obtain a considerable improvement in the order of magnitude of the estimates for γ_k.

The case $k = 1$ of the above result was used by Bourgain [2] in obtaining inequalities for maximal functions of functions of several variables.

It should be remarked that these bounds on the ratio of volumes of sections of convex sets generalize, to the setting of arbitrary convex sets, results proved for the cube $[-\frac{1}{2}, \frac{1}{2}]^n \subset \boldsymbol{R}^n$ by Hensley [3], Vaaler [7] and the present author [1]. They can also be regarded as complementing the wellknown result of F. John on the Banach–Mazur distance of an n-dimensional normed space from l_2^n.

Hensley showed that there is a constant M such that if Q_n is the central unit cube in $\mathbf{R}^n$ then for every $(n-1)$-dimensional subspace H of $\mathbf{R}^n$,

$$1 \leqslant |H \cap Q_n| \leqslant M.$$

The best possible constant here, namely $\sqrt{2}$, was obtained in [1], in which there is also a simple proof of the lower bound 1. The paper of Vaaler extends the lower bound to sections of Q_n by subspaces of arbitrary dimension.

Given that for $n \geqslant 2$ there are obvious $(n-1)$-dimensional sections of Q_n with volumes 1 and $\sqrt{2}$, the fact that $1 \leqslant |H \cap Q_n| \leqslant \sqrt{2}$ for all H of dimension $n-1$ can be restated as follows: for any two $(n-1)$-dimensional subspaces H and K of $\mathbf{R}^n$, $n \geqslant 2$,

$$\frac{|H \cap Q_n|}{|K \cap Q_n|} \leqslant \sqrt{2}.$$

The classical result of John can also be stated in terms of volumes: every n-dimensional normed space E can be represented on $\mathbf{R}^n$ (i.e. given a Euclidean structure), with unit ball C, say, so that if H and K are any two 1-dimensional subspaces of $\mathbf{R}^n$ then

$$\frac{|H \cap C|}{|K \cap C|} \leqslant \sqrt{n}.$$

2. Definitions and preliminary results. Recall that a function $f\colon \mathbf{R}^k \to [0, \infty)$ is said to be *logarithmically concave* (log. concave) if the function $\log f\colon \mathbf{R}^k \to [-\infty, \infty)$ is concave (with the usual convention regarding $-\infty$).

The appearance of such functions in connection with convex sets is a consequence of the Brunn–Minkowski inequality which we state as a lemma.

LEMMA 1. *If A and B are nonempty measurable sets in $\mathbf{R}^n$ and $\lambda \in [0, 1]$ then*

$$|\lambda A + (1-\lambda) B|^{1/n} \geqslant \lambda |A|^{1/n} + (1-\lambda)|B|^{1/n}. \quad \blacksquare \tag{1}$$

The convexity of the exponential function yields as an immediate consequence the slightly weaker inequality

$$|\lambda A + (1-\lambda) B| \geqslant |A|^{\lambda} |B|^{1-\lambda}. \tag{2}$$

The following easy lemma is a restatement of inequality (2) which motivates the consideration of log. concave functions.

LEMMA 2. *Let C be a compact convex set in $\mathbf{R}^n$, H an $(n-k)$-dimensional subspace of $\mathbf{R}^n$ (for some $1 \leqslant k < n$) and $e_1, \ldots, e_k$ an orthonormal basis of $H^{\perp}$. Define $f\colon \mathbf{R}^k \to [0, \infty)$ by*

$$f(\lambda_1, \ldots, \lambda_k) = \left|(H + \sum \lambda_i e_i) \cap C\right|.$$

Then f is log. concave. $\blacksquare$

In all our results concerning the volumes of sections of unit balls of normed spaces we shall always use a particular representation of an n-dimensional normed space E on $\mathbf{R}^n$. To specify this we make the following definition.

DEFINITION. A measurable set $C \subset \mathbf{R}^n$ will be called *isotropic* if there is a constant M_C such that

$$\int_C \langle x, y\rangle^2 \, dm(y) = M_C |x|^2 \quad \text{for all } x \in \mathbf{R}^n$$

where m is the Lebesgue measure on $\mathbf{R}^n$, i.e. if C has isotropic inertia tensor. Similarly a function $f\colon \mathbf{R}^k \to [0, \infty)$ will be called *isotropic* if there is a constant M_f such that

$$\int_{\mathbf{R}^k} \langle x, y\rangle^2 f(y) \, dm(y) = M_f |x|^2 \quad \text{for all } x \in \mathbf{R}^k.$$

Plainly a convex set $C \subset \mathbf{R}^n$ is isotropic if and only if its characteristic function $\chi_C\colon \mathbf{R}^n \to [0, \infty)$ is isotropic.

We may reformulate the condition for isotropy in coordinate form as follows: a set C is isotropic if and only if there is a constant M_C such that

$$\int_C y_i y_j \, dm(y) = M_C \delta_{ij} \quad \text{for all } i \text{ and } j$$

where δ_{ij} is the Kronecker symbol. A similar reformulation may be made for functions $f\colon \mathbf{R}^k \to [0, \infty)$.

That any n-dimensional symmetric convex set C has a linear transformation which is isotropic may be observed as follows. Define an operator $T\colon \mathbf{R}^n \to \mathbf{R}^n$ by

$$Tx = \int_C \langle x, y\rangle y \, dm(y).$$

Then T is a strictly positive Hermitian operator and so has a strictly positive, and hence invertible, square root S, say. The symmetric convex set $S^{-1}C$ is isotropic since

$$\begin{aligned}\int_{S^{-1}C} \langle x, y\rangle^2 \, dm(y) &= \int_C \langle x, S^{-1}z\rangle^2 \, dm(S^{-1}z)\\ &= (\det S)^{-1} \int_C \langle x, S^{-1}z\rangle^2 \, dm(z)\\ &= (\det S)^{-1} \int_C \langle S^{-1}x, z\rangle^2 \, dm(z)\end{aligned}$$

$$= (\det S)^{-1} \langle S^{-1}x, \int_C \langle S^{-1}x, z\rangle z\,dm(z)\rangle$$
$$= (\det S)^{-1} \langle S^{-1}x, TS^{-1}x\rangle$$
$$= (\det S)^{-1} \langle x, S^{-1}TS^{-1}x\rangle$$
$$= (\det S)^{-1} |x|^2.$$

So any n-dimensional normed vector space has a representation on $\boldsymbol{R}^n$ with isotropic unit ball. Similarly, if $f\colon \boldsymbol{R}^k \to [0, \infty)$ is a function for which

$$\int_{\boldsymbol{R}^k} \langle x, y\rangle^2 f(y)\,dm(y) < \infty \quad \text{for all } x \in \boldsymbol{R}^k$$

then there is a linear transformation S such that the function $x \mapsto f(Sx)$ is isotropic.

Before proceeding to the principal results we shall prove some lemmas concerning log. concave functions on $\boldsymbol{R}$.

LEMMA 3. *Suppose $p \geqslant 0$, $\varphi\colon [0, \infty) \to [0, \infty)$ is convex with $\varphi(0) = 0$, $g\colon [0, \infty) \to [0, \infty)$ is decreasing and integrable and*

$$\int_0^\infty g(\varphi(x))x^p\,dx = \int_0^\infty g(x)x^p\,dx.$$

Then for all $t \geqslant 0$,

$$\int_t^\infty g(\varphi(x))x^p\,dx \leqslant \int_t^\infty g(x)x^p\,dx.$$

Proof. Fix $t \geqslant 0$. By the convexity of φ and the fact that g is decreasing we have

$$\int_t^\infty g(\varphi(x))x^p\,dx \leqslant \int_t^\infty g\left(\frac{\varphi(t)}{t}x\right)x^p\,dx = \left[\frac{t}{\varphi(t)}\right]^{p+1} \int_{\varphi(t)}^\infty g(x)x^p\,dx. \tag{3}$$

Similarly

$$\int_0^t g(\varphi(x))x^p\,dx \geqslant \left[\frac{t}{\varphi(t)}\right]^{p+1} \int_0^{\varphi(t)} g(x)x^p\,dx. \tag{4}$$

If $\varphi(t) \geqslant t$ then by (3),

$$\int_t^\infty g(\varphi(x))x^p\,dx \leqslant \int_{\varphi(t)}^\infty g(x)x^p\,dx \leqslant \int_t^\infty g(x)x^p\,dx.$$

If $\varphi(t) \leqslant t$ then let $G(r) = r^{-p-1}\int_0^r g(x)x^p\,dx$ for $r > 0$. Since g is decreasing on $[0, \infty)$, so is G, and hence by (4),

$$\int_0^t g(\varphi(x))x^p\,dx \geqslant t^{p+1}G(\varphi(t)) \geqslant t^{p+1}G(t) = \int_0^t g(x)x^p\,dx.$$

Combining this with the hypothesis we obtain

$$\int_t^\infty g(\varphi(x))x^p\,dx \leqslant \int_t^\infty g(x)x^p\,dx$$

as required. ∎

In the next lemma we convert the distributional inequality of Lemma 3 into an inequality between L_p-integrals.

LEMMA 4. *Let $f\colon [0, \infty) \to [0, \infty)$ be decreasing with $\log f$ concave. Then for $0 \leqslant p < q < \infty$,*

$$f(0)^q\,\Gamma(p+1)^{q+1}\Big[\int_0^\infty f(x)x^q\,dx\Big]^{p+1} \leqslant f(0)^p\,\Gamma(q+1)^{p+1}\Big[\int_0^\infty f(x)x^p\,dx\Big]^{q+1}.$$

Proof. We may write $f(x) = f(0)e^{-\psi(x)}$ where ψ is convex, increasing on $[0, \infty)$ and $\psi(0) = 0$. Let $g(x) = e^{-\lambda x}$ where

$$\lambda = \Big[f(0)\,\Gamma(p+1)\Big/\int_0^\infty f(x)x^p\,dx\Big]^{1/(p+1)}$$

is chosen so that

$$\int_0^\infty e^{-\psi(x)}x^p\,dx = \int_0^\infty e^{-\lambda x}x^p\,dx.$$

Then $f(x)/f(0) = g(\lambda^{-1}\psi(x))$ and $\int_0^\infty g(\lambda^{-1}\psi(x))x^p\,dx = \int_0^\infty g(x)x^p\,dx$. Hence, taking $\varphi = \lambda^{-1}\psi$ and applying Lemma 3 we obtain

$$\int_t^\infty f(x)x^p\,dx \leqslant f(0)\int_t^\infty e^{-\lambda x}x^p\,dx \quad \text{for all } t \leqslant 0.$$

But since for any function $h \geqslant 0$ we may write

$$\int_0^\infty h(x)x^q\,dx = (q-p)\int_0^\infty t^{q-p-1}\int_t^\infty h(x)x^p\,dx\,dt$$

this inequality gives

$$\int_0^\infty f(x)x^q\,dx \leqslant f(0)\int_0^\infty e^{-\lambda x}x^q\,dx = f(0)\,\Gamma(q+1)/\lambda^{q+1},$$

and the desidred result follows by substituting for λ. ∎

3. Norms generated by log. concave functions. This section is devoted to the principal theorem concerning log. concave functions.

THEOREM 5. *Suppose $f\colon \mathbf{R}^k \to [0, \infty)$ is an even log. concave function with $0 < \int_{\mathbf{R}^k} f < \infty$ and $p \geqslant 1$. Then*

$$\|x\| = \begin{cases} \left[\int_0^\infty f(rx)\, r^{p-1}\, dr\right]^{-1/p}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

defines a norm on $\mathbf{R}^k$.

Proof. It is easily checked that $0 < \|x\| < \infty$ if $x \neq 0$ by virtue of the facts that f is log. concave and $0 < \int f < \infty$. The homogeneity of $\|\cdot\|$ is a consequence of the evenness of f and the form of the integral.

So it suffices to show that if $x, y \neq 0$,

$$\|\tfrac{1}{2}(x+y)\| \leqslant \tfrac{1}{2}\|x\| + \tfrac{1}{2}\|y\|.$$

Define functions $g, h, m\colon [0, \infty) \to [0, \infty)$ by

$$g(r) = f(rx), \qquad h(r) = f(ry), \qquad m(r) = f(\tfrac{1}{2} r(x+y)),$$

and let

$$(5) \qquad A = \left[\int_0^\infty g(r)\, r^{p-1}\, dr\right]^{-1/p}, \quad B = \left[\int_0^\infty h(r)\, r^{p-1}\, dr\right]^{-1/p}, \quad C = \left[\int_0^\infty m(r)\, r^{p-1}\, dr\right]^{-1/p}.$$

We wish to show that $2/C \leqslant 1/A + 1/B$, i.e. that

$$(6) \qquad C \geqslant \frac{2AB}{A+B}.$$

Now if $r, s, t \in (0, \infty)$ and $2/t = 1/r + 1/s$ then with $\lambda = s/(r+s)$ we have $\frac{1}{2}t = \lambda r = (1-\lambda)s$ and hence $\frac{1}{2}t(x+y) = \lambda rx + (1-\lambda)sy$. Since f is log. concave, $f(\frac{1}{2}t\,(x+y)) \geqslant f(rx)^\lambda f(sy)^{1-\lambda}$, i.e.

$$m(t) \geqslant g(r)^\lambda h(s)^{1-\lambda}.$$

So g, h and m satisfy the inequality

$$(7) \qquad m(t) \geqslant \sup\{g(r)^{s/(r+s)} h(s)^{r/(r+s)}\colon\ 1/r + 1/s = 2/t\}.$$

We claim that for any functions g, h, m on $[0, \infty)$ satisfying (7), inequality (6) holds where A, B and C are defined by (5). To prove this claim, we may assume that g, h and m are bounded, have compact support in $(0, \infty)$ and are not almost everywhere zero.

Let θ be the number defined by

$$\sup_r g(r)\, r^{p+1} = \theta^{p+1} \sup_r h(r)\, r^{p+1}$$

and define $G, H\colon (0, \infty) \to [0, \infty)$ by

$$G(u) = g(u^{-1})\, u^{-p-1}, \qquad H(u) = h(\theta^{-1} u^{-1})\, u^{-p-1}.$$

Then $\sup G = \sup H = a$, say, and we have

$$\int_0^\infty G(u)\, du = \int_0^\infty g(u^{-1})\, u^{1-p} \frac{du}{u^2} = \int_0^\infty g(r)\, r^{p-1}\, dr = A^p,$$

$$\int_0^\infty H(u)\, du = (\theta B)^p.$$

Now suppose $u, v, w \in (0, \infty)$ and $u + \theta v = 2w$. Then

$$\begin{aligned} G(u)^{u/(u+\theta v)} H(v)^{\theta v/(u+\theta v)} &= g(u^{-1})^{u/(u+\theta v)} h(\theta^{-1} v^{-1})^{\theta v/(u+\theta v)} \left[\left[\frac{1}{u}\right]^{u/(u+\theta v)} \left[\frac{1}{v}\right]^{\theta v/(u+\theta v)}\right]^{p+1} \\ &= g(r)^{s/(r+s)} h(s)^{r/(r+s)} \left[r^{s/(r+s)} (\theta s)^{r/(r+s)}\right]^{p+1} \end{aligned}$$

where $r = 1/u$, $s = 1/(\theta v)$.

Now setting $\lambda = s/(r+s)$ we have $r^\lambda (\theta s)^{1-\lambda} \leqslant \lambda r + (1-\lambda)\theta s$ and so the above expression is dominated by

$$\begin{aligned} g(r)^{s/(r+s)} h(s)^{r/(r+s)} \left[(1+\theta)\frac{rs}{r+s}\right]^{p+1} &= g(r)^{s/(r+s)} h(s)^{r/(r+s)} \left[\frac{1+\theta}{2}\right]^{p+1} \frac{1}{w^{p+1}} \\ &\leqslant \left[\frac{1+\theta}{2}\right]^{p+1} m(w^{-1}) \frac{1}{w^{p+1}}, \end{aligned}$$

since $1/r + 1/s = 2/w^{-1}$.

So defining $M\colon (0, \infty) \to [0, \infty)$ by

$$M(w) = \left[\frac{1+\theta}{2}\right]^{p+1} m(w^{-1})\, w^{-p-1},$$

we have

$$M(w) \geqslant \sup\{G(u)^{u/(u+\theta v)} H(v)^{\theta v/(u+\theta v)}\colon\ u + \theta v = 2w\},$$

$$\int_0^\infty M(w)\, dw = \left[\frac{1+\theta}{2}\right]^{p+1} C^p.$$

Now suppose $0 \leqslant z < a = \sup G = \sup H$. If u and v are such that

$G(u)$, $H(v) \geqslant z$, then $M(\frac{1}{2}(u+\theta v)) \geqslant z$. Hence

$$\{w\colon M(w) \geqslant z\} \supset \tfrac{1}{2}\{u\colon G(u) \geqslant z\} + \tfrac{1}{2}\theta\{v\colon H(v) \geqslant z\}$$

(where the addition and scalar multiplication of sets is performed in the usual Minkowski fashion). Since both sets on the right-hand side are nonempty, we may apply the (1-dimensional) Brünn–Minkowski inequality to obtain

$$\mu(\{w\colon M(w) \geqslant z\}) \geqslant \tfrac{1}{2}\mu(\{u\colon G(u) \geqslant z\}) + \tfrac{1}{2}\theta\mu(\{v\colon H(v) \geqslant z\})$$

where μ is the Lebesgue measure on the line. Hence

$$\int_0^\infty M(w)\,dw = \int_0^a \mu(M \geqslant z)\,dz \geqslant \tfrac{1}{2}\int_0^a \mu(G \geqslant u)\,du + \tfrac{1}{2}\theta\int_0^a \mu(H \geqslant v)\,dv$$

$$= \tfrac{1}{2}\int_0^\infty G(u)\,du + \tfrac{1}{2}\theta\int_0^\infty H(v)\,dv.$$

Equivalently, in terms of A, B and C:

$$\left[\frac{1+\theta}{2}\right]^{p+1} C^p \geqslant \tfrac{1}{2}A^p + \tfrac{1}{2}\theta(\theta B)^p.$$

So

$$C^p \geqslant \frac{2^p}{(1+\theta)^{p+1}}(A^p + \theta(\theta B)^p) = \left[\frac{2}{1+\theta}\right]^p \left[\frac{A^p + \theta(\theta B)^p}{1+\theta}\right]$$

$$\geqslant \left[\frac{2}{1+\theta}\right]^p \left[\frac{A + \theta(\theta B)}{1+\theta}\right]^p,$$

the last inequality being a consequence of Hölder's inequality since $p \geqslant 1$. Hence

$$C \geqslant \frac{2}{1+\theta}\left[\frac{A+\theta^2 B}{1+\theta}\right] = \frac{2AB}{A+B} + \frac{2(A-\theta B)^2}{(A+B)(1+\theta)^2} \geqslant \frac{2AB}{A+B}$$

which is the desired inequality (6). ∎

4. Volumes of sections of convex sets. The principal result in this section will be deduced from the existence of bounds (depending only on k) on the expression

$$f(0)^{2/k} \int_{\mathbf{R}^k} |x|^2 f(x)\,dm(x)$$

valid for all isotropic even log. concave functions $f\colon \mathbf{R}^k \to [0, \infty)$ which satisfy

$$\int_{\mathbf{R}^k} f(x)\,dm(x) = 1.$$

Note that an even log. concave function f attains its maximum value at 0 since for any $x \in \mathbf{R}^k$,

$$f(0) \geqslant \sqrt{f(x)}\sqrt{f(-x)} = f(x).$$

The lower bound on the above expression will depend only on this fact and has been observed by Hensley [4]. For completeness we include a simple proof.

LEMMA 6. *Let $f\colon \mathbf{R}^k \to [0, \infty)$ be measurable with*

$$\int_{\mathbf{R}^k} f(x)\,dm(x) = 1, \quad f(x) \leqslant f(0) \quad \text{for all } x \in \mathbf{R}^k.$$

Then

$$f(0)^{1/k}\Big[\int_{\mathbf{R}^k} |x|^2 f(x)\,dm(x)\Big]^{1/2} \geqslant \frac{\sqrt{k}}{\sqrt{k+2}\cdot v_k^{1/k}}$$

where v_k is the volume of the Euclidean ball of radius 1 *in $\mathbf{R}^k$.*

Proof. Define a probability P on $\mathbf{R}^k$ by

$$P(A) = \int_A f\,dm.$$

Then since $f(x) \leqslant f(0)$ for all x, $P(|x| \leqslant t) \leqslant v_k t^k f(0)$, for all $t \geqslant 0$. Define $F\colon \mathbf{R} \to [0, 1]$ by

$$F(t) = \begin{cases} 0, & t \leqslant 0, \\ v_k t^k f(0), & 0 \leqslant t \leqslant s, \\ 1, & s \leqslant t, \end{cases}$$

where $s = (v_k f(0))^{-1/k}$. Then $P(|x| \leqslant t) \leqslant F(t)$ for all t and so

$$\int_{\mathbf{R}^k} |x|^2 f(x)\,dm(x) = 2\int_0^\infty tP(|x| \geqslant t)\,dt$$

$$\geqslant 2\int_0^\infty t(1-F(t))\,dt = 2\int_0^s t(1 - v_k t^k f(0))\,dt$$

$$= s^2 - v_k f(0)\frac{2}{k+2}s^{k+2} = \frac{k}{k+2}(v_k f(0))^{-2/k}$$

as required. ∎

We now move to the problem of obtaining a reverse inequality. It is readily checked that, for $k \geqslant 1$, the expression

$$f(0)^{1/k}\Big[\int_{\mathbf{R}^k} |x|^2 f(x)\,dm(x)\Big]^{1/2}$$

may be arbitrarily large for even log. concave functions satisfying

$$\int_{R^k} f\,dm = 1,$$

so for us to obtain a uniform bound, some extra condition on f is required. Now for a positive linear map S of determinant 1, the function $f_S = f\circ S$ is even and log. concave with

$$\int_{R^k} f_S\,dm = 1.$$

It is elementary that the expression

$$f_S(0)^{1/k}\big[\int_{R^k} |x|^2 f_S(x)\,dm(x)\big]^{1/2}$$

is minimized over positive linear maps S with determinant 1 when S is chosen so that f_S is isotropic. The isotropy condition is thus the one desired.

With regard to obtaining our bounds we define the following two constants. Let A_k be the least number such that for any isotropic symmetric convex set C in R^k,

$$\int_C |x|^2\,dm(x) \leqslant kA_k|C|^{1+2/k}, \tag{8}$$

and let B_k be the least number such that for any isotropic even log. concave function f: $R^k\to[0,\infty)$,

$$f(0)^{2/k}\int_{R^k} |x|^2 f(x)\,dm(x) \leqslant kB_k\big[\int_{R^k} f(x)\,dm(x)\big]^{1+2/k}. \tag{9}$$

Observe that (8) may be rewritten

$$v_k\int_{S^{k-1}} \|\theta\|^{-k-2}\,d\sigma_{k-1}(\theta) \leqslant (k+2)A_k\big[v_k\int_{S^{k-1}} \|\theta\|^{-k}\,d\sigma_{k-1}(\theta)\big]^{1+2/k} \tag{10}$$

(where σ_{k-1} is the normalized rotation invariant measure on the unit sphere S^{k-1} in R^k) for any norm on R^k which gives rise to an isotropic unit ball. Theorem 5 and Lemma 4 together determine precisely the relationship between A_k and B_k.

THEOREM 7. *With the above notation*

$$B_k = \frac{(k+1)(k+2)}{(k!)^{2/k}}A_k < e^2 A_k.$$

Proof. It is easily checked that

$$B_k \geqslant \frac{(k+1)(k+2)}{(k!)^{2/k}}A_k$$

using the fact that for a norm $\|\cdot\|$ on R^k, the function f: $R^k\to[0,\infty)$ defined by $f(x)=\exp(-\|x\|)$ is log. concave. (This will also be clear from our proof of the reverse inequality and the fact that there is equality in Lemma 4 for functions of the form $t\mapsto e^{-\lambda t}$ on $[0,\infty)$.)

For the reverse inequality, suppose that f: $R^k\to[0,\infty)$ is an isotropic even log. concave function. Using Theorem 5 define a norm on R^k by

$$\|x\| = \big[(k+2)\int_0^\infty f(rx)r^{k+1}\,dr\big]^{-1/(k+2)} \quad \text{for } x\neq 0.$$

It is an immediate consequence of the isotropy of f that this norm yields an isotropic unit ball and hence satisfies inequality (10). So

$$\begin{aligned}
f(0)^{2/k}\int_{R^k} |x|^2 f(x)\,dm(x) &= f(0)^{2/k}kv_k\int_{S^{k-1}}\int_0^\infty f(r\theta)r^{k+1}\,dr\,d\sigma_{k-1}(\theta)\\
&= f(0)^{2/k}\frac{kv_k}{k+2}\int_{S^{k-1}} \|\theta\|^{-k-2}\,d\sigma_{k-1}(\theta)\\
&\leqslant kA_k f(0)^{2/k}\big[v_k\int_{S^{k-1}} \|\theta\|^{-k}\,d\sigma_{k-1}\big]^{1+2/k}\\
&= kA_k f(0)^{2/k}\Big(v_k\int_{S^{k-1}}\big[(k+2)\int_0^\infty f(r\theta)r^{k+1}\,dr\big]^{k/(k+2)}d\sigma_{k-1}\Big)^{(k+2)/k}\\
&= k(k+2)A_k\Big(v_k\int_{S^{k-1}}\big(f(0)^{2/k}\int_0^\infty f(r\theta)r^{k+1}\,dr\big)^{k/(k+2)}d\sigma_{k-1}\Big)^{(k+2)/k}.
\end{aligned}$$

Now, the function $r\mapsto f(r\theta)$ is decreasing and log. concave on $[0,\infty)$ and so by Lemma 4 with $p=k-1$ and $q=k+1$ we have

$$f(0)^{2/k}\int_0^\infty f(r\theta)r^{k+1}\,dr \leqslant \frac{k(k+1)}{((k-1)!)^{2/k}}\Big(\int_0^\infty f(r\theta)r^{k-1}\,dr\Big)^{(k+2)/k}.$$

Hence

$$\begin{aligned}
&f(0)^{2/k}\int_{R^k} |x|^2 f(x)\,dm(x)\\
&\leqslant k(k+2)A_k\left(v_k\int_{S^{k-1}}\left(\frac{k(k+1)}{((k-1)!)^{2/k}}\right)^{k/(k+2)}\int_0^\infty f(r\theta)r^{k-1}\,dr\,d\sigma_{k-1}\right)^{(k+2)/k}\\
&= \frac{k(k+1)(k+2)A_k}{(k!)^{2/k}}\Big(kv_k\int_{S^{k-1}}\int_0^\infty f(r\theta)r^{k-1}\,dr\,d\sigma_{k-1}\Big)^{(k+2)/k}\\
&= \frac{k(k+1)(k+2)A_k}{(k!)^{2/k}}\Big(\int_{R^k} f(x)\,dm(x)\Big)^{1+2/k}.
\end{aligned}$$

Hence

$$B_k \leqslant \frac{(k+1)(k+2)}{(k!)^{2/k}}A_k. \ \blacksquare$$

In the next lemma we obtain a simple upper bound on A_k. The resulting bound on B_k is then used in Theorem 9 to obtain an improvement of Hensley's result on volumes of sections of convex sets. For the purposes of this lemma it is most convenient to drop the isotropy condition on the convex set C and deal instead with an expression which is invariant under linear transformations and reduces to

$$\int_C |x|^2\, dm(x)$$

when C is isotropic.

LEMMA 8. *If C is a symmetric convex set in $\mathbf{R}^k$ with $|C| = 1$ then*

$$\int_C \ldots \int_C |\operatorname{conv}\{\pm x_1, \ldots, \pm x_k\}|^2 \prod_{i=1}^{k} dm(x_i) \leqslant \left(\frac{k}{k+2}\right)^k,$$

where $\operatorname{conv}\{\pm x_1, \ldots, \pm x_k\}$ *is the convex hull of the* $2k$ *vectors* $\pm x_1, \ldots, \pm x_k$.

In consequence,

$$A_k \leqslant \frac{k(k!)^{1/k}}{4(k+2)}.$$

Proof. Let $\|\cdot\|$ be the norm on $\mathbf{R}^k$ determined by C. Suppose $\theta_1, \ldots, \theta_k \in S^{k-1}$. Then the set

$$\operatorname{conv}\left\{\pm\frac{\theta_1}{\|\theta_1\|}, \ldots, \pm\frac{\theta_k}{\|\theta_k\|}\right\}$$

is a subset of C and hence has volume at most 1. Therefore

$$\frac{|\operatorname{conv}\{\pm\theta_1, \ldots, \pm\theta_k\}|^2}{\prod_i \|\theta_i\|^2} \leqslant 1.$$

Hence

$$\begin{aligned}
&\int_C \ldots \int_C |\operatorname{conv}\{\pm x_1, \ldots, \pm x_k\}|^2 \prod dm(x_i)\\
&= \left[\frac{k}{k+2}\right]^k v_k^k \int_{(S^{k-1})^k} \frac{|\operatorname{conv}\{\pm\theta_1, \ldots, \pm\theta_k\}|^2}{\prod \|\theta_i\|^{k+2}} \prod d\sigma_{k-1}(\theta_i)\\
&\leqslant \left[\frac{k}{k+2}\right]^k v_k^k \int_{(S^{k-1})^k} \|\theta_i\|^{-k} \prod d\sigma_{k-1}(\theta_i)\\
&= \left[\frac{k}{k+2}\right]^k |C|^k = \left[\frac{k}{k+2}\right]^k.
\end{aligned}$$

This completes the first part of the lemma.

Now suppose C is isotropic. For $x_1, \ldots, x_k \in C$,

$$|\operatorname{conv}\{\pm x_1, \ldots, \pm x_k\}| = \frac{2^k}{k!}|\det X|$$

where $X = (x_{ij})$ is the matrix with entries $x_{ij} = (x_i)_j$. So

$$\begin{aligned}
I &= \int_{C^k} |\operatorname{conv}\{\pm x_1, \ldots, \pm x_k\}|^2 \prod dm(x_i)\\
&= \frac{4^k}{(k!)^2} \sum_{\sigma\in S_k}\sum_{\tau\in S_k} \varepsilon(\sigma\tau) \int_{C^k} \prod (x_i)_{\sigma(i)}(x_i)_{\tau(i)} \prod dm(x_i).
\end{aligned}$$

If σ and τ are distinct permutations in the symmetric group S_k then there is an index i for which $\sigma(i) \neq \tau(i)$ and hence by the coordinate form of the isotropy condition

$$\int_C (x_i)_{\sigma(i)}(x_i)_{\tau(i)}\, dm(x_i) = 0.$$

Hence

$$\begin{aligned}
I &= \frac{4^k}{(k!)^2} \sum_{\sigma\in S_k} \int_{C^k} \prod (x_i)^2_{\sigma(i)} \prod dm(x_i)\\
&= \frac{4^k}{(k!)^2} \sum_{\sigma\in S_k} \Big(k^{-1}\int_C |x|^2\, dm(x)\Big)^k = \frac{4^k}{k!}\Big(k^{-1}\int_C |x|^2\, dm(x)\Big)^k.
\end{aligned}$$

Hence by the first part of the lemma,

$$k^{-1}\int_C |x|^2\, dm(x) \leqslant \frac{(k!)^{1/k}}{4}\left(\frac{k}{k+2}\right).$$

The fact that this holds for an arbitrary isotropic symmetric convex set C of volume 1 is exactly the statement that

$$A_k \leqslant \frac{(k!)^{1/k}k}{4(k+2)}. \ \blacksquare$$

THEOREM 9. *If C is an isotropic symmetric convex set in $\mathbf{R}^n$, $k < n$ and H and K are k-codimensional subspaces of $\mathbf{R}^n$ then*

$$\frac{|H\cap C|}{|K\cap C|} \leqslant \frac{(k(k+1)(k+2))^{k/2} v_k}{2^k (k!)^{1/2}} < (\tfrac{1}{2}\pi e^2 k)^{k/2}.$$

Proof. Assume without loss of generality that $|C| = 1$. Let M be such that

$$\int_C \langle a, x\rangle^2\, dm(x) = M^2|a|^2 \quad \text{for all } a\in\mathbf{R}^n.$$

For H a k-codimensional subspace of $\mathbf{R}^n$, let $e_1, \ldots, e_k$ be an orthonormal basis of $H^\perp$ and define $f = f_H$: $\mathbf{R}^k \to [0, \infty)$ by

$$f(\lambda_1, \ldots, \lambda_k) = \left|\left(H + \sum \lambda_i e_i\right) \cap C\right|.$$

Then f is an isotropic even log. concave function with

$$\int_{\mathbf{R}^k} f(x)\,dm(x) = |C| = 1, \quad f(0) = |H \cap C|.$$

By Lemma 6 we have

$$f(0)^{1/k} \sqrt{k} \cdot M \geqslant \frac{\sqrt{k}}{\sqrt{k+2}\, v_k^{1/k}},$$

i.e.

$$|H \cap C| \cdot M^k \geqslant \frac{1}{(k+2)^{k/2} v_k}. \tag{11}$$

By Theorem 7 and Lemma 8

$$f(0)^{2/k} M^2 \leqslant B_k = \frac{(k+1)(k+2)}{(k!)^{2/k}} A_k \leqslant \frac{k(k+1)}{4(k!)^{1/k}},$$

i.e.

$$|H \cap C| \cdot M^k \leqslant \frac{(k(k+1))^{k/2}}{2^k (k!)^{1/2}}. \tag{12}$$

Now since M is independent of the original choice of subspace H, we may combine inequalities (11) and (12) to conclude that for any two k-codimensional subspaces H and K,

$$\frac{|H \cap C|}{|K \cap C|} \leqslant \frac{(k(k+1)(k+2))^{k/2} v_k}{2^k (k!)^{1/2}}. \ \blacksquare$$

5. Concluding remarks. We remark that using the full strength of the Brunn–Minkowski inequality, we find that the function f defined in the proof of Theorem 9 has the property that $f^{1/(n-k)}$ is concave on its support. The functions $r \mapsto f(rx)$ then also have this property and applying Lemma 3 to the function g given by $g(x) = (1-\lambda x)^{n-k}$ for suitable λ, instead of the function $x \mapsto e^{-\lambda x}$, we may obtain a slightly stronger inequality between the expressions

$$\int_0^\infty x^{k+1} f(rx)\,dx \quad \text{and} \quad \int_0^\infty x^{k-1} f(rx)\,dx$$

than that used in Theorem 7. This in turn gives a slightly better constant in

Theorem 9. In particular, for $k = 1$ we may obtain the bound

$$\frac{\sqrt{6} \cdot n}{\sqrt{(n+1)(n+2)}} < \sqrt{6}.$$

Thus for $n = 2$ the bound is $\sqrt{2}$ and we recover the result of John in this very special case.

It is natural to ask whether we can find absolute bounds on the volumes of sections of isotropic convex sets (of a given volume) rather than bounding them relative to one another. As an immediate consequence of the above we have, for example:

PROPOSITION 10. *Let C be an isotropic symmetric convex set in $\mathbf{R}^n$ of volume* 1. *Then for any $(n-1)$-dimensional subspace H of $\mathbf{R}^n$,*

$$\left(\frac{2}{3n}\right)^{1/2} < |H \cap C| < \sqrt{\pi e}.$$

Proof. By the $k = 1$ case of (11) and (12), we have

$$\frac{1}{2\sqrt{3}} \leqslant |H \cap C| \cdot M \leqslant \frac{1}{\sqrt{2}}$$

where $M|a| = \left(\int_C \langle a, x\rangle^2\,dm(x)\right)^{1/2}$ for all $a \in \mathbf{R}^n$. But by Lemma 6 and Lemma 8

$$\frac{1}{\sqrt{n+2} \cdot v_n^{1/n}} \leqslant M \leqslant \left[\frac{n}{n+2}\right]^{1/2} \frac{(n!)^{1/(2n)}}{2}.$$

Combining these we obtain

$$\left(\frac{2}{3n}\right)^{1/2} < \frac{1}{\sqrt{3}} \left(\frac{n+2}{n}\right)^{1/2} \frac{1}{(n!)^{1/(2n)}} \leqslant |H \cap C| \leqslant \frac{\sqrt{n+2} \cdot v_n^{1/n}}{\sqrt{2}} < \sqrt{\pi e}. \ \blacksquare$$

The lower bound in the above result, depending as it does on n, seems likely to be far from best possible. An improvement would result from (and imply) an improvement in the bound for A_k given in Lemma 8. Such an improvement would automatically transfer to give stronger estimates in Theorem 9. We conjecture that in fact $(A_k)_k$ is bounded by some constant; possibly the number $A_1 = \frac{1}{12}$.

Using Theorem 9, this conjecture may be reformulated in a number of ways. The following are examples:

1) There is a constant $\delta > 0$ such that for every symmetric convex set C in $\mathbf{R}^n$ of volume 1 there is a 1-codimensional subspace H of $\mathbf{R}^n$ such that $|H \cap C| > \delta$.

2) There is a constant $\delta > 0$ such that if C is an isotropic symmetric

convex set in $\mathbf{R}^n$ of volume 1 then for every 1-codimensional subspace H of $\mathbf{R}^n$, $|H \cap C| > \delta$.

3) There is a constant M such that if C is an isotropic symmetric convex set in $\mathbf{R}^n$ of volume 1 then

$$m(C \cap B(M\sqrt{n})) \geqslant \tfrac{1}{2},$$

where $B(M\sqrt{n})$ is the Euclidean ball of radius $M\sqrt{n}$.

4) There is a constant M such that for every symmetric convex set C in $\mathbf{R}^n$ of volume 1 there is an ellipsoid $\mathscr{E}$ of volume at most M^n such that

$$|\mathscr{E} \cap C| \geqslant \tfrac{1}{2}.$$

We may remark that such bounds do hold uniformly for the unit balls of spaces with a 1-unconditional basis. This follows from the observation that such a space can be represented on $\mathbf{R}^n$ with an isotropic unit ball C, say, and with the unconditional basis vectors orthogonal. In this situation the section of C perpendicular to a basis vector is also the projection of C onto the orthogonal complement of that vector.

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On the strong maximal function and rearrangements

by

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Abstract. We provide sufficient conditions for almost everywhere finiteness, integrability and membership in weak L^1 of the strong maximal function on $\mathbf{T}^2$. These are the weakest possible conditions which are invariant under all measure-preserving transformations of $\mathbf{T}^2$ which preserve the product structure. We also give examples showing that the conditions are not necessary.

1. Introduction. There are many points of contact between probability theory and harmonic analysis. One of the more striking concerns the connections between the Hardy–Littlewood maximal operator and its probabilistic counterpart. In this paper we explore similar connections between the strong maximal operator and a two-parameter probabilistic maximal operator. The differences between the two maximal operators are related to their behavior relative to rearrangements which preserve the product structure.

Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables on some probability space $(\Omega, \mathfrak{F}, P)$. Suppose also each X_i has a uniform on $[0, 1)$ $(U(0, 1))$ distribution. For Borel functions f on $[0, 1)$ let

$$s_n(f) = \sum_{i=1}^{n} f(X_i), \qquad s^*(f) = \sup_{1 \leqslant n < \infty} (|s_n(f)|/n).$$

Then by classical results of Khinchin and Kolmogorov and of Marcinkiewicz and Zygmund we have

$$s^*(f) < \infty \text{ a.s.} \quad \text{if and only if} \quad \|f\|_1 < \infty, \tag{1.1}$$

and

$$Es^*(f) \approx |f|_{L\log_+ L} := \int_0^1 |f(x)| \left(1 + \log_+ \frac{|f(x)|}{\|f(x)\|_1}\right) dx. \tag{1.2}$$

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